What Is the Wilson Matrix?

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The 4×4 matrix

$$W = \begin{bmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{bmatrix}$$

appears in a 1946 paper by Morris, in which it is described as having been "devised by Mr. T. S. Wilson." The matrix is symmetric positive definite with determinant 1 and inverse

$$W^{-1} = \begin{bmatrix} 68 & -41 & -17 & 10 \\ -41 & 25 & 10 & -6 \\ -17 & 10 & 5 & -3 \\ 10 & -6 & -3 & 2 \end{bmatrix},$$

so it is moderately ill conditioned with $\kappa_2(W) = \|W\|_2 \|W^{-1}\|_2 \approx 2.98409 \times 10^3$. This little matrix has been used as an example and for test purposes in many research papers and books over the years, in particular by John Todd, who described it as "the notorious matrix W of T. S. Wilson".

Rutishauser (1968) stated that "the famous Wilson matrix is not a very striking example of an ill-conditioned matrix", on the basis that $\kappa_2(A) \le 40{,}000$ for a "positive definite symmetric 4×4 matrix with integer elements not exceeding 10" and he gave the positive definite matrix

$$A_0 = \begin{bmatrix} 10 & 1 & 4 & 0 \\ 1 & 10 & 5 & -1 \\ 4 & 5 & 10 & 7 \\ 0 & -1 & 7 & 9 \end{bmatrix}, \quad A_0^{-1} = \begin{bmatrix} 105 & 167 & -304 & 255 \\ 167 & 266 & -484 & 406 \\ -304 & -484 & 881 & -739 \\ 255 & 406 & -739 & 620 \end{bmatrix},$$

for which $\kappa_2(A_0) = 3.57924 \times 10^4$. The matrix A_0 is therefore a factor 12 more ill conditioned than W. Rutishauser did not give a proof of the stated bound.

Moler (2018) asked how ill-conditioned W is relative to matrices in the set

$$\mathcal{S} = \{\, A \in \mathbb{R}^{4\times 4} : A = A^T \text{ is nonsingular with integer entries between 1 and 10} \,\}.$$

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He generated one million random matrices from $\mathcal{S}$ and found that about 0.21 percent of them had a larger condition number than W. The matrix with the largest condition number was the indefinite matrix

$$A_1 = \begin{bmatrix} 1 & 3 & 10 & 10 \\ 3 & 4 & 8 & 9 \\ 10 & 8 & 3 & 9 \\ 10 & 9 & 9 & 3 \end{bmatrix}, \quad A_1^{-1} = \begin{bmatrix} 573 & -804 & 159 & 25 \\ -804 & 1128 & -223 & -35 \\ 159 & -223 & 44 & 7 \\ 25 & -35 & 7 & 1 \end{bmatrix},$$

for which $\kappa_2(A_1) \approx 4.80867 \times 10^4$. How far is this matrix from being a worst case?

As the Wilson matrix is positive definite, we are also interested in how ill conditioned a matrix in the set

$$\mathcal{P} = \{\, A \in \mathbb{R}^{4\times 4} : A = A^T \text{ is symmetric positive definite with integer entries between 1 and 10} \,\}$$

can be.

Condition Number Bounds

We first consider bounds on $\kappa_2(A)$ for $A \in \mathcal{S}$. It is possible to obtain a bound from first principles by using the relation $A^{-1} = \operatorname{adj}(A)/\det(A)$, where $\operatorname{adj}(A)$ is the adjugate matrix, along with the fact that $|\det(A)| \geq 1$ since A has integer entries. Higham and Lettington (2021) found that the smallest bound they could obtain came from a bound of Merikoski et al. (2007): for nonsingular $B \in \mathbb{R}^{n\times n}$,

$$\kappa_2(B) \leq \left(\frac{1+x}{1-x}\right)^{1/2}, \quad x = \sqrt{1 - (n/\|B\|_F^2)^n |\det(B)|^2}.$$

Applying this bound to $A \in \mathcal{S}$, using the fact that $(1+x)/(1-x)$ is monotonically increasing for $x \in (0,1)$, gives

$$\kappa_2(A) \leq 2.97606\cdots \times 10^5 =: \beta_S, \quad A \in \mathcal{S}. \qquad (1)$$

Another result from Merikoski et al. (2007) gives, for symmetric positive definite $C \in\in \mathbb{R}^{n\times n}$,

$$\kappa_2(C) \leq \frac{1+x}{1-x}, \quad x = \sqrt{1 - (n/\operatorname{trace}(C))^n \det(C)}.$$

For $A \in \mathcal{P}$, since $\det(A) \geq 1$ we have $x \leq \sqrt{1-(1/10)^4}$, and hence

$$\kappa_2(A) \leq 3.99980 \times 10^4 =: \beta_P, \quad A \in \mathcal{P}. \qquad (2)$$

Recall that Rutishauser's bound is 4×10^4. The bounds (1) and (2) remain valid if we modify the definitions of $\mathcal{S}$ and $\mathcal{P}$ to allow zero elements (note that Rutishauser's matrix A_0 has a zero element).

Experiment

The sets $\mathcal{S}$ and $\mathcal{P}$ are large: $\mathcal{S}$ has on the order of 10^{10} elements. Exhaustively searching over the sets in reasonable time is possible with a carefully optimized code. Higham and Lettington (2021) use a MATLAB code that loops over all symmetric matrices with integer elements between 1 and 10 and

- evaluates $\det(A)$ from an explicit expression (exactly computed for such matrices) and discards A if the matrix is singular;
- computes the eigenvalues λ_i of A and obtains the condition number as $\kappa_2(A) = \max_i |\lambda_i| / \min_i |\lambda_i|$ (since A is symmetric); and
- for $\mathcal{P}$, checks whether A is positive definite by checking whether the smallest eigenvalue is positive.

The code is available at `https://github.com/higham/wilson-opt`.

The maximum over $\mathcal{S}$ is attained for

$$A_2 = \begin{bmatrix} 2 & 7 & 10 & 10 \\ 7 & 10 & 10 & 9 \\ 10 & 10 & 10 & 1 \\ 10 & 9 & 1 & 9 \end{bmatrix}, \quad A_2^{-1} = \begin{bmatrix} 640 & -987 & 323 & 240 \\ -987 & 1522 & -498 & -370 \\ 323 & -498 & 163 & 121 \\ 240 & -370 & 121 & 90 \end{bmatrix},$$

which has $\kappa_2(A_2) \approx 7.6119 \times 10^4$. and determinant -1. The maximum over $\mathcal{P}$ is attained for

$$A_3 = \begin{bmatrix} 9 & 1 & 1 & 5 \\ 1 & 10 & 1 & 9 \\ 1 & 1 & 10 & 1 \\ 5 & 9 & 1 & 10 \end{bmatrix}, \quad A_3^{-1} = \begin{bmatrix} 188 & 347 & -13 & -405 \\ 347 & 641 & -24 & -748 \\ -13 & -24 & 1 & 28 \\ -405 & -748 & 28 & 873 \end{bmatrix}.$$

which has $\kappa_2(A_3) \approx 3.5529 \times 10^4$ and determinant 1. Obviously, symmetric permutations of these matrices are also optimal.

The following table summarizes the condition numbers of the matrices discussed and how close they are to the bounds.

Matrix A	Comment	$\kappa_2(A)$	$\beta_S/\kappa_2(A)$	$\beta_P/\kappa_2(A)$
W	Wilson matrix	2.98409×10^3	99.73	13.40
A_9	Rutishauser's matrix	3.57924×10^4	8.31	1.12
A_1	By random sampling	4.80867×10^4	6.19	–
A_2	Optimal matrices in $\mathcal{S}$	7.61190×10^4	3.91	–
A_3	Optimal matrices in $\mathcal{P}$	3.55286×10^4	8.38	1.13

Clearly, the bounds are reasonably sharp.

We do not know how Wilson constructed his matrix or to what extent he tried to maximize the condition number subject to the matrix entries being small integers. One possibility is that he constructed it via the factorization in the next section.

Integer Factorization

The Cholesky factor of the Wilson matrix is

$$R = \begin{bmatrix} \sqrt{5} & \frac{7\sqrt{5}}{5} & \frac{6\sqrt{5}}{5} & \sqrt{5} \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & 0 & \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad (W = R^T R).$$

Apart from the zero $(2,4)$ element, it is unremarkable. If we factor out the diagonal then we obtain the LDL^T factorization, which has rational elements:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{7}{5} & 1 & 0 & 0 \\ \frac{6}{5} & -2 & 1 & 0 \\ 1 & 0 & \frac{3}{2} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad (W = LDL^T).$$

Suppose we drop the requirement of triangularity and ask whether the Wilson matrix has a factorization $W = Z^T Z$ with a 4×4 matrix Z of integers. It is known that every symmetric positive definite $n \times n$ matrix A of integers with determinant 1 has a factorization $A = Z^T Z$ with Z an $n \times n$ matrix of integers as long as $n \le 7$, but examples are known for $n = 8$ for which the factorization does not exist. This result is mentioned by Taussky (1961) and goes back to Hermite, Minkowski, and Mordell. Higham and Lettington (2021) found the integer factor

$$Z_0 = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

of W, which is block upper triangular so can be thought of as a block Cholesky factor. Higham, Lettington, and Schmidt (2021) draw on recent research that links the existence of such factorizations to number-theoretic considerations of quadratic forms to show that for the existence of an integer solution Z to $A = Z^T Z$ it is necessary that a certain quadratic equation in n variables has an integer solution. In the case of the Wilson matrix the equation is

$$2w^2 + x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2 = 952.$$

The authors solve this equation computationally and find Z_1 and two rational factors:

$$Z_1 = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 1 \\ \frac{3}{2} & 2 & 3 & 3 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 2 & 1 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \frac{3}{2} & 2 & 2 & 2 \\ \frac{3}{2} & 2 & 2 & 1 \\ \frac{1}{2} & 1 & 1 & 2 \\ -\frac{1}{2} & -1 & 1 & 1 \end{bmatrix}.$$

They show that these matrices are the only factors $Z \in \frac{1}{16}\mathbb{Z}$ of W up to left multiplication by integer orthogonal matrices.

Conclusions

The Wilson matrix has provided sterling service throughout the digital computer era as a convenient symmetric positive definite matrix for use in textbook examples and for testing algorithms. The recent discovery of its integer factorization has led to the development of new theory on when general $n \times n$ integer matrices A can be factored as $A = Z^T Z$ (when A is symmetric positive definite) or $A = Z^2$ (a problem also considered in Higham, Lettington, and Schmidt (2021)), with integer Z.

Olga Taussky Todd wrote in 1961 that "matrices with integral elements have been studied for a very long time and an enormous number of problems arise, both theoretical and practical." We wonder what else can be learned from the Wilson matrix and other integer test matrices.

References

This is a minimal set of references, which contain further useful references within.

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