What Is an Orthogonal Matrix?

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A real, square matrix Q is orthogonal if $Q^TQ = QQ^T = I$ (the identity matrix). Equivalently, $Q^{-1} = Q^T$. The columns of an orthogonal matrix are orthonormal, that is, they have 2-norm (Euclidean length) 1 and are mutually orthogonal. The same is true of the rows.

Important examples of orthogonal matrices are rotations and reflectors. A 2×2 rotation matrix has the form

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \quad c^2 + s^2 = 1.$$

For such a matrix, $c = \cos\theta$ and $s = \sin\theta$ for some θ, and the multiplication $y = Qx$ for a 2×1 vector x represents a rotation through an angle θ radians. An $n \times n$ rotation matrix is formed by embedding the 2×2 matrix into the identity matrix of order n.

A Householder reflector is a matrix of the form $H = I - 2uu^T/(u^Tu)$, where u is a nonzero n-vector. It is orthogonal and symmetric. When applied to a vector it reflects the vector about the hyperplane orthogonal to v. For $n = 2$, such a matrix has the form

$$\begin{bmatrix} c & s \\ s & -c \end{bmatrix}, \quad c^2 + s^2 = 1.$$

Here is the 4×4 Householder reflector corresponding to $v = [1, 1, 1, 1]^T/2$:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

This is 1/2 times a Hadamard matrix.

Various explicit formulas are known for orthogonal matrices. For example, the $n \times n$ matrices with (i, j) elements

$$q_{ij} = \frac{2}{\sqrt{2n+1}} \sin\left(\frac{2ij\pi}{2n+1}\right)$$

and

$$q_{ij} = \sqrt{\frac{2}{n}} \cos\left(\frac{(i-1/2)(j-1/2)\pi}{n}\right)$$

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are orthogonal. These and other orthogonal matrices, as well as diagonal scalings of orthogonal matrices, are constructed by the MATLAB function `gallery('orthog',...)`.

Here are some properties of orthogonal matrices.

- All the eigenvalues are on the unit circle, that is, they have modulus 1.
- All the singular values are 1.
- The 2-norm condition number is 1, so orthogonal matrices are perfectly conditioned.
- Multiplication by an orthogonal matrix preserves Euclidean length: $\|Qx\|_2 = \|x\|_2$ for any vector x.
- The determinant of an orthogonal matrix is ± 1. A rotation has determinant 1 while a reflection has determinant -1.

Orthogonal matrices can be generated from skew-symmetric ones. If S is skew-symmetric ($S = -S^T$) then $\exp(S)$ (the matrix exponential) is orthogonal and the Cayley transform $(I - S)(I + S)^{-1}$ is orthogonal as long as S has no eigenvalue equal to -1.

Unitary matrices are complex square matrices Q for which $Q^*Q = QQ^* = I$, where Q^* is the conjugate transpose of Q. They have analogous properties to orthogonal matrices.

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