What Is a Vector Norm?

Nicholas J. Higham[*]

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A vector norm measures the size, or length, of a vector. For complex n-vectors, a vector norm is a function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ satisfying

- $\|x\| \ge 0$ with equality if and only if $x = 0$,
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$, $x \in \mathbb{C}^n$,
- $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in \mathbb{C}^n$ (the triangle inequality).

An important class of norms is the Hölder p-norms

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad p \ge 1. \qquad (1)$$

The ∞-norm is interpreted as $\lim_{p\to\infty} \|x\|_p$ and is given by

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

Other important special cases are

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i|, && \text{“Manhattan” or “taxi cab” norm,} \\ \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = (x^* x)^{1/2}, && \text{Euclidean length.} \end{aligned}$$

A useful concept is that of the *dual norm* associated with a given vector norm, which is defined by

$$\|y\|^D = \max_{x \ne 0} \frac{|x^* y|}{\|x\|}.$$

The maximum is attained because the definition is equivalent to $\|y\|^D = \max\{\, |x^* y| : \|x\| = 1 \,\}$, in which we are maximizing a continuous function over a compact set. This definition immediately gives the attainable inequality

$$|x^* y| \le \|x\| \, \|y\|^D, \qquad (2)$$

which is the generalized Hölder inequality.

[*]Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (`nick.higham@manchester.ac.uk`).

The dual of the p-norm is the q-norm, where $p^{-1} + q^{-1} = 1$, so for the p-norms the inequality (2) becomes the (standard) Hölder inequality,

$$|x^*y| \le \|x\|_p \, \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

An important special case is the Cauchy–Schwarz inequality,

$$|x^*y| \le \|x\|_2 \, \|y\|_2.$$

The notation $\|x\|_0$ is used to denote the number of nonzero entries in x, even though it is not a vector norm and is not obtained from (1) with $p = 0$. In portfolio optimization, if x_k specifies how much to invest in stock k then the inequality $\|x\|_0 \le k$ says "invest in at most k stocks".

In numerical linear algebra, vector norms play a crucial role in the definition of a subordinate matrix norm, as we will explain in the next post in this series.

All norms on $\mathbb{C}^n$ are equivalent in the sense that for any two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ there exist positive constants ν_1 and ν_2 such that

$$\nu_1 \|x\|_\alpha \le \|x\|_\beta \le \nu_2 \|x\|_\alpha \quad \text{for all } x \in \mathbb{C}^n.$$

For example, it is easy to see that

$$\begin{aligned} \|x\|_2 &\le \|x\|_1 \le \sqrt{n}\|x\|_2, \\ \|x\|_\infty &\le \|x\|_2 \le \sqrt{n}\|x\|_\infty, \\ \|x\|_\infty &\le \|x\|_1 \le n\|x\|_\infty. \end{aligned}$$

The 2-norm is invariant under unitary transformations: if $Q^*Q = I$, then $\|Qx\|^2 = x^*Q^*Qx = x^*x = \|x\|_2^2$.

Care must be taken to avoid overflow and (damaging) underflow when evaluating a vector p-norm in floating-point arithmetic for $p \ne 1, \infty$. One can simply use the formula $\|x\|_p = \|(x/\|x\|_\infty)\|_p \|x\|_\infty$, but this requires two passes over the data (the first to evaluate $\|x\|_\infty$). For more efficient one-pass algorithms for the 2-norm see Higham (2002, Sec. 21.8) and Harayama et al. (2021).

References

This is a minimal set of references, which contain further useful references within.

- Takeyuki Harayama, Shuhei Kudo, Daichi Mukunoki, Toshiyuki Imamura, and Daisuke Takahashi, A Rapid Euclidean Norm Calculation Algorithm that Reduces Overflow and Underflow, in Computational Science and Its Applications–ICCSA 2021, O. Gervasi et al., eds, 95–110, Springer, 2021.
- Nicholas J. Higham, Accuracy and Stability of Numerical Algorithms, second edition, Society for Industrial and Applied Mathematics,

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