What Is a Permutation Matrix?

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A *permutation matrix* is a square matrix in which every row and every column contains a single 1 and all the other elements are zero. Such a matrix, P say, is orthogonal, that is, $P^TP = PP^T = I_n$, so it is nonsingular and has determinant ± 1. The total number of $n \times n$ permutation matrices is $n!$.

Premultiplying a matrix by P reorders the rows and postmultiplying by P reorders the columns. A permutation matrix P that has the desired reordering effect is constructed by doing the same operations on the identity matrix.

Examples of permutation matrices are the identity matrix I_n, the reverse identity matrix J_n, and the shift matrix K_n (also called the cyclic permutation matrix), illustrated for $n = 4$ by

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad J_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Pre- or postmultiplying a matrix by J_n reverses the order of the rows and columns, respectively. Pre- or postmultiplying a matrix by K_n shifts the rows or columns, respectively, one place forward and moves the first one to the last position—that is, it cyclically permutes the rows or columns. Note that J_n is a symmetric Hankel matrix and K_n is a circulant matrix.

An *elementary permutation matrix* P differs from I_n in just two rows and columns, i and j, say. It can be written $P = I_n - (e_i - e_j)(e_i - e_j)^T$, where e_i is the ith column of I_n. Such a matrix is symmetric and so satisfies $P^2 = I_n$, and it has determinant -1. A general permutation matrix can be written as a product of elementary permutation matrices $P = P_1P_2 \dots P_k$, where k is such that $\det(P) = (-1)^k$.

It is easy to show that $\det(\lambda I - K_n) = \lambda^n - 1$, which means that the eigenvalues of K_n are $1, w, w^2, \dots, w^{n-1}$, where $w = \exp(2\pi \mathrm{i}/n)$ is the nth root of unity. The matrix K_n has two diagonals of 1s, which move up through the matrix as it is powered: $K_n^i \neq I$ for $i < n$ and $K_n^n = I$. The following animated gif superposes MATLAB spy plots of K_{64}, K_{64}^2, …, $K_{64}^{64} = I_{64}$. (Only one frame is shown in this LaTeX document.)

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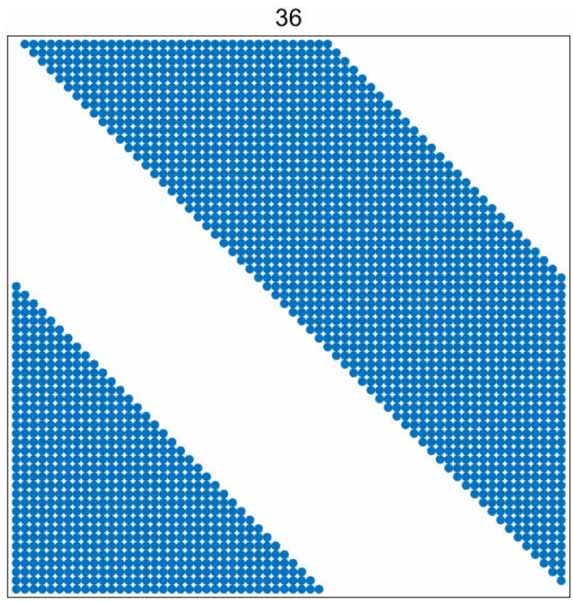

The shift matrix K_n plays a fundamental role in characterizing irreducible permutation matrices. Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is irreducible if there does not exist a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square, nonempty submatrices.

Theorem 1. For a permutation matrix $P \in \mathbb{R}^{n\times n}$ the following conditions are equivalent.

- P is irreducible.
- There exists a permutation matrix Q such that $Q^{-1}PQ = K_n$
- The eigenvalues of P are $1, w, w^2, \ldots, w^{n-1}$.

One consequence of Theorem 1 is that for any irreducible permutation matrix P, $\operatorname{rank}(P - I) = \operatorname{rank}(K_n - I) = n - 1$.

The next result shows that a reducible permutation matrix can be expressed in terms of irreducible permutation matrices.

Theorem 2. Every reducible permutation matrix is permutation similar to a direct sum of irreducible permutation matrices.

Another notable permutation matrix is the vec-permutation matrix, which relates $A \otimes B$ to $B \otimes A$, where $\otimes$ is the Kronecker product.

A permutation matrix is an example of a *doubly stochastic matrix*: a nonnegative matrix whose row and column sums are all equal to 1. A classic result characterizes doubly stochastic matrices in terms of permutation matrices.

Theorem 3 (Birkhoff). A matrix is doubly stochastic if and only if it is a convex combination of permutation matrices.

In coding, memory can be saved by representing a permutation matrix P as an integer vector p, where p_i is the column index of the 1 within the ith row of P. MATLAB functions that return permutation matrices can also return the permutation in vector form. Here is an example with the MATLAB `lu` function that illustrates how permuting a matrix can be done using the vector permutation representation.

```
>> A = gallery('frank',4), [L,U,P] = lu(A); P
A =
     4     3     2     1
     3     3     2     1
     0     2     2     1
     0     0     1     1
P =
     1     0     0     0
     0     0     1     0
     0     0     0     1
     0     1     0     0
>> P*A
ans =
     4     3     2     1
     0     2     2     1
     0     0     1     1
     3     3     2     1
>> [L,U,p] = lu(A,'vector'); p
p =
     1     3     4     2
>> A(p,:)
ans =
     4     3     2     1
     0     2     2     1
     0     0     1     1
     3     3     2     1
```

For more on handling permutations in MATLAB see section 24.3 of MATLAB Guide.

Notes

For proofs of Theorems 1–3 see Zhang (2011, Sec. 5.6). Theorem 3 is also proved in Horn and Johnson (2013, Thm. 8.7.2).

Permutations play a key role in the fast Fourier transform and its efficient implementation; see Van Loan (1992).

References

- Desmond Higham and Nicholas Higham, MATLAB Guide, third edition, SIAM, 2017.
- Roger A. Horn and Charles R. Johnson, Matrix Analysis, second edition, Cambridge University Press, 2013. My review of the second edition.

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