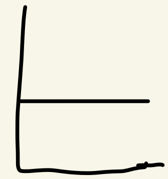
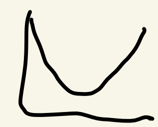
$\text{Beta}(a,b) = C x^{a-1}(1-x)^{b-1}$ $\quad 0<x<1$

- Flexible family of distributions on $(0,1)$
- Often used as a prior in $(0,1)$
- Conjugate Prior to Binomial & many connections to interdisciplinary fields

with $(a=1, b=1)$, beta =



with $(a=2, b=2)$ beta =



Conjugate Prior for Binomial

$X|p$ is $\text{Bin}(n,p)$ -> if we know p, then probability is binomial

When p is not known, p is $\text{Beta}(a,b)$ ← prior (not based on data)

Get evidence, update using bayes rule -> Find Posterior $p|X$

$$f(p|X=k) = \frac{P(X=k|p)\, f(p)}{P(X=k)} = \frac{\binom{n}{k} p^{k}(1-p)^{n-k} \cdot c\, p^{a-1}(1-p)^{b-1}}{P(X=k)}$$

given p, what is probability $X=k$?

density function Beta

Does not depend on the data (no p)

Ignore constants by using proportions: $f(p|X=k) \propto p^{k+a-1}(1-p)^{n-p+b-1}$

$$p(p|x) \sim \text{Beta}(a+x,\ b+n-x) \quad (x=k-1)$$

Updating parameters stays as beta function

Intuition: Let our old data have $\text{Beta}(a,b)$

Then with x new successes and $n-x$ new failures

our Beta is updated to $\text{Beta}(a+x, b+n-x)$

(Stays in the Beta family) -> very convenient

Gamma Distribution and Poisson Process

Given a sequence $0, 1, 2, x$: what could x be?

x can be 3, e, $\frac{1}{e}$, π, $720!$, whatever. Any sequence is ok.

Real Question: How do we extend a sequence of nums?

Or, what is pi factorial?

Sterlings formula $\to$ $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Gamma Function

- Closely connected to gamma function
- Has a gamma distribution

$$\Gamma(a) = \int_0^\infty x^a e^{-x} \frac{dx}{x} \quad \text{for } a > 0$$

Behavior of Γ:

At 0, $\int \frac{dx}{x}$ is $\ln|x|$ which is $-\infty$ as $x \to 0$

At 1, still $\int \frac{dx}{x^2}$ is $-x^{-1}$ which is 0

At large a, it is just $\int e^{-x}$ because e^{-x} overrules x^a

$\Gamma(n) = (n-1)!$ for n a positive integer

$\Gamma(x+1) = x\Gamma(x)$

What is $\Gamma(\frac{1}{2})$? $\to \sqrt{\pi}$

$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

Generating a PDF

$$1 = \int_0^\infty \frac{1}{\Gamma(a)} x^a e^{-x} \frac{dx}{x}$$

Gamma $(a, 1)$ PDF

Gamma distribution also related to exponential Distribution

If x is randomly distributed with Beta(a,b)

$$\text{PDF} = f(x) = \frac{1}{\text{Beta}(a,b)} x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$$

$$\text{Beta}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$\text{Beta}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Beta distribution is ideal model proportions/probability

$$E(x) = \frac{a}{a+b}, \quad E(x^2) = \frac{(a)(a+1)}{(a+b)(a+b+1)}$$

$$\text{Var}(x) = \frac{ab}{(a+b)^2(a+b+1)}$$

Kth moment

$$E(x^k) = \int_0^1 x^k p(x)\,dx = \int_0^1 x^k \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \int_0^1 x^{a+k-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a+k+b)} \cdot \int_0^1 \frac{\Gamma(a+k+b)}{\Gamma(a+k)\Gamma(b)} x^{a+k-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)\,\Gamma(a+k)}{\Gamma(a)\,\Gamma(a+k+b)} \cdot 1$$

PDF, integral of PDF is always 1

$$\frac{(a+b-1)! \cdot (a+k-1)!}{(a-1)! \cdot (a+b+k-1)!} = \text{for } k=1, \quad \frac{(a+b-1)! \cdot a!}{(a-1)! \cdot (a+b)!}$$

$$\boxed{\frac{a}{a+b}}$$ for 1st moment

Moments

What's a moment? → Has to do with Expectation

$E(x)$ is first moment

$E(x^2)$ is second moment

$E(x^3)$ is third moment

$E(x^k)$ is kth moment

kth central moment is $E\left((x-E(x))^k\right)$

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \underbrace{f(x)}_{\text{PDF}} \, dx$$

Expectation of values must exist for MGF to exist

$$E(x^n) = \frac{d^n}{dt^n} M_x(t) \Big|_{t=0}$$

nth order moment

why is MGF defined as $E(e^{tx})$?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} \dots$$

$$E(e^{tx}) = 1 + tE(x) + \frac{t^2E(x^2)}{2!} + \dots$$

$$\frac{d}{dt}E(e^{tx})\Big|_{t=0} = 0 + E(x) + tE(x^2) + \dots\Big|_{t=0}$$

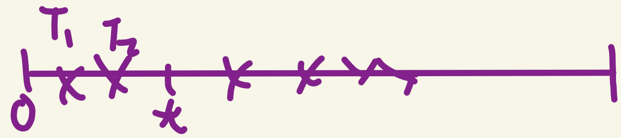
Since t is evaluated at 0, $\frac{d}{dt}E(e^{tx}) = 0 + E(x) + 0 + \dots$

$$\frac{d^2}{dt^2}E(e^{tx})\Big|_{t=0} = 0 + 0 + E(x^2) + 0 \dots$$

Therefore $E(e^{tx})$ gives moment gen function

and deriving MGF gives the kth order moment

Poisson process   Gamma-expo connection



N_t = # emails up to $t \sim Pois(\lambda t)$

Assume arrivals in disjoint intervals are independent

We draw x everytime we get an email

Let T_1 be time of first email

$$P(T_1 > t) = P(N_t = 0) = e^{-\lambda t}$$

T_1 is exponential, T_2 is also exponential

fish?

POISSON

DISTRIBUTION

What is it?

why?

Poisson Distribution:

Suppose we are counting # of occurrences in a given unit of time, distance, area/volume

Example: # of car accidents in a day

The # is a RV that may or may not follow the poisson Distribution

Suppose Events are indepent

The probability does not changes through time

Then X, the # of events, has the Poisson distribution

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$ Probability mass function

or $P(x)$

$\mu = \lambda$ (mean of the poisson dist)

$\sigma^2 = \lambda$ (standard deviation squared = variance)

One nanogram of Plutonium has an average of 2.3 decays per second, and # of decays follows a poisson distribution

Let X be # decays in 2 seconds find $P(X=3)$

So: $\lambda = 2 \cdot 2.3 = 4.6$

$$P(X=3) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{4.6^3 e^{-4.6}}{3!} = 0.163$$

"Truncated" Poisson

Recall Poisson $(X=x) = P(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$ where λ(lambda) $= EV(f)$

Say we had a party of n people. Find the probability that there is a group of 3 people with the same birthday. Assume a poisson distribution.

By Poisson: The first moment $EV = \lambda$, $Var = \lambda$

Lambda is Expected value of getting a group of 3 people w/same bday

$$\binom{n}{3} \text{ people} \cdot \frac{1}{1} \cdot \frac{1}{365} \cdot \frac{1}{365} = \binom{n}{3} \cdot \frac{1}{365^2} = \lambda$$

Probability: We want to find $P(X=x \mid x>0) = \frac{P(x>0 \mid X=x) \cdot P(X=x)}{P(x>0)}$

$$\frac{1 \cdot P(X=x)}{P(x>0)}$$ ← poisson

← change to comp

$$\frac{P(X=x)}{1-P(X=0)} = \frac{\lambda^x e^{-\lambda}}{x!} \% \left(1 - \frac{\lambda^0 e^{-\lambda}}{0!}\right)$$

$$\frac{\lambda^x e^{-\lambda}}{x!} \% \left(1 - e^{-\lambda}\right)$$

$$= \frac{\lambda^x e^{-\lambda}}{x!(1-e^{-\lambda})} = \boxed{\frac{\lambda^x}{x!(e^{\lambda}-1)}}$$

Poisson Max Likelihood Estimator (MLE)

Suppose X_i = # of textbooks a person buys

$X_1 \dots X_{50}$, $\sum X_i = 150$

λ is mean, $X_i \sim \text{Poisson}(\lambda)$

$f(x_i \mid \lambda)$ is the PDF for a given x_i

Recall Poisson$(x) = \dfrac{\lambda^x \cdot e^{-\lambda}}{x!}$

So $f(x_i \mid \lambda) \to$ pdf of x_i given λ

$$\frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!}$$