MATH-253-YJH-CRN82680 Exam 3

David Yang

TOTAL POINTS

64 / 70

QUESTION 1

1 Spherical **10 / 10**

✓ **- 0 pts Correct**

QUESTION 2

2 Change of variable **10 / 10**

✓ **- 0 pts Correct**

QUESTION 3

3 Triple integral for volume **7 / 10**

✓ **- 3 pts Bad bounds**

QUESTION 4

4 Volume 2 **10 / 10**

✓ **- 0 pts Correct**

QUESTION 5

5 Cartesian to polar **10 / 10**

✓ **- 0 pts Correct**

QUESTION 6

Surface 10 pts

6.1 Tangent plane **5 / 5**

✓ **- 0 pts Correct**

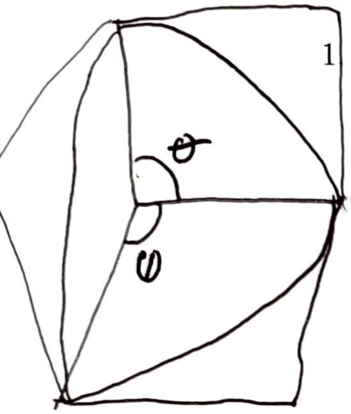
6.2 Normal line **5 / 5**

✓ **- 0 pts Correct**

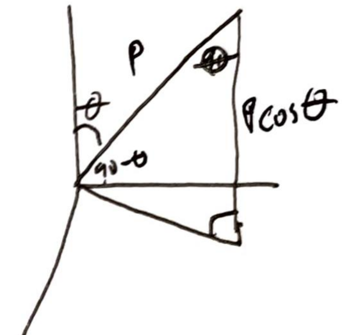
QUESTION 7

7 Directional derivatives **7 / 10**

✓ **- 3 pts Bad algebra**



1. E is 3-D solid in the first octant ($x \geq 0, y \geq 0, z \geq 0$) enclosed by the xy plane, the yz plane, the xz plane and the quarter sphere $x^2+y^2+z^2=9$.
Find $\iiint\limits_E z\ dV$



$x^2+y^2+z^2=\rho^2, \rho=3$

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \rho \leq 3$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho\cos\theta \cdot \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

$$= \rho^3 \cos\theta \sin\theta \, d\rho \, d\theta \, d\phi$$

$$\frac{\rho^4}{4}\cos\theta\sin\theta \Big]_0^3 = \frac{81}{4}\cos\theta\sin\theta$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{81}{4}\cos\theta\sin\theta \, d\theta \, d\phi$$

$$\frac{81}{4}\ \frac{(\sin\theta)^2}{2}\Big]_0^{\pi/2} = \frac{1}{2}\cdot\frac{81}{4} = \frac{81}{8} \quad \cancel{\frac{(\sin x)^2}{2} = \frac{2\sin x \cos x}{2}}$$

$$\int_0^{\pi/2} \frac{81}{8}\, d\phi = \frac{81\pi}{16}$$

$$\boxed{\frac{81\pi}{16}}$$

1 Spherical 10 / 10

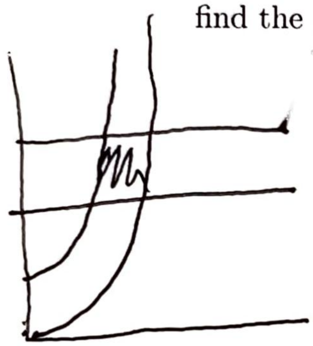
✓ - 0 pts Correct

$y=x^2 \qquad y=x^2+1$

$x=\sqrt{y} \qquad x^2=y-1$

$x=\sqrt{y-1}$

2. Region D in Quadrant I of the xy plane is bounded by $y=2, y=3, y=x^2$, and $y=x^2+1$. Use the change of variable, $s=y, t=x^2-y$ to find the area of D.



$$\leftarrow \int_2^3 \int_{\sqrt{y-1}}^{\sqrt{y}} 1 \, dx\, dy$$

Convert to st plane

Jacobian

$x=\sqrt{t+s} \qquad \frac{\partial x}{\partial s}=\frac{1}{2\sqrt{t+s}} \qquad \frac{\partial x}{\partial t}=\frac{1}{2\sqrt{t+s}}$

$y=s \qquad \frac{\partial y}{\partial s}=1 \qquad \frac{\partial y}{\partial t}=0$

$$\det\begin{bmatrix} \frac{1}{2\sqrt{t+s}} & \frac{1}{2\sqrt{t+s}} \\ 1 & 0 \end{bmatrix} = -\frac{1}{2\sqrt{t+s}}$$

$$\iint_D 1 \cdot \left| -\frac{1}{2\sqrt{t+s}} \right| ds\, dt$$

D is defined as

$2 \le s \le 3$

$-1 \le t \le 0$

$t=x^2-y$

$y=x^2, \; y-x^2=0, \; x^2-y=0$

$y=x^2+1, \; y-x^2=+1, \; x^2-y=-1$

$$\int_{-1}^{0}\int_2^3 \frac{1}{2\sqrt{t+s}}\, ds\, dt = \int_{-1}^{0} \left[\sqrt{t+s}\right]_2^3 dt = \int_{-1}^{0} \sqrt{t+3}-\sqrt{t+2}\, dt$$

$$= \frac{2}{3}\left[(t+3)^{3/2} - (t+2)^{3/2}\right]_{-1}^{0} = \frac{2}{3}\left(\left(3^{3/2}-2^{3/2}\right)-\left(2^{3/2}-2\right)\right)$$

$$= \frac{2}{3}\left(\sqrt{3}^3 - 2\cdot 2^{3/2} + 2\right)$$

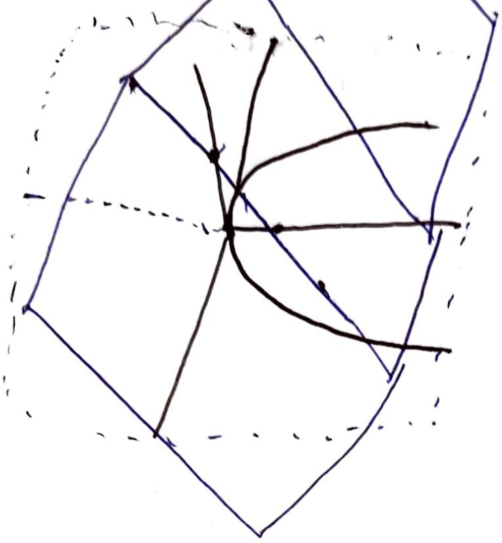
$$= \frac{2}{3}\left(\sqrt{3}^3 - 2^{7/2} + 2\right)$$

$$\approx 0.3595$$

2 Change of variable 10 / 10

✓ - 0 pts Correct

3. The solid E is bounded by $y = x^2$, $z = 1 - y$, and the xy plane. Use a triple integral to find the volume. (Hint: x can be negative.)



~~30~~

$0 \le z \le 1-y$

$\boxed{0 \le y \le 1}$ at $y=1$, $z=0$, bounded by xy plane

$-1 \le x \le 1$

$(-1)^2 = 1 \qquad 1^2 = 1$

$$V = \int_{-1}^{1} \boxed{\int_{0}^{1}} \int_{0}^{1-y} 1 \, dz \, dy \, dx$$

$$= \int_{-1}^{1} \int_{0}^{1} 1 - y \, dy \, dx$$

$$= \int_{-1}^{1} y - \frac{y^2}{2} \Big]_0^1 = 1 - \frac{1}{2}$$

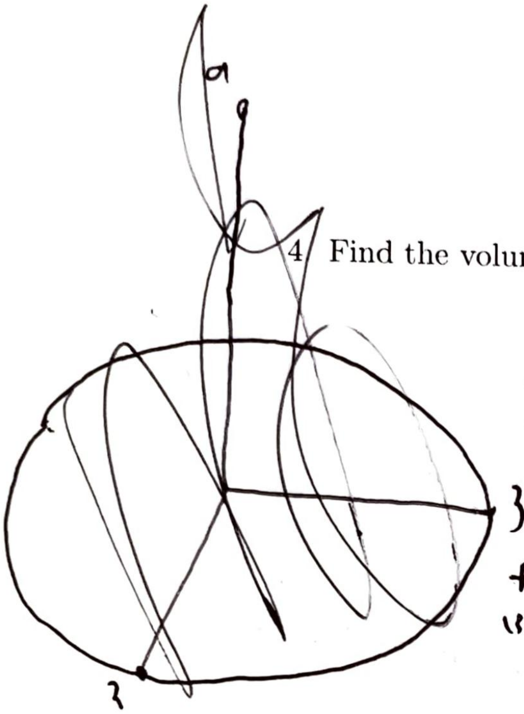
$$\int_{-1}^{1} 1/2 \, dx = \frac{1}{2} x \Big]_{-1}^{1} = \frac{1}{2} - -\frac{1}{2} = 1$$

$$\boxed{V = 1}$$

3 Triple integral for volume 7 / 10

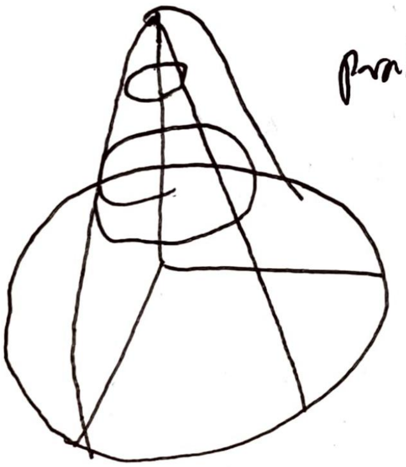
✓ **- 3 pts** Bad bounds

4. Find the volume enclosed by $z = 9 - x^2 - y^2$ and the xy plane.



rip

my is right



paraboloid

Convert to polar

$z = 9 - r^2$

$0 \le \theta \le 2\pi$

$0 \le r \le 3$

$$\int_0^{2\pi}\int_0^{3} (9-r^2)\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{3} 9r - r^3\,dr\,d\theta$$

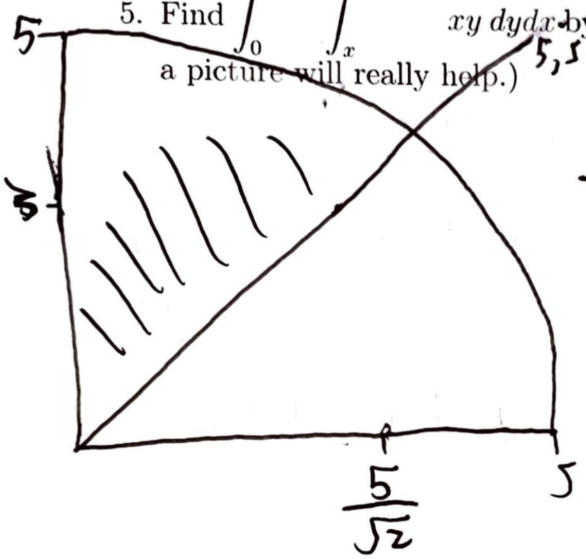
$$= \int_0^{2\pi} \frac{9}{2}r^2 - \frac{r^4}{4}\Big]_0^3 \,d\theta = \frac{81}{2} - \frac{81}{4}$$

$$\int_0^{2\pi} 81/4 \,d\theta = \frac{81\cdot 2\pi}{4} = \boxed{\frac{81\pi}{2}}$$

4 Volume 2 10 / 10

✓ - 0 pts Correct

5. Find $\int_0^{5/\sqrt{2}} \int_x^{\sqrt{25-x^2}} xy\, dy dx$ by converting to polar coordinates. (Hint: a picture will really help.)



$x = \sqrt{25-x^2}$

$x^2 = 25 - x^2 \quad 2x^2 = 25$

$x^2 = \frac{25}{2}$

$x = \sqrt{\frac{25}{2}}$ or $\frac{5}{\sqrt{2}}$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 5$

$\int_{\pi/4}^{\pi/2} \int_0^5 (r^2 \sin\theta \cos\theta) r \, dr d\theta$

$= \iint_R r^3 \sin\theta \cos\theta \, dr d\theta$

$= \int_{\pi/4}^{\pi/2} \left.\frac{r^4}{4}\right]_0^5 \sin\theta \cos\theta \, d\theta$

$= \frac{625}{4} \int_{\pi/4}^{\pi/2} \sin\theta \cos\theta \, d\theta$

$= \frac{625}{4} \cdot (\sin\theta)^2 \Big]_{\pi/4}^{\pi/2}$

$\frac{\sin\theta^2}{2}$

$\frac{625}{4} \cdot \frac{\left(1^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right)}{2}$ two

$\frac{625}{4} \cdot \frac{1}{4} = \frac{625}{16}$

Final answer

$\frac{625}{16}$

5 Cartesian to polar **10 / 10**

✓ **- 0 pts** **Correct**

6. A surface is given by $x^2y + xz = yz$.

(a) Find a Cartesian equation of the tangent plane at $(1, 2, 2)$.

$x^2y + xz - yz = f(x,y,z)$

$f_x = 2xy + z$, at $(1,2,2) = 4+2 = 6$

$f_y = x^2 - z$, at $(1,2,2) = -1$

$f_z = x - y$, at $(1,2,2) = -1$

$\vec{N} = \nabla f(1,2,2) = \langle 6, -1, -1 \rangle$

$\vec{N} \cdot \langle x-1, y-2, z-2 \rangle = 0$

$= 6(x-1) - (y-2) - (z-2) =$

$= 6x - 6 - y + 2 - z + 2 = 0$

$= \boxed{6x - y - z = 2}$

(b) Find parametric equations of the normal line at $(1, 2, 2)$.

$\langle 1, 2, 2 \rangle + t\langle 6, -1, -1 \rangle$.

$$\boxed{\begin{aligned} x(t) &= 6t + 1 \\ y(t) &= -t + 2 \\ z(t) &= -t + 2 \end{aligned}}$$

6.1 Tangent plane 5 / 5

✓ - 0 pts Correct

6. A surface is given by $x^2y + xz = yz$.

(a) Find a Cartesian equation of the tangent plane at $(1, 2, 2)$.

$x^2y + xz - yz = f(x,y,z)$

$f_x = 2xy + z$, at $(1,2,2) = 4+2 = 6$

$f_y = x^2 - z$, at $(1,2,2) = -1$

$f_z = x - y$, at $(1,2,2) = -1$

$\vec{N} = \nabla f(1,2,2) = \langle 6, -1, -1 \rangle$

$\vec{N} \cdot \langle x-1, y-2, z-2 \rangle = 0$

$= 6(x-1) - (y-2) - (z-2) =$

$= 6x - 6 - y + 2 - z + 2 = 0$

$= \boxed{6x - y - z = 2}$

(b) Find parametric equations of the normal line at $(1, 2, 2)$.

$\langle 1,2,2 \rangle + t\langle 6,-1,-1 \rangle$.

$$\boxed{\begin{aligned} x(t) &= 6t + 1 \\ y(t) &= -t + 2 \\ z(t) &= -t + 2 \end{aligned}}$$

6.2 Normal line 5 / 5

✓ **- 0 pts** Correct

7. $\dfrac{df}{d\vec{u}}(1,2) = \dfrac{6}{5}$ in the direction of $< 4, 3 >$, and $\dfrac{df}{d\vec{u}}(1,2) = \dfrac{3}{\sqrt{5}}$ in the direction of $< 1, 2 >$. Find $\nabla f(1,2)$.

$\vec{u}\langle 4,3\rangle = |\langle 4,3\rangle| = 5$

$\frac{4}{5}f_x + \frac{3}{5}f_y = \frac{6}{5}$, $\quad 4f_x + 3f_y = 6$

$\vec{u}\langle 1,2\rangle = |\langle 1,2\rangle| = \sqrt{5}$

$\frac{1}{\sqrt{5}}f_x + \frac{2}{\sqrt{5}}f_y = \frac{3}{\sqrt{5}}$, $\quad f_x + 2f_y = 3$

$4(f_x + 2f_y = 3)$

$-\ 4f_x + 3f_y = 6$

$\cancel{8}\ 5f_y = 6$

$f_y = \frac{5}{6}$

$f_y + \frac{10}{6} = \cancel{3}\ \frac{18}{6}$

$f_x = \frac{8}{6}$

$$\boxed{\nabla f(1,2) = \boxed{\left\langle \frac{8}{6}, \frac{5}{6} \right\rangle}}$$

7 Directional derivatives 7 / 10

✓ **- 3 pts** **Bad algebra**