What Is the Matrix Exponential?

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The exponential of a square matrix A is defined by the power series (introduced by Laguerre in 1867)

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

That the series converges follows from the convergence of the series for scalars. Various other formulas are available, such as

$$e^A = \lim_{s\to\infty} (I + A/s)^s.$$

The matrix exponential is always nonsingular and $(e^A)^{-1} = e^{-A}$.

Much interest lies in the connection between e^{A+B} and $e^A e^B$. It is easy to show that $e^{A+B} = e^A e^B$ if A and B commute, but commutativity is not necessary for the equality to hold. Series expansions are available that relate e^{A+B} to $e^A e^B$ for general A and B, including the Baker–Campbell–Hausdorff formula and the Zassenhaus formula, both of which involve the commutator $[A, B] = AB - BA$. For Hermitian A and B the inequality $\operatorname{trace}(e^{A+B}) \le \operatorname{trace}(e^A e^B)$ was proved independently by Golden and Thompson in 1965.

Especially important is the relation

$$e^A = \left(e^{A/2^s}\right)^{2^s},$$

for integer s, which is used in the scaling and squaring method for computing the matrix exponential.

Another important property of the matrix exponential is that it maps skew-symmetric matrices to orthogonal ones. Indeed if $A = -A^T$ then

$$\left(e^A\right)^{-1} = e^{-A} = e^{A^T} = \left(e^A\right)^T.$$

This is a special case of the fact that the exponential maps elements of a Lie algebra into the corresponding Lie group.

The matrix exponential plays a fundamental role in linear ordinary differential equations (ODEs). The vector ODE

$$\frac{dy}{dt} = Ay, \quad y(0) = c$$

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has solution $y(t) = \mathrm{e}^{At}c$, while the solution of the ODE in $n \times n$ matrices

$$\frac{dY}{dt} = AY + YB, \quad Y(0) = C$$

is $Y(t) = \mathrm{e}^{At}Ce^{Bt}$.

In control theory, the matrix exponential is used in converting from continuous time dynamical systems to discrete time ones. Another application of the matrix exponential is in centrality measures for nodes in networks.

Many methods have been proposed for computing the matrix exponential. See the references for details.

References

This is a minimal set of references, which contain further useful references within.

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