Notes

$y'' - 4y' + 4y = 0 \qquad y(0) = 12,\ y'(0) = -3$

if $y = e^{rx},\ y' = re^{rx},\ y'' = r^2 e^{rx}$

$e^{rx}(r^2 - 4r + 4) = 0$

$(r-2)^2 = 0,$

try

$x^2 y'' + 3x y' + y = 0,\ y_1 = \frac{1}{x}$

$y = u y_1$

$y = \frac{u(x)}{x}$

$y' = \frac{u'}{x} - \frac{u}{x^2}$

$y'' = \frac{u''}{x} - \frac{u'}{x^2} - \left(\frac{u'}{x^2} - \frac{2u}{x^3}\right) = \frac{u''}{x} - \frac{2u'}{x^2} + \frac{2u}{x^3}$

$x^2(y'') + 3x(y') + (y) = 0$

$x^2\left(\frac{u''}{x} - \frac{2u'}{x^2} + \frac{2u}{x^3}\right) + 3x\left(\frac{u'}{x} - \frac{u}{x^2}\right) + \frac{u}{x} = 0$

$xu'' - 2u' + \frac{2u}{x} + 3u' - \frac{3u}{x} + \frac{u}{x}$

$xu'' + u' = 0$

$xv' + v = 0$

Let $v = u'$

1. Homogenous $\to$ Sep
2. Non-H $\to$ Linear

$\frac{dv}{dx} x + v = 0$

$\frac{dv}{dx} = \frac{-v}{x}, \int \frac{1}{v} dv = \int -\frac{1}{x} dx$

$= \ln(|v|) = -\ln(|x|) + C$

$v = e^{-\ln x + c}$

$u' = v$, so $u = \int v$

3.

$x^2 y'' - 9xy' + 25y = 0, \ y_1 = x^5$

$y_2 = u(x) y_1 \to$ find u

$y_2 = ux^5$

$y' = u'x^5 + 5ux^4$

$y'' = u''x^5 + 5u'x^4 + 5u'x^4 + 20ux^3 = u''x^5 + 10u'x^4 + 20ux^3$

plug in

$x^2(u''x^5 + 10u'x^4 + 20ux^3) - 9x(u'x^5 + 5ux^4) + 25(ux^5)$

$u''x^7 + 10u'x^6 + 20ux^5 - 9u'x^6 - 45ux^5 + 25ux^5 = 0$

$u''x^7 + u'x^6 = 0$

Let $v = u'$

$x^7 v' + x^6 v = 0$

$\frac{dv}{dx} x^7 + x^6 v = 0$, $\frac{dv}{dx} x^7 = -x^6 v$, $\frac{dv}{dx} = \frac{-v}{x}$

$\int \frac{1}{v} \frac{dv}{dx} dx = \int -\frac{1}{x} dx = e^{\ln|v|} = e^{-\ln|x|} + C$

$v = C \cdot \frac{1}{x}$

$u' = \frac{C}{x}$, $u = \int \frac{C}{x} = C\ln|x|$

$y_2 = y_1 \cdot u$, $y_2 = x^5 \cdot C\ln|x|$

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$y_2 = y_1(x) \cdot \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2} dx$

$(1-x^2)y'' + 2xy' = 0$, $y_1 = 1$

$y'' + \frac{2x}{1-x^2} y' = 0$

$-\int P(x)\,dx = \int -\frac{2x}{1-x^2} dx$ $\quad (\ln|1-x^2|)' = \frac{1}{1-x^2} \cdot -2x$

$= \ln|1-x^2|$

$$e^{-\int P(x)\,dx} = 1 - x^2$$

$$\int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{1-x^2}{1^2} = \int 1 - x^2\,dx$$

$y_1 = 1$ $\quad\quad u = x - \frac{x^3}{3} + C$

$$y_2 = y_1 \cdot u, \quad x - \frac{x^3}{3} + C \quad :)$$

5.

$y'' - 16y = 3$, $\quad y_1 = e^{-4x}$ is sol of homogeneous eq

$y_2(x) = ?$

$y_p(x) = ?$ of the non H equation

$y'' - 16y = 0$, sol has y_1 and y_2

$y_2 = u\,y_1$, $u\,e^{-4x}$

$y' = u'e^{-4x} - 4ue^{-4x}$

$y'' = u''e^{-4x} - 4u'e^{-4x} - 4u'e^{-4x} + 16ue^{-4x}$

$\quad = u''e^{-4x} - 8u'e^{-4x} + 16ue^{-4x}$

$y'' - 16y = 0$, $\quad u''e^{-4x} - 8u'e^{-4x} + 16ue^{-4x} - 16ue^{-4x} = 0$

$u'' e^{-4x} - 8 u' e^{-4x} = 0$, Let $v = u'$

$v' e^{-4x} - 8 v e^{-4x} = 0$

$u'' - 8u' = 0$

$v' - 8v = 0$

$\frac{dv}{dx} e^{-4x} - 8 v e^{-4x} = 0 \rightarrow \frac{dv}{dx} = 8v$ $\frac{dv}{dx} = 8v$

$\int \frac{1}{v} dv = \int 8 dx$, $\ln|v| = 8x + C$, $v = Ce^{8x}$

$u' = Ce^{8x}$, $u = \int u'$, $\int Ce^{8x} = \frac{C}{8} e^{8x} + D$

$y_2 = y_1 \cdot u$, $y_2 = e^{-4x} \cdot (Ce^{8x} + D)$

$= Ce^{4x} + De^{-4x}$

Solve $y'' - 16y = 3$

$Ce^{4x} + De^{-4x} = Y_{complementary}$

$Y_p(t) = A$ $\longrightarrow$ $y'' - 16y = 3$

$Y_p' = 0$, $Y'' = 0$ $0 - 16A = 3$

$A = \frac{-3}{16}$

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$y''-3y'+2y=7e^{3x}$, $y_1=e^x$

$y''-3y'+2y=0$ Solve Homogeneous equation!

$y_2=uy_1=e^x\cdot u$

$y''=u'e^x+ve^x$

$y''=v''e^x+v'e^x+v'e^x+ve^x$

$v''e^x+2v'e^x+ve^x-3v'e^x-3ve^x+2ve^x=0$

$u''e^x-u'e^x=0$

$u''-u'=0$, $v=u'$

$v'-v=0$, $v'=v$

$\frac{dv}{dx}=v$, $\int\frac{1}{v}dv=\int dx$

$\ln|v|=x+C$

$e \quad e$

$v=Ce^x$

$u'=Ce^x$, $u=\int u'$, $u=\int Ce^x=Ce^x+D$

$y_2=y_1\cdot u$, $e^x(Ce^x+D)=\underbrace{Ce^{2x}+De^x}_{Y_{comp}}$

$y''-3y'+2y=7e^{3x}$

solution in form of Ae^{3x}...

$y=Ae^{3x}$

$y'=3Ae^{3x}$

$y''=9Ae^{3x}$

$9Ae^{3x}-9Ae^{3x}+2Ae^{3x}=7e^{3x}$

$Ae^{3x}=3.5e^{3x}$

$A=3.5$

$y=3.5e^{3x}$

2. Verify $y_1(x) = x$ is a solution of $xy'' - xy' + y = 0$

$$y'' - y' + \frac{y}{x} = 0 \to -1 + 1 = 0$$

Use Reduction to find y_2 in the form of an inf series

$$y_2 = u y_1 = ux$$

$$y' = u'x + u$$

$$y'' = u''x + u' + u' = u''x + 2u'$$

$$u''x + 2u' - u'x - u + \frac{ux}{x} = 0$$

$$u''x + 2u' - u'x = 0 \qquad v = u'$$

$$v'x + 2v - vx = 0$$

$$\frac{dv}{dx}x + 2v - vx = 0, \quad \frac{dv}{dx}x = v(x-2)$$

$$\int \frac{1}{v}\,dv = \int \frac{x-2}{x}\,dx, \quad \ln(|v|) = -2\ln(x) + x + C$$

$$\int uv' = uv - \int u'v$$

$$\int (x-2)\left(\frac{1}{x}\right) = (x-2)(\ln(x)) - \int \ln(x)$$

$$= x\ln(x) - 2\ln(x) - x\ln(x) + x = -2\ln(x) + x$$

$$\ln(|v|) = -2\ln(x) + x + C$$

$$v = \frac{Ce^x}{e^{2\ln(x)}} = \frac{Ce^x}{x^2}$$

$$u' = \frac{Ce^x}{x^2}$$ not easily integrable :(

Taylor series! :)

$$\frac{e^x}{x^2} = \sum \frac{x^n}{n!} \cdot \frac{1}{x^2} = \sum \frac{x^{n-2}}{n!}$$

$$u' = \sum \frac{x^{n-2}}{n!}$$

$$u = \int \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \frac{x^3}{120} \dots \frac{x^n}{(n+2)!}$$

$$u = \sum \frac{x^{n+1}}{(n+1)(n+2)!} = \sum \frac{x^n}{(n)(n+1)!}$$

$$y_2 = u\,y_1 = x \cdot \sum \frac{x^n}{n(n+1)!} = \sum \frac{x^{n+1}}{(n)(n+1)!} = \sum \frac{x^n}{(n-1)(n)!}$$

$$y_2 = -1 + \frac{1}{0} + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3!} + \frac{x^4}{3 \cdot 4!} \dots$$

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