MATH-253-YJH-CRN82680 Exam 2

David Yang

TOTAL POINTS

79 / 79

QUESTION 1

1 Order of integration **10 / 10**

✓ **- 0 pts** Correct

QUESTION 2

Directional derivative 9 pts

2.1 Particular directional derivative **3 / 3**

✓ **- 0 pts** Correct

2.2 Greatest directional derivative **3 / 3**

✓ **- 0 pts** Correct

2.3 No vertical change **3 / 3**

✓ **- 0 pts** Correct

QUESTION 3

3 Volume **10 / 10**

✓ **- 0 pts** Correct

QUESTION 4

4 Find gradient from directional derivative **10 / 10**

✓ **- 0 pts** Correct

QUESTION 5

For a surface 10 pts

5.1 Tangent plane **5 / 5**

✓ **- 0 pts** Correct

5.2 Normal line **5 / 5**

✓ **- 0 pts** Correct

QUESTION 6

Double integrals 10 pts

6.1 Constant bounds **5 / 5**

✓ **- 0 pts** Correct

6.2 Variable bounds **5 / 5**

✓ **- 0 pts** Correct

QUESTION 7

7 Complex dot product **10 / 10**

✓ **- 0 pts** Correct

QUESTION 8

8 Equation of a plane **10 / 10**

✓ **- 0 pts** Correct

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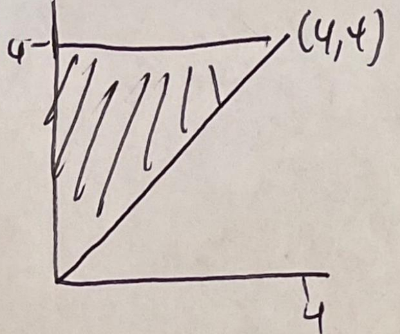
1. Reverse the order of integration for $12-2y$

$$\int_0^4\int_0^y f(x,y)\,dxdy + \int_4^6\int_0^{2(6-y)} f(x,y)\,dxdy$$



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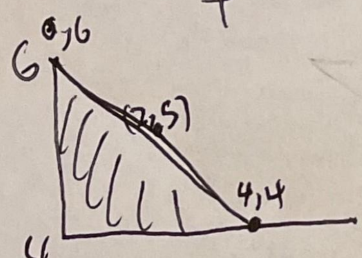

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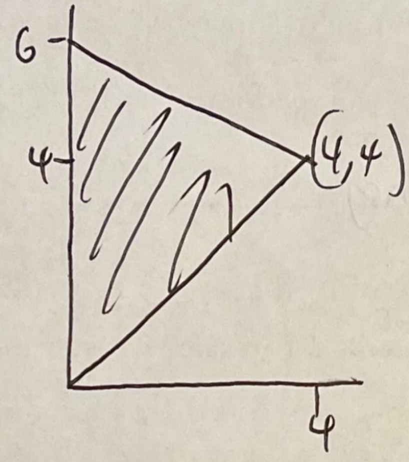

$$= \int_0^6\int_y^{12-2y} f(x,y)\,dx\,dy$$

$0 \le y \le 6$

$y \le x \le 12-2y$

reverse):

lower	upper
$x=y$	$x=12-2y$
$y=x$	x^{-1}: $12-2x=y$
$x=y$	$12-y=2x$
	$x=\frac{12-y}{2}=6-\frac{y}{2}$

$0 \le x \le 4$

$x \le y \le 6-\frac{x}{2}$

$$\boxed{\int_0^4\int_x^{6-\frac{x}{2}} f(x,y)\,dydx}$$

1 Order of integration 10 / 10

✓ - 0 pts Correct

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2. $f(x,y) = x^2y$ Let $\vec{v} = \langle 2,5 \rangle$

(a) Find $\frac{df}{d\vec{u}}(1,2)$ if $\vec{u}$ is in the direction of $< 2, 5 >$.

$\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial x}(1,2) = 4$, $\frac{\partial f}{\partial y} = x^2$, $\frac{\partial f}{\partial y}(1,2) = 1$

$\nabla f(1,2) = \langle 4,1 \rangle$.

Directional Derivative $= \nabla f \cdot \vec{v}$

$|\vec{v}| = \sqrt{2^2+5^2} = \sqrt{4+25} = \sqrt{29}$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$

$\langle 4,1 \rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle = \frac{8}{\sqrt{29}} + \frac{5}{\sqrt{29}}$

$= \boxed{\frac{13}{\sqrt{29}}}$

(b) Find the greatest directional derivative at $(3, 1, 9)$.

$\max\left(\frac{df}{d\vec{u}}\right) = |\nabla f|$.

$\frac{\partial f}{\partial x}(3,1,9) = \underset{2\cdot 3\cdot 1}{6}$, $\frac{\partial f}{\partial y}(3,1,9) = \underset{3^2}{9}$

$\nabla f(3,1,9) = \langle 6,9 \rangle$. $|\langle 6,9 \rangle| = \sqrt{6^2+9^2} = \sqrt{117}$

(c) Find (as vectors) the two directions from $(3, 1, 9)$ where the surface has no vertical change.

Since $\frac{df}{d\vec{v}} = |\nabla f|\,|\vec{v}| \cos\theta$, and no vertical change means $\frac{df}{d\vec{v}} = 0$, then $\cos(\theta) = 0$. This occurs when $\theta = 90$ or $270°$) so, we can use dot product because the vectors must be perpendicular to ∇f.

$\nabla f(3,1,9) = \langle 6,9 \rangle$.

$\langle 6,9 \rangle \cdot \langle -9,6 \rangle = 0$

$\langle 6,9 \rangle \cdot \langle 9,-6 \rangle = 0$.

The surface has no vertical change in the directions $\langle -9,6 \rangle$ and $\langle 9,-6 \rangle$

2.1 Particular directional derivative 3 / 3

✓ **- 0 pts** Correct

2. $f(x, y) = x^2y$ Let $\vec{v} = \langle 2,5 \rangle$

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2.2 Greatest directional derivative 3 / 3

✓ **- 0 pts** **Correct**

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2.3 No vertical change 3 / 3

✓ **- 0 pts** **Correct**

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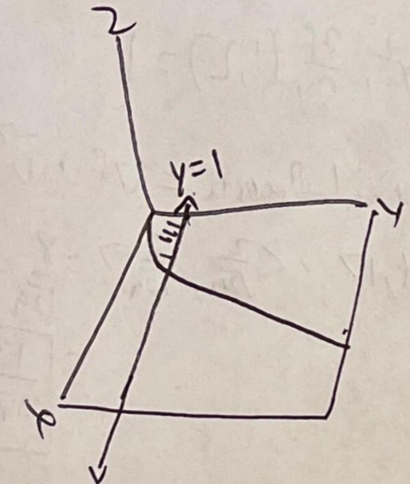
3. Find the volume under the surface $z = 1 - y$ above the region in the xy plane enclosed by $y = \sqrt{x}$, $y = 1$, the y axis, and the xy plane.

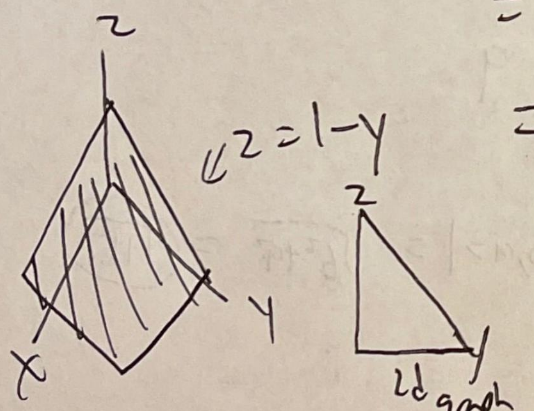


$$\int_0^1 \int_{\sqrt{x}}^1 1-y \, dy \, dx$$

$$= \int_0^1 y - \frac{y^2}{2}\Big]_{\sqrt{x}}^1 dx = \int_0^1 \left(1-\frac{1}{2}\right) - \left(\sqrt{x} - \frac{x}{2}\right) dx$$

$$= \int_0^1 \frac{1}{2} + \frac{1}{2}x - \sqrt{x} \, dx$$

$$= \frac{1}{2}x + \frac{1}{4}x^2 - x^{3/2} \cdot \frac{2}{3}\Big]_0^1$$



$$= \left(\frac{1}{2}(1) + \frac{1}{4}(1) - (1)^{3/2} \cdot \frac{2}{3}\right) - (0+0-0)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{2}{3}$$

$$= \frac{6}{12} + \frac{3}{12} - \frac{8}{12} = \frac{1}{12}$$

3 Volume 10 / 10

✓ - 0 pts Correct

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4. For the function $f(x, y)$ the directional derivative at $(1, 2)$ in the direction of $< 2, 1 >$ is $-\frac{1}{\sqrt{5}}$. And the directional derivative at $(1, 2)$ in the direction of $< -3, 2 >$ is $\sqrt{13}$. Find $\nabla f(1, 2)$

$|<2,1>| = \sqrt{2^2+1^2} = \sqrt{5}$

$\frac{2}{\sqrt{5}} f_x + \frac{1}{\sqrt{5}} f_y = -\frac{1}{\sqrt{5}}$, $\left(\frac{2}{\sqrt{5}} f_x + \frac{1}{\sqrt{5}} f_y\right)\sqrt{5} = \left(\frac{-1}{\sqrt{5}}\right)\sqrt{5} = 2f_x + 1f_y = -1$

$|<-3,2>| = \sqrt{(-3)^2+2^2} = \sqrt{13}$

$\frac{-3}{\sqrt{13}} f_x + \frac{2}{\sqrt{13}} f_y = \sqrt{13}$, $\left(\frac{-3}{\sqrt{13}} f_x + \frac{2}{\sqrt{13}} f_y\right)\sqrt{13} = (\sqrt{13})\sqrt{13} = -3f_x + 2f_y = 13$

$2f_x + f_y = -1$, $4f_x + 2f_y = -2$

$-3f_x + 2f_y = 13$

$$\begin{array}{r} 4f_x + 2f_y = -2 \\ - \; -3f_x + 2f_y = 13 \\ \hline 7f_x = -15 \end{array}$$

$f_x = \frac{-15}{7}$

$2 \cdot \frac{-15}{7} + f_y = -1$

$\frac{-30}{7} + f_y = -1$

$f_y = -1 + \frac{30}{7}$

$f_y = \frac{23}{7}$

$\rightarrow \boxed{\nabla f(1,2) = <\frac{-15}{7}, \frac{23}{7}>}$

4 Find gradient from directional derivative 10 / 10

✓ **- 0 pts** Correct

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5. The equation of a hyperboloid of one sheet is $2x^2 - 3y^2 + 2z^2 = 1$.

(a) Find the equation of the tangent plane at $(1, 1, 1)$.

$f(x,y,z) = 2x^2 - 3y^2 + 2z^2$

$f_x(1,1,1) = 4x = 4$

$f_y(1,1,1) = -6y = -6$

$f_z(1,1,1) = 4z = 4$

$\vec{N} = \nabla f(1,1,1) = \langle 4, -6, 4 \rangle$

$\vec{N} \bullet \langle x-1, y-1, z-1 \rangle = 0$

$4(x-1) - 6(y-1) + 4(z-1) = 0$

$\boxed{4x - 6y + 4z = 2}$

(b) Find parametric equations of the normal line at $(1, 1, 1)$.

Whole line = $\langle 1,1,1 \rangle + t\langle 4,-6,4 \rangle$

$$\boxed{\begin{aligned} x(t) &= 1 + 4t \\ y(t) &= 1 - 6t \\ z(t) &= 1 + 4t \end{aligned}}$$

5.1 Tangent plane **5 / 5**

✓ **- 0 pts** Correct

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5. The equation of a hyperboloid of one sheet is $2x^2 - 3y^2 + 2z^2 = 1$.

(a) Find the equation of the tangent plane at $(1, 1, 1)$.

$f(x,y,z) = 2x^2 - 3y^2 + 2z^2$

$f_x(1,1,1) = 4x = 4$

$f_y(1,1,1) = -6y = -6$

$f_z(1,1,1) = 4z = 4$

$\vec{N} = \nabla f(1,1,1) = \langle 4, -6, 4 \rangle$

$\vec{N} \cdot \langle x-1, y-1, z-1 \rangle = 0$

$4(x-1) - 6(y-1) + 4(z-1) = 0$

$\boxed{4x - 6y + 4z = 2}$

(b) Find parametric equations of the normal line at $(1, 1, 1)$.

Whole line = $\langle 1,1,1 \rangle + t\langle 4,-6,4 \rangle$

$x(t) = 1 + 4t$

$y(t) = 1 - 6t$

$z(t) = 1 + 4t$

5.2 Normal line 5 / 5

✓ **- 0 pts** Correct

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6. Find

(a) $\int_0^1 \int_2^3 x^2 y \; dydx$

$$\int_0^1 \int_2^3 x^2 y \, dy \, dx$$

$$\int_0^1 \frac{x^2y^2}{2}\Big]_2^3 dx = \int_0^1 \left(x^2 \cdot \frac{9}{2}\right) - \left(x^2 \cdot \frac{4}{2}\right) dx = \int_0^1 \frac{5}{2}x^2 \, dx$$

$$\int_0^1 \frac{5}{2}x^2 \, dx = \frac{5}{6}x^3\Big]_0^1 = \frac{5}{6}(1)^3 - \frac{5}{6}(0)^3 = \frac{5}{6}$$

(b) $\int_0^1 \int_{x^3}^{x^2} x^2 y \; dydx$

$$\int_0^1 \int_{x^3}^{\cancel{x^3}\, x^2} x^2 y \, dy dx = \int_0^1 \frac{x^2y^2}{2}\Big]_{x^3}^{x^2} dx = \int_0^1 \left(x^2 \frac{(x^2)^2}{2}\right) - \left(x^2 \frac{(x^3)^2}{2}\right) dx$$

$$= \int_0^1 \left(x^2 \cdot \frac{x^4}{2} - x^2 \cdot \frac{x^6}{2}\right) dx = \int_0^1 \left(\frac{x^6}{2} - \frac{x^8}{2}\right) dx = \frac{x^7}{14} - \frac{x^9}{18}\Big]_0^1$$

$$= \left(\frac{1^7}{14} - \frac{1^9}{18}\right) - \left(\frac{0}{14} - \frac{0}{18}\right) = \frac{1}{14} - \frac{1}{18} = \frac{9}{126} - \frac{7}{126} = \frac{2}{126} = \frac{1}{63}$$

2 | 14 18
  7 9

63
×2
126

6.1 Constant bounds 5 / 5

✓ - 0 pts Correct

gradescope

6. Find

(a) $\int_0^1 \int_2^3 x^2y \, dydx$

$$\int_0^1 \int_2^3 x^2 y \, dy \, dx$$

$$\int_0^1 \frac{x^2y^2}{2}\Big]_2^3 dx = \int_0^1 \left(x^2 \cdot \frac{9}{2}\right) - \left(x^2 \cdot \frac{4}{2}\right) dx = \int_0^1 \frac{5}{2}x^2 \, dx$$

$$\int_0^1 \frac{5}{2}x^2 \, dx = \frac{5}{6}x^3\Big]_0^1 = \frac{5}{6}(1)^3 - \frac{5}{6}(0)^3 = \frac{5}{6}$$

(b) $\int_0^1 \int_{x^3}^{x^2} x^2y \, dydx$

$$\int_0^1 \int_{x^3}^{\cancel{x^3}\,x^2} x^2 y \, dy dx = \int_0^1 \frac{x^2y^2}{2}\Big]_{x^3}^{x^2} dx = \int_0^1 \left(x^2 \frac{(x^2)^2}{2}\right) - \left(x^2 \frac{(x^3)^2}{2}\right) dx$$

$$= \int_0^1 \left(x^2 \cdot \frac{x^4}{2} - x^2 \cdot \frac{x^6}{2}\right) dx = \int_0^1 \left[\frac{x^6}{2} - \frac{x^8}{2}\right) dx = \frac{x^7}{14} - \frac{x^9}{18}\Big]_0^1$$

$$= \left(\frac{1^7}{14} - \frac{1^9}{18}\right) - \left(\frac{0}{14} - \frac{0}{18}\right) = \frac{1}{14} - \frac{1}{18} = \frac{9}{126} - \frac{7}{126} = \frac{2}{126} = \frac{1}{63}$$

2 | 14 18

7 9

63
× 2
126

6.2 Variable bounds 5 / 5

✓ **- 0 pts** Correct

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$\vec{a}\cdot\vec{a} = |a||a|\cos 0 = |a|^2$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

7. $|\vec{a}| = 3$, $|\vec{b}| = 4$, and the angle between the vectors is $60°$. Find

$$(2\vec{a} + 3\vec{b}) \cdot (3\vec{a} - \vec{b})$$

$6\vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + 9\vec{b}\cdot\vec{a} - 3\vec{b}\cdot\vec{b}$

$6|a|^2 - 2\vec{a}\cdot\vec{b} + 9\vec{a}\cdot\vec{b} - 3|b|^2$

$6\cdot 3^2 - 3\cdot 4^2 + 7\vec{a}\cdot\vec{b}$

$54 - 48 + 7(|a||b|\cos(60°)) = 6 + 7\cdot 3\cdot 4\cdot \frac{1}{2} = 48$

7 Complex dot product 10 / 10

✓ **- 0 pts** **Correct**

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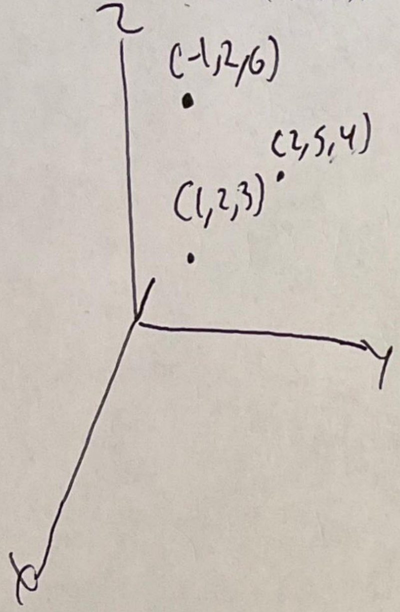
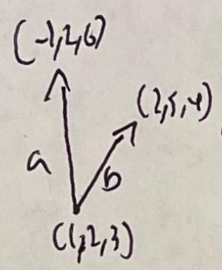
8. Find a Cartesian equation of the plane through the points $(1, 2, 3)$, $(2, 5, 4)$, and $(-1, 2, 6)$





~~x-x~~

$x-x_0, y-y_0, z-z_0$

$a = \langle -1-1, 2-2, 6-3 \rangle = \langle -2, 0, 3 \rangle$

$b = \langle 2-1, 5-2, 4-3 \rangle = \langle 1, 3, 1 \rangle$

$$\vec{N} = a \times b = \begin{matrix} i & j & k \\ -2 & 0 & 3 \\ 1 & 3 & 1 \end{matrix} = -9i - (-2-3)j + (-6)k$$

$$= -9i + 5j - 6k$$

$$\vec{N} = \langle -9, 5, -6 \rangle$$

$$\vec{N} \cdot \langle x-1, y-2, z-3 \rangle = 0$$

$$-9(x-1) + 5(y-2) - 6(z-3) = 0$$

$$-9x + 9 + 5y - 10 - 6z + 18 = 0$$

$$-9x + 5y - 6z + 17 = 0$$

$$\boxed{-9x + 5y - 6z = -17}$$

8 Equation of a plane 10 / 10

✓ **- 0 pts** **Correct**

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