Abstract Vector Spaces

$M_{2\times 2}$ = Space of 2x2 matrices with real number entries

Ex: Are $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$ Linearly Independent?

Solve $x\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + y\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + z\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} = 0$ has non trivial solutions

$$= \begin{bmatrix} x+y+z & 2x+2y+2z \\ 3x+2y+4z & 4x+y+7z \end{bmatrix} = 0$$

Therefore each entry in matrix is 0,

$x+y+z=0$
$2x+2y+2z=0$
$3x+2y+4z=0$
$4x+y+7z=0$

Solve
$$\begin{matrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 1 & 7 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

so $x = -2z$
$y = z$

Dependence Relation

$$-2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = 0$$

Is $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \ a, b \text{ any number in } \mathbb{R} \right\}$ a subspace

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$ has span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

so it has a subspace

P_m = Set of all polynomials of degree m

Let $S = \{ p(x) \text{ in } P_2 \mid \int_0^1 p(x)\,dx = 0 \}$

Is S a subspace?

Since $p(x)$ is in P_2, then it must satisfy membership in $p(x) = ax^2 + bx + c$

$$\int_0^1 p(x)\,dx = \int_0^1 ax^2 + bx + c \,dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx \Big|_0^1 = \frac{a}{3} + \frac{b}{2} + c = 0$$

$c = -\frac{a}{3} - \frac{b}{2}$, so $p(x) = ax^2 + bx - \frac{a}{3} - \frac{b}{2} = 0$

$a\left(x^2 - \frac{1}{3}\right) + b\left(x - \frac{1}{2}\right) = 0$ so $S = \text{Span}\left\{x^2 - \frac{1}{3},\ x - \frac{1}{2}\right\} = 0$

Reminder:

To show that something is not a subspace, show 1 of the following

1. objects a, b are in span, thus $a+b$ is in span.
2. a is in span, so ca is in span
3. The 0 object exists in the span

$M_{2\times2}$: Span of 2x2 real matrices

P_m = Span of Polynomials of at least degree m

Coordinates

A basis for $\mathbb{R}^n$ is a set of vectors that span $\mathbb{R}^n$ and are Linearly Ind.

Ordered Basis for $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$ where the order of vectors is specified

$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ a basis for $\mathbb{R}^2$? Yes.

$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a different ordered basis bc order of vectors is different

Dumb example of order mattering: $(1,2)$ is $x=1, y=2$, $(2,1)$ is $x=2, y=1$

$\begin{bmatrix}3\\4\end{bmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{bmatrix}1\\2\end{bmatrix}$ can be written as $\begin{bmatrix}2\\1\end{bmatrix}$ or $\widehat{(2,1)}$

Consider as matrix math: where A is the linear transformation of the coordinate sys

$\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}$, so $\begin{bmatrix}2\\1\end{bmatrix} = A^{-1}\begin{bmatrix}3\\4\end{bmatrix}$ or $\begin{bmatrix}3\\4\end{bmatrix}_{B} = A^{-1}\begin{pmatrix}3\\4\end{pmatrix}$