MATH-253-YJH-CRN82680 Exam 5

David Yang

TOTAL POINTS

48 / 70

QUESTION 1

1 Flux **7 / 10**

✓ **- 1 pts** **Bad algebra**

✓ **- 2 pts** **Bad antiderivative**

QUESTION 2

2 Stokes **7 / 10**

✓ **- 3 pts** **Bad $$\vec{u}_x\times\vec{u}_y$$**

QUESTION 3

3 Surface area **5 / 10**

✓ **- 5 pts** **Bad parameterization**

QUESTION 4

4 Green's Theorem **6 / 10**

✓ **- 4 pts** **Interior of C is a triangle, not a square.**

QUESTION 5

5 Fundamental Theorem of Line Integrals **10 / 10**

✓ **- 0 pts** **Correct**

QUESTION 6

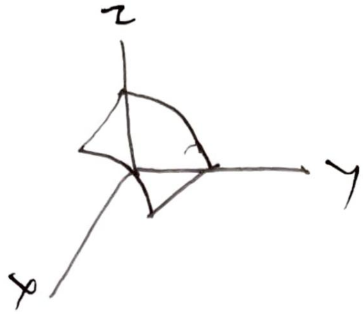
6 Change of variable **3 / 10**

✓ **- 7 pts** **Bad algebra**

QUESTION 7

7 Line integral of a scalar function **10 / 10**

✓ **- 0 pts** **Correct**



1. The surface S is the part of the cylinder $z = \sqrt{4 - y^2}$ in the first octant with $0 \leq x \leq 1$. S is oriented up. The vector field $\vec{F}$ is $\vec{F}(x, y, z) = < x, 0, y >$. Find the flux of $\vec{F}$ through S.

Use Cylindrical Coord...

$x = r, \; y = 2\cos(\theta), \; z = 2\sin(\theta)$

$\vec{r}(r,\theta) = \langle r, 2\cos(\theta), 2\sin(\theta) \rangle$ $\qquad 0 \leq r \leq 1$, $\; 0 \leq \theta \leq \frac{\pi}{2}$

$$\text{Flux} = \iint_S \vec{F} \cdot \frac{r_u \times r_v}{|r_u \times r_v|} \cdot |r_u \times r_v| \, du \, dv = \iint_S \vec{F} \cdot (r_u \times r_v) \, du \, dv$$

$$= \iint_S \vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta) \, r \, dr \, d\theta$$

$\vec{r}_r = \langle 1, 0, 0 \rangle$, $\vec{r}_\theta = \langle 0, -2\sin(\theta), 2\cos(\theta) \rangle$, $\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -2\sin(\theta) & 2\cos(\theta) \end{vmatrix}$

$= \langle 0, -2\cos\theta, -2\sin(\theta) \rangle$

$\vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta) = 0 + 0 + -2\sin(\theta) \cdot 2\cos(\theta) = -4\sin(\theta)\cos(\theta) = -2\sin(2\theta)$

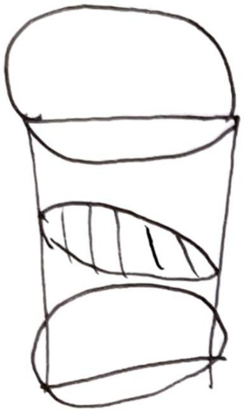
$$\int_0^{\pi/2}\int_0^1 -2\sin(2\theta) \, r \, dr \, d\theta = \int_0^{\pi/2} -2\sin(2\theta) \frac{r^2}{2}\Big]_0^1 d\theta = \int_0^{\pi/2} -\sin(2\theta) \, d\theta$$

$$= \boxed{\frac{\cos(2\theta)}{2}}\Bigg]_0^{\pi/2} = \frac{\cos(\pi)}{2} - \frac{\cos(0)}{2} = \frac{-1-1}{2} = \frac{-2}{2} = \boxed{-1}$$

1 Flux 7 / 10

- ✓ - 1 pts Bad algebra
- ✓ - 2 pts Bad antiderivative

2. Surface S is the part of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 1$, oriented up. C is the boundary curve of S traced counter clockwise as seen from above. $\vec{F}$ is the vector field $\vec{F}(x, y, z) = < z, x, y >$. Use Stokes' Theorem to find $\int_C \vec{F} \cdot d\vec{r}$.



$z = 1 - x - y$

$z_x = z_y = -1$

$z_x \times z_y = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = \langle -1, -1, 1 \rangle$

$\cancel{|z_x \times z_y| = |\langle -1, -1, 1 \rangle| = \sqrt{3}}$

$\vec{F} = \langle \overset{P}{z}, \overset{Q}{x}, \overset{R}{y} \rangle, \quad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$\text{curl}(\vec{F}) =$

$\langle 1, 1, 1 \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot (z_x \times z_y)\, dx\, dy = \iint_S \langle 1, 1, 1 \rangle \cdot \boxed{\langle -1, -1, 1 \rangle}$

$\iint_S -1 + -1 + 1\, dx\, dy = \iint_S -1\, dx\, dy = \int_0^{2\pi}\int_0^1 -1\, r\, dr\, d\theta = \int_0^{2\pi} \left. \frac{-r^2}{2} \right|_0^1 = \int_0^{2\pi} -\frac{1}{2}$

$= \boxed{-\pi}$

2 Stokes 7 / 10

✓ **- 3 pts** **Bad $$\vec{u}_x\times\vec{u}_y$$**

3. Surface S is the part of the paraboloid $z = x^2 + y^2$ that is inside the cylinder $x^2 + y^2 = 1$. Find the surface area.

AKA parabola x^2+y^2 from $0 \le z \le 1$

$$z = \vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$$

$$z = \vec{r}(r,\theta) = \boxed{\langle \cos(\theta), \sin(\theta), r \rangle} \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$\vec{r}_r = \langle 0, 0, 1 \rangle$$

$$\vec{r}_\theta = \langle -\sin(\theta), \cos(\theta), 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{matrix} i & j & k \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{matrix} = \langle -\cos\theta, \sin\theta, 0 \rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\iint_S dS = \iint_S |\vec{r}_r \times \vec{r}_\theta| \, dA = \iint_S 1 \, dA$$

$$= \int_0^{2\pi}\int_0^1 r \, dr \, d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big]_0^1 d\theta = \int_0^{2\pi} \frac{1}{2} \, d\theta = \boxed{\pi}$$

3 Surface area **5 / 10**

✓ **- 5 pts** **Bad parameterization**

4. Curve C consists of line segments from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,0)$.
Use Green's Theorem to find $\int_C 2y\ dx + xy\ dy$.

Greens Stuces

$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA \qquad \begin{aligned} Q &= \cancel{2x}\ xy \\ P &= \cancel{xy}\ 2y \end{aligned}$$

$$= \iint_D y-2\,dA = \int_0^1 \boxed{\int_0^1} y-2\,dy\,dx = \int_0^1 \frac{y^2}{2} - 2y\Big)_0^1 dx = \int_0^1 \frac{-3}{2}\,dx$$

$$= \int_0 -\frac{3}{2}\,dx = \boxed{\frac{-3}{2}}$$

Integrate this

4 Green's Theorem **6 / 10**

✓ **- 4 pts** **Interior of C is a triangle, not a square.**

5. Use the Fundamental Theorem for Line Integrals to find

$$\int_C (3x^2y + 2x)\,dx + (x^3 + 4y)\,dy$$

for the curve C that is $y = x^2$ from $(0,0)$ to $(1,1)$.

$\nabla f = \vec{F} = \langle 3x^2y + 2x, x^3 + 4y \rangle$ ~~assume $\vec{F}$ is conservative~~

$f = x^3y + x^2 + 2y^2$ thus $\vec{F}$ is conservative

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

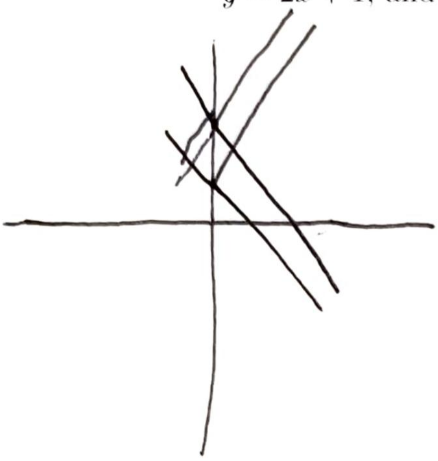
$$= f(1,1) - f(0,0)$$

$$= (1+1+2) - (0) = 4$$

5 Fundamental Theorem of Line Integrals **10 / 10**

✓ **- 0 pts** Correct

6. Region D in the xy plane is enclosed by $y = -x + 1$, $y = -x + 2$, $y = 2x + 1$, and $y = 2x + 2$. Find $\iint_D (x + y)(2x - y)^2 \, dydx$.



$1 \le y \le 2$

$-x + 2 = 2x + 1$

$3x = 1$

$x = \frac{1}{3}$

$-x + 1 = 2x + 2$

$3x = -1$

$x = -\frac{1}{3}$

Let $F = (x+y)(2x-y)^2$

$$\iint_D F \, dydx = \iint_D F \, dxdy = \iint_{D_1} F \, dxdy + \iint_{D_2} F \, dxdy$$

D_1

~~x~~ $1 \le y \le 2$

$1 + 2y \le x \le 2 - y$

D_2

$1 \le y \le 2$

$1 - y \le x \le -1 + \frac{1}{2}y$

$$\iint_{D_1} F \, dA = \int_1^2 \int_{1+2y}^{2-y} (x+y)(2x-y)^2 \, dxdy = \int_1^2 \int_{1+2y}^{2-y} 4x^3 - 3xy^2 + y^3 \, dxdy = \int_1^2 \left. x^4 - \frac{3}{2}x^2y^2 + xy^3 \right]_{1+2y}^{2-y}$$

$$= \int_1^2 \left[(2-y)^4 - \frac{3}{2}(2-y)^2 y^2 + (2-y)y^3 - \left((1+2y)^4 - \frac{3}{2}(1+2y)^2 y^2 + (1+2y)y^3 \right) \right]$$

	$2x$	$-y$
$2x$	$4x^2$	$2xy$
$-y$	$-2xy$	y^2

	$4x^2$	$-4xy$	y^2
x	$4x^3$	\	xy^2
y	\	~~$-4xy^2$~~	y^3

$= 4x^3 - 3xy^2 + y^3$

$$= \int_1^2 -18y^4 - 39y^3 - 40y + \frac{9y^4}{2} + \overset{12y^3}{\cancel{}} - \frac{9y^2}{2} + 15 \, dy$$

$$\text{let } f = \left[-\frac{18}{5}y^5 - \frac{39}{4}y^4 - 20y^2 + \frac{9y^5}{10} + 3y^4 - \frac{3y^3}{2} + 15y \right]_1^2$$

$f(2) = -256$

$f(1) = -15.95$

$f(2) - f(1) = -256 - -15.95 = \boxed{-240.45}$

6 Change of variable 3 / 10

✓ - 7 pts Bad algebra

7. Find $\int_C x \, ds$ if C is the curve $y = x^2$ from $(0, 0)$ to $(2, 4)$.

Parameterize into t

$x = t$ $\quad \langle t, t^2 \rangle \quad 0 \le t \le 2$

$y = t^2$

$\vec{r}(t) = \langle t, t^2 \rangle$

$r'(t) = \langle 1, 2t \rangle$

$|r'(t)| = \sqrt{4t^2+1}$

$$\int_C x \, ds = \int_0^2 t \cdot |r'(t)| \, dt =$$

$$\int_0^1 t \cdot \sqrt{4t^2+1} \, dt = \int_0^2 t(4t^2+1)^{0.5} \, dt$$

$$\left. \frac{(4t^2+1)^{1.5}}{12} \right]_0^2 =$$

$$\frac{t(4t^2+1)^{1.5}}{1.5} \cdot 2t \cdot 4 = \frac{(4t^2+1)^{1.5} t}{12t}$$

$$= \frac{17^{\frac{3}{2}}}{12} - \frac{1^{3/2}}{12}$$

$$= \boxed{\frac{17^{3/2} - 1}{12}}$$

7 Line integral of a scalar function 10 / 10

✓ **- 0 pts** **Correct**