What Is the Second Difference Matrix?

Nicholas J. Higham[*]

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The second difference matrix is the tridiagonal matrix T_n with diagonal elements 2 and sub- and superdiagonal elements -1:

$$T_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

It arises when a second derivative is approximated by the central second difference $f''(x) \approx (f(x+h) - 2f(x) + f(x-h))/h^2$. (Accordingly, the second difference matrix is sometimes defined as $-T_n$.) In MATLAB, T_n can be generated by `gallery('tridiag',n)`, which is returned as a aparse matrix.

This is Gil Strang's favorite matrix. The photo, from his home page, shows a birthday cake representation of the matrix.

The second difference matrix is symmetric positive definite. The easiest way to see this is to define the full rank rectangular matrix

$$L_n = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times n}$$

[*]Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (`nick.higham@manchester.ac.uk`).

and note that $T_n = L_n^T L_n$. The factorization corresponds to representing a central difference as the product of a forward difference and a backward difference. Other properties of the second difference matrix are that it is diagonally dominant, a Toeplitz matrix, and an M-matrix.

Cholesky Factorization

In an LU factorization $A = LU$ the pivots u_{ii} are 2, 3/2, 4/3, ..., $(n+1)/n$. Hence the pivots form a decreasing sequence tending to 1 as $n \to \infty$. The diagonal of the Cholesky factor contains the square roots of the pivots. This means that in the Cholesky factorization $A = R^*R$ with R upper bidiagonal, $r_{nn} \to 1$ and $r_{n,n-1} \to -1$ as $n \to \infty$.

Determinant, Inverse, Condition Number

Since the determinant is the product of the pivots, $\det(T_n) = n+1$.

The inverse of T_n is full, with (i,j) element $i(n-j+1)/(n+1)$ for $j \geq i$. For example,

$$T_5^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

The 2-norm condition number satisfies $\kappa_2(T_n) \sim 4n^2/\pi^2$ (as follows from the formula (1) below for the eigenvalues).

Eigenvalues and Eigenvectors

The eigenvalues of T_n are

$$\mu_k = 4\sin^2\Big(\frac{k\phi}{2}\Big), \quad k = 1:n, \qquad (1)$$

where $\phi = \pi/(n+1)$, with corresponding eigenvector

$$v_k = \big[\sin(k\phi), \quad \sin(2k\phi), \quad \dots, \quad \sin(nk\phi)\big]^T.$$

The matrix Q with

$$q_{ij} = \Big(\frac{2}{n+1}\Big)^{1/2} \sin\Big(\frac{ij\pi}{n+1}\Big)$$

is therefore an eigenvector matrix for T_n: $Q^*AQ = \mathrm{diag}(\mu_k)$.

Variations

Various modifications of the second difference matrix arise and similar results can be derived. For example, consider the matrix obtained by changing the (n,n) element to 1:

$$\widetilde{T}_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

It can be shown that $\widetilde{T}_n^{-1}$ has (i,j) element $\min(i,j)$ and eigenvalues $4\cos(j\pi/(2n+1))^2$, $j=1:n$.

If we perturb the $(1,1)$ of $\widetilde{T}_n$ to 1, we obtain a singular matrix, but suppose we perturb the $(1,1)$ element to 3:

$$\widehat{T}_n = \begin{bmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

The inverse is $\widehat{T}_n^{-1} = G/2$, where G with (i,j) element $2\min(i,j)-1$ is a matrix of Givens, and the eigenvalues are $4\cos((2j-1)\pi/(4n))^2$, $j=1:n$.

Notes

The factorization $T_n = L_n^T L_n$ is noted by Strang (2012).

For a derivation of the eigenvalues and eigenvectors see Todd (1977, p. 155 ff.). For the eigenvalues of $\widetilde{T}_n$ see Fortiana and Cuadras (1997). Givens's matrix is described by Newman and Todd (1958) and Todd (1977).

References

This is a minimal set of references, which contain further useful references within.

- J. Fortiana and C. N. Cuadras, A Family of Matrices, the Discretized Brownian Bridge, and Distance-Based Regression, Linear Algebra Appl. 264, 173–188, 1997.
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- John Todd, Basic Numerical Mathematics, Vol. 2: Numerical Algebra, Birkhäuser, Basel, and Academic Press, New York, 1977.

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