Solve Homogeneous 2nd order DE

Let's say our equation is $y'' + By' + Cy = 0$

We assume equation is of the form e^{rx}.

Thus, $y' = re^{rx}$, $y'' = r^2 e^{rx}$, so $r^2 e^{rx} + Bre^{rx} + Ce^{rx} = 0$

Divide out by 0 because $e^{rx} \neq 0$ for all r, x

$r^2 + Br + C = 0$. We have a quadratic! yay

Three cases:

If the discriminant $\sqrt{B^2 - 4ac} > 0$: Two roots r_1, r_2

Discriminant $= 0$, 1 root, Discriminant < 0, Two complex roots

Case 1: Solution is $Ce^{r_1 x} + De^{r_2 x}$

Case 2: $y_1 = e^{rx}$. Find y_2 by using $y_2 = y_1 \cdot u(x)$ and plugging in

Case 3: Let the roots be $a \pm Bi$. Sol is $e^{ax}[C\cos(Bx) + D\sin(Bx)]$

For case 2, $y_2 = uy_1$, $y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

Plug into $y'' + By' + Cy = 0$

Should get something in the form without u

So integrate both sides using sep or first order linear ü

Solve Non-homogeneous Eq

Given the DE: $y'' - 4y' - 12y = xe^{4x}$

Solve homogenous $y'' - 4y' - 12y = 0$, so $r^2 - 4r - 12 = 0$, $(r-6)(r+2) = 0$

The solution to the homogeneous equation is called y complementary

$y_c = Ce^{6x} + De^{-2x}$. Now we need to guess a y particular.

For complex RHS terms, like xe^{4x}, we solve the two parts differently

y_p solves $e^{4x}(Ax+B)$, we find values of A, B ...

How?

Assume $y = y_p$, so then $y' = 4e^{4x}(Ax+B) + e^{4x} \cdot A$

$y'' = 16e^{4x}(Ax+B) + 4Ae^{4x} + 4Ae^{4x} = 16Axe^{4x} + 16Be^{4x} + 8Ae^{4x}$

Plug in: $16Axe^{4x} + 16Be^{4x} + 8Ae^{4x} - 4[4Axe^{4x} + 4Be^{4x} - Ae^{4x}]$
$-12[Axe^{4x} + Be^{4x}]$

Simplify: $16\cancel{Ax}e^{4x} + 16\cancel{B}e^{4x} + 8Ae^{4x} \cancel{-4[4Axe^{4x}} + 4\cancel{Be^{4x}}] - Ae^{4x}$
$-12[Axe^{4x} + Be^{4x}]$

$4Ae^{4x} - 12Axe^{4x} - 12Be^{4x} = xe^{4x} + 0e^{4x}$

$-12Axe^{4x} = xe^{4x}$, $A = -\frac{1}{12}$

$4 \cdot \frac{-1}{12} e^{4x} - 12Be^{4x} = 0$, $\frac{-4}{12} - 12B = 0$

$-4 - 144B = 0$, $B = \frac{-1}{36}$

Since solution is $y_p = e^{4x}(Ax+B)$, $e^{4x}\left(-\frac{1}{12}x - \frac{1}{36}\right)$

And $y_{general} = y_{comp} + y_{hom}$

$$y = Ce^{6x} + De^{-2x} + e^{4x}\left(-\frac{1}{12}x - \frac{1}{36}\right)$$

Variation of Parameters. :)

1. Solve Homogeneous solution
2. y_p is of the form $y_p = u_1 y_1 + u_2 y_2$, assume $u_1' y_1 + u_2' y_2 = 0$
3. generate y_p', y_p'', plug into original

In our example... $y'' - 4y' - 12y = xe^{4x}$

$y_c = Ce^{6x} + De^{-2x}$

Let $y_1 = e^{6x}$, $y_2 = e^{-2x}$, $y_p = u_1 e^{6x} + u_2 e^{-2x}$

$y_p' = \cancel{u_1' e^{6x}} + 6u_1 e^{6x} + \cancel{u_2'} e^{-2x} - 2u_2 e^{-2x} = 6u_1 e^{6x} - 2u_2 e^{-2x}$

$y_p'' = 6u_1' e^{6x} + 36u_1 e^{6x} - 2u_2' e^{-2x} + 4u_2' e^{-2x}$

Plug in: $6u_1' e^{6x} - 2u_2' e^{-2x} + 36u_1 e^{6x} + 4u_2' e^{-2x} - 24u_1 e^{6x} + 8u_2 e^{-2x}$
$- 12u_1 e^{6x} - 12u_2 e^{-2x}$

$xe^{4x} = 6u_1' e^{6x} - 2u_2' e^{-2x}$
$0 = u_1' e^{6x} + u_2' e^{-2x}$ → Linear system

Reduce to get u_1' so $8u_1' e^{6x} = xe^{4x}$, $u_1' = \frac{x}{8} e^{-2x}$

Integrate $\int u_1' = u_1 \ldots \frac{1}{8}\int xe^{-2x} = \left[-\frac{x}{16} - \frac{1}{32}\right] e^{-2x}$

$\left(\frac{-x}{16} - \frac{1}{32}\right) e^{4x} + u_2' \cdot e^{-2x} = xe^{4x}$

$\left[\frac{-x}{16} - \frac{1}{32}\right] + u_2' \cdot e^{-6x} = x$

$$u_2 = e^{6x}(Ax + B)$$

$$\frac{-x}{16} - \frac{1}{32} + Ax + B = x$$

$$-2x - 1 + 32Ax + 32B = 32x$$

$$B = \frac{1}{32}, \quad A = \frac{32}{30} = \frac{16}{15}$$

$$u_2 = e^{6x}\left(\frac{16}{15}x + \frac{1}{32}\right)$$

So: $y_p = u_1 y_1 + u_2 y_2 =$

u_2 can be wronskianed

$$\int \frac{y_1\, g(x)}{W(y_1, y_2)} = \frac{e^{6x} \cdot xe^{4x}}{-8e^{4x}} = \int \frac{xe^{6x}}{-8} = \left[\frac{-x}{48} + \frac{1}{288}\right]e^{6x}$$

$$\text{wronskian} = \begin{vmatrix} e^{6x} & e^{-2x} \\ 6e^{6x} & -2e^{-2x} \end{vmatrix} = -2e^{4x} - 6e^{4x} = -8e^{4x}$$

Using wronskian method: $y_p = \left[-\frac{x}{16} - \frac{1}{32}\right]e^{4x} + \left[\frac{-x}{48} + \frac{1}{288}\right]e^{4x}$

$$y_p = \left[-\frac{1}{12} - \frac{1}{36}\right]e^{4x}$$

$$-12A + 36(Ax+B) = 24x+2$$

$$-12A + 36B = 2$$

$$\frac{24}{36} = A = \frac{2}{3}$$

$$-8 + 36B = 2$$

$$36B = 10$$

$$B = \frac{10}{36} = \boxed{\frac{5}{18}}$$

$$y_p(x) = Ax + B$$

$$\frac{2}{3}x + \frac{5}{18}$$

2.

$$y'' + 2y = 48x^2 e^{2x}$$

Solve Homo

$$y'' + 2y = 0$$

$$\cancel{r^2 + 2r = 0}$$

$$\cancel{r(r+2) = 0}$$

$$\cancel{r = -2 \to e^{-2x}}$$

$\to$

$$\cancel{y_2 = ue^{-2x}}$$

$$\cancel{y' = u'e^{-2x} - 2ue^{-2x}}$$

$$\cancel{y'' = u''e^{-2x} - 2u'e^{-2x} - 2u'e^{-2x} + 4ue^{-2x}}$$

$$\cancel{u''e^{-2x} - 4u'e^{-2x} + 6ue^{-2x} = 0}$$

$$\cancel{u'' - 4u' + 6u = 0}$$

~~unsolvable oop~~

$$r^2 + 2 = 0$$

imaginary... $r = 2i, -2i$

For 2 complex roots $y_1 \neq y_2$

where $y_1 = a + iB$, $y_2 = a - iB$

the solution is $y = e^{ax}(C\cos(Bx) + D\sin(Bx))$

$$e^{0}('') = y = C\cos(\sqrt{2}x) + D\sin(\sqrt{2}x)$$

Solve particular

$$y'' + 2y = -18x^2e^{2x}$$

solution $y_p = Axe^{2x}$, $y' = A(e^{2x} + 2xe^{2x}$

Oops too hard

Crayon de pomme

$$y'' - y = -5$$

$$r^2 - 1 = 0, \quad r = 1, -1$$

$$e^x, e^{-x}$$

Solution to $y_p = A$

$$-A = -5, \quad A = 5$$

Try $y'' + y = \sec x$

Solve Homo $\to$ $y'' + y = 0$, $r^2 + 1 = 0$, $r = \pm i$

$r_1 = 0 + i$

$r_2 = 0 - i$

$$Y_c = e^{0x}\left(c_1 \cos(x) + c_2 \sin(x)\right) = c_1 \cos(x) + c_2 \sin(x)$$

Using Variation of Parameters

$$Y_p = u_1 y_1 + u_2 y_2$$

$$Y_c = \overset{u_1}{C_1}\ \overset{y_1}{\cos(x)} + \overset{u_2}{C_2}\ \overset{y_2}{\sin(x)}$$

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' \cos(x) + u_2' \sin(x) = 0$$

$$Y_p = u_1 \cos(x) + u_2 \sin(x)$$

$$y_p' = u_1'\cos(x) - u_1\sin(x) + u_2'\sin(x) + u_2\cos(x)$$

$$= 0$$

$$y_p' = u_2\cos(x) - u_1\sin(x)$$

$$y_p'' = u_2'\cos(x) - u_2\sin(x) - u_1'\sin(x) - u_1\cos(x)$$

Solve $\to$ $y'' + y = \sec x$

$$\sec(x) = u_2'\cos(x) - u_1'\sin(x) - u_2\sin(x) - u_1\cos(x) + u_1\cos(x) + u_2\sin(x)$$

$$u_2'\cos(x) - u_1'\sin(x) = \sec x$$

$$u_2'\cos^2(x) - u_1'\sin(x)\cos(x) = 1$$

Since $u_1'\cos(x) + u_2'\sin(x) = 0$

So $u_1'\sin(x)\cos(x) + u_2'\sin^2(x) = 0$

Add highlighting together: $u_1'\sin(x)\cos(x) - u_1'\sin(x)\cos(x) + u_2'\sin^2(x) + u_2'\cos^2(x) = 1$

$\boxed{u_2' = 1} \to \boxed{u_2 = x}$

$$u_1'\cos(x) + \sin(x) = 0$$

$$u_1' = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

$$\int u_1' = \int -\tan(x)$$

$$\int -\frac{\sin(x)}{\cos(x)}\,dx = \ln(\cos(x))$$

$$u_1 = \ln|\cos x|$$

$$y_p = \ln|\cos(x)|\cos(x) + x\sin x$$

$$y = y_c + y_p$$

$$y = C\cos(x) + D\sin(x) + \cos(x)\cdot\ln|\cos x| + x\sin(x)$$

$$y = [C + \ln(\cos(x))]\cos(x) + [D + x]\sin(x)$$

Consider the DE $y'' + q(t)y' + r(t)y = g(t)$

Assume y_1, y_2 are the fundamental set of solutions for the homo eq

Then a particular solution the the non-homo equation is

$$y_p(t) = -y_1\int\frac{y_2\,g(t)}{W(y_1,y_2)}\,dt + y_2\int\frac{y_1\,g(t)}{W(y_1,y_2)}\,dt$$

Wronskian Reminder: $\det\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix}$

Solve $2y'' + 18y = 6\tan(3t)$

$y'' + 9y = 3\tan(3t) \to$ homo $\to y'' + 9y = 0 \to r^2 + 9 = 0,\ r = \pm 3i$

$y_c = e^{0x}[C\cos(3x) + D\sin(3x)] = C\cos(3x) + D\sin(3x)$

Cannot guess using undetermined coefficients :(

Use V of P! & make assumption $u_1' y_1 + u_2' y_2 = 0$

so $y_1 = \cos(3x),\ y_2 = \sin(3x)$

$y_p = u_1 y_1 + u_2 y_2 = u_1\cos(3x) + u_2\sin(3x)$

$y_p' = \cancel{u_1'\cos(3x)} - 3u_1\sin(3x) + \cancel{u_2'\sin(x)} + 3u_2\cos(3x)$

$= 0$

$y_p' = 3u_2\cos(3x) - 3u_1\sin(3x)$, find y_p''

$y_p'' = 3u_2'\cos(3x) - 9u_2\sin(3x) - 3u_1'\sin(3x) - 9u_1\cos(3x)$

Plug into $y'' + 9y = 3\tan(3t)$

$3u_2'\cos(3x) - 3u_1'\sin(3x) - 9u_2\sin(3x) - 9u_1\cos(3x) + 9u_1\cos(3x) + 9u_2\sin(3x)$

$= 3u_2'\cos(3x) - 3u_1'\sin(3x) = 3\tan(3t)$

$u_2'\cos(3t) - u_1'\sin(3x) = \tan(3t)$

Remember $u_1'\cos(3x) + u_2'\sin(3x) = 0$

so $u_1'\sin(3x)\cos(3x) + u_2'\sin^2(x) = 0$

$u_2'\cos^2(3x) - u_1'\sin(3x)\cos(3x) = \sin(3x)$

Add pink together $u_2' \cos^2(3x) + u_2' \sin^2(x) = \sin(3x)$

$u_2' = \sin(3x)$, $u_2 = \int \sin(3x) = -\frac{\cos(3x)}{3}$

Plug back in

$u_1' \cos(3x) + u_2' \sin(3x) = 0$

$u_1' \cos(3x) + \sin^2(3x) = 0$

$u_1' \cos(3x) = -\sin^2(3x)$, $u_1' = \underline{-\sin(3x)\tan(3x)}$

$$\int -\sin(3x)\tan(3x) \to \int -\frac{\sin^2(3x)}{\cos(3x)}$$

$$\int \frac{\sin^2(3x)}{\cos(3x)} = \int \sec(3x)\sin^2(3x) \to \int \sec(3x) - \cos(3x)$$

$$= \frac{\sin(3x) - \ln|\tan(3x) + \sec(3x)|}{3} = u_1$$

$y_p = u_1 y_1 + u_2 y_2 =$

$$\frac{\sin(3x) - \ln|\tan(3x) + \sec(3x)|}{3}(\cos 3x) + \frac{-\cos(3x)}{3}\sin(3x)$$

$y = y_p + y_c =$

$$\boxed{y = C\cos(3x) + D\sin(3x) - \frac{\cos(3x)}{3} \cdot \ln|\tan(3x) + \sec(3x)|}$$

$$y = C\cos(3x) + D\sin(3x) + \frac{\cancel{\sin(3x)} - \ln|\tan(3x) + \sec(3x)|}{3}(\cos 3x) + \cancel{\frac{-\cos(3x)}{3}\sin(3x)}$$