What Is a Submatrix?

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A *submatrix* of a matrix is another matrix obtained by forming the intersection of certain rows and columns, or equivalently by deleting certain rows and columns. More precisely, let A be an $m \times n$ matrix and let $1 \le i_1 < i_2 < \cdots < i_p \le m$ and $1 \le j_1 < j_2 < \cdots < j_q \le n$. Then the $p \times q$ matrix B with $b_{rs} = a_{i_r j_s}$ is the submatrix of A comprising the elements at the intersection of the rows indexed by $i_1, \dots, i_p$ and the columns indexed by $j_1, \dots, j_q$. For example, for the matrix

$$\begin{bmatrix} \boxed{1} & \boxed{2} & 3 \\ 4 & 5 & 6 \\ \boxed{7} & \boxed{8} & 9 \end{bmatrix} = \begin{bmatrix} \boxed{1} & \boxed{2} & 3 \\ \boxed{4} & 5 & 6 \\ 7 & \boxed{8} & 9 \end{bmatrix},$$

shown with four elements highlighted in two different ways,

$$\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

is a submatrix (the intersection of rows 1 and 3 and columns 1 and 2, or what is left after deleting row 2 and column 3), but

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

is *not* a submatrix.

Submatrices include the mn matrix elements and the matrix itself, but there are many of intermediate size: an $m \times n$ matrix has $(2^m - 1)(2^n - 1)$ submatrices in total (counting both square and nonsquare submatrices).

If $p = q$ and $i_k = j_k$, $k = 1\colon p$, then B is a *principal submatrix* of A, which is a submatrix symmetrically located about the diagonal. If, in addition, $i_k = k$, $k = 1\colon p$, then B is a *leading principal submatrix* of A, which is one situated in the top left corner of A.

The determinant of a square submatrix is called a *minor*. The Laplace expansion of the determinant expresses the determinant as a weighted sum of minors.

The Colon Notation

In various programming languages, notably MATLAB, and in numerical linear algebra, a colon notation is used to denote submatrices consisting of contiguous rows and columns.

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For integers p and q we denote by $p:q$ the sequence $p, p+1, \dots, q$. Thus $i = 1:n$ is another way of writing $i = 1, 2, \dots, n$.

We write $A(p:q, r:s)$ for the submatrix of A comprising the intersection of rows p to q and columns r to s, that is,

$$A(p:q, r:s) = \begin{bmatrix} a_{pr} & \cdots & a_{ps} \\ \vdots & \ddots & \vdots \\ a_{qr} & \cdots & a_{qs} \end{bmatrix}.$$

We can think of $A(p:q, r:s)$ as a projection of A using the corresponding rows and columns of the identity matrix:

$$A(p:q, r:s) = I(p:q,:)\, A\, I(:, r:s).$$

As special cases, $A(k,:)$ denotes the kth row of A and $A(:,k)$ the kth column of A.

Here are some examples of using the colon notation to extract submatrices in MATLAB. Rows and columns can be indexed by a range using the colon notation or by specifying the required indices in a vector. The matrix used is from the Anymatrix collection.

```
>> A = anymatrix('core/beta',5)
A =
     1     2     3     4     5
     2     6    12    20    30
     3    12    30    60   105
     4    20    60   140   280
     5    30   105   280   630

>> A(3:4, [2 4 5]) % Rectangular submatrix.
ans =
    12    60   105
    20   140   280

>> A(1:2,4:5)      % Square, but nonprincipal, submatrix.
ans =
     4     5
    20    30

>> A([3 5],[3 5])  % Principal submatrix.
ans =
    30   105
   105   630
```

Block Matrices

Submatrices are intimately associated with block matrices, which are matrices in which the elements are themselves matrices. For example, a 4×4 matrix A can be regarded as a block 2×2 matrix, where each element is a 2×2 submatrix of A:

$$A = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} = A(1\!:\!2, 1\!:\!2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and likewise for the other three blocks.

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- What Is a Block Matrix?
- What Is the Determinant of a Matrix?

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