Vector Basis

In $\mathbb{R}^n$ $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots \vec{v}_n$ is a basis iff
1. The $\vec{v}$s are linearly ind.
2. Therefore they span $\mathbb{R}^n$

A subspace W of $\mathbb{R}^n$ is a mdi $\mathbb{R}^n$:

1. if $\vec{x}$ and $\vec{y}$ are in W, then $\vec{x}+\vec{y}$ is in W (closure under vector addition)
2. if $\vec{x}$ is in W, then if c is a constant, $c\vec{x}$ is in W (closure under scalar multiplication)
3. $\vec{0}$ is in W

Anything that is a span, is a subspace

To show W is a subspace, show W is a span

To show W is not a subspace, show a violation of rule 1 or 2

Example: Show $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$ in $\mathbb{R}^3 \mid \underset{\text{membership rule}}{x+2y-z=0}$ is a subspace

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in W, Then $x+2y-z=0$, $x = -2y+z$

so $\begin{matrix} x \\ y \\ z \end{matrix} = \begin{pmatrix} -2y+z \\ y \\ z \end{pmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

W is a span of $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Show $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbb{R}^3 \mid x+2y-z=1 \right\}$

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ in W so $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ should be in span $\quad -1+4-1=1$, $2 \neq 1$

Show $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbb{R}^3 \mid x+y=0, z=0 \right\}$ is a subspace

find span: $x=-y$, $z=0$, span is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Special Subspaces

- The null space of A (Nul A) or $N(A)$ is the set of all vectors x such that $Ax = \vec{0}$

Find Null space of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$

$x\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + y\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + z\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \vec{0}$ $\quad \begin{matrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{matrix} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] = \begin{matrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{matrix} = \begin{matrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{matrix} = \begin{matrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{matrix}$$

$$= \begin{matrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{matrix}$$

$$\text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right) \quad z\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \leftarrow \begin{matrix} y = -2z \\ x = z \end{matrix}$$

The column space of matrix A is the span of columns of A (col A)

Find Basis for $\text{col}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \text{col}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Independent columns

$\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$ are independent, so $\left\{\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}\right\}$ are the column span

Non $\mathbb{R}^n$ vector spaces

$M_{2\times2}$ = Vector Basis of 2x2 matrices with real number entries

Vectors in $M_{2\times2}$ are $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

etc :)

Find Determinent of

$$\begin{matrix} -4 & 3 & -4 \\ -4 & -3 & 3 \\ -4 & -2 & -4 \end{matrix} \to \begin{matrix} -4 & 3 & -4 \\ 0 & -6 & 7 \\ 0 & -5 & 0 \end{matrix} \quad -4 \cdot \begin{vmatrix} -6 & 7 \\ -5 & 0 \end{vmatrix} = -4 \cdot 35$$

If $A = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{matrix}$, find $\det(A)$

$1 \cdot (1-0) = 1$

$-1 \times 3 \times 81 = -243$

Find determinant of the matrix

$$\begin{matrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & -3 & 3 & 0 & 0 \end{matrix} \to \begin{matrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & -3 & 3 & 0 & 0 \end{matrix}$$

$$1 \times \det \begin{pmatrix} 0 & 3 & -1 & 0 \\ -3 & 0 & 0 & 1 \\ 0 & 0 & -3 & -3 \\ -3 & 3 & 0 & 0 \end{pmatrix} \to \begin{matrix} 0 & 3 & -1 & 0 \\ -3 & 0 & 0 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 3 & 0 & -1 \end{matrix}$$

$$1 \times -3 \times \det \begin{pmatrix} 3 & -1 & 0 \\ 0 & -3 & -3 \\ 3 & 0 & -1 \end{pmatrix} \to \det \begin{pmatrix} 3 & -1 & 0 \\ 0 & -3 & -3 \\ 0 & 1 & -1 \end{pmatrix} \to 3 \times \begin{vmatrix} -3 & -3 \\ 1 & -1 \end{vmatrix}$$

$(3+3) = 6$

$6 \times 3 \times 3 \times 1 = 54$