What Is the Inertia of a Matrix?

Nicholas J. Higham*

December 6, 2022

The *inertia* of a real symmetric $n \times n$ matrix A is a triple, written $\mathrm{In}(A) = (i_+(A), i_-(A), i_0(A))$, where $i_+(A)$ is the number of positive eigenvalues of A, $i_-(A)$ is the number of negative eigenvalues of A, and $i_0(A)$ is the number of zero eigenvalues of A.

The rank of A is $i_+(A) + i_-(A)$. The difference $i_+(A) - i_-(A)$ is called the *signature*.

In general it is not possible to determine the inertia by inspection, but some deductions can be made. If A has both positive and negative diagonal elements then $i_+(A) > 1$ and $i_-(A) > 1$. But in general the diagonal elements do not tell us much about the inertia. For example, here is a matrix that has positive diagonal elements but only one positive eigenvalue (and this example works for any n):

```
>> n = 4; A = -eye(n) + 2*ones(n), eigA = eig(sym(A))'
A =
     1     2     2     2
     2     1     2     2
     2     2     1     2
     2     2     2     1
eigA =
[-1, -1, -1, 7]
```

A *congruence transformation* of a symmetric matrix A is a transformation $A \to X^TAX$ for a nonsingular matrix X. The result is clearly symmetric. *Sylvester's law of inertia* (1852) says that the inertia is preserved under congruence transformations.

Theorem 1 (Sylvester's law of inertia). *If $A \in \mathbb{R}^{n\times n}$ is symmetric and $X \in \mathbb{R}^{n\times n}$ is nonsingular then* $\mathrm{In}(A) = \mathrm{In}(X^TAX)$.

Sylvester's law gives a way to determine the inertia without computing eigenvalues: find a congruence transformation that transforms A to a matrix whose inertia can be easily determined. A factorization $PAP^T = LDL^T$ does the job, where P is a permutation matrix, L is unit lower triangular, and D is diagonal Then $\mathrm{In}(A) = \mathrm{In}(D)$, and $\mathrm{In}(D)$ can be read off the diagonal of D. This factorization does not always exist, and if it does exist is can be numerically unstable. A block LDL^T factorization, in which D is block diagonal with diagonal blocks of size 1 or 2, always exists, and its computation is numerically stable with a suitable pivoting strategy such as symmetric rook pivoting.

For the matrix above we can compute a block LDL^T factorization using the MATLAB `ldl` function:

*Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (`nick.higham@manchester.ac.uk`).

```
>> [L,D,P] = ldl(A); D
D =
   1.0000e+00   2.0000e+00            0            0
   2.0000e+00   1.0000e+00            0            0
            0            0  -1.6667e+00            0
            0            0            0  -1.4000e+00
```

Since the leading 2-by-2 block of D has negative determinant and hence one positive eigenvalue and one negative eigenvalue, it follows that A has one positive eigenvalue and three negative eigenvalues.

A congruence transformation preserves the signs of the eigenvalues but not their magnitude. A result of Ostrowski (1959) bounds the ratios of the eigenvalues of the original and transformed matrices. Let the eigenvalues of a symmetric matrix be ordered $\lambda_n \le \cdots \le \lambda_1$.

> **Theorem** (Ostrowski). *For a symmetric $A \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$,*
>
> $$\lambda_k(X^*AX) = \theta_k \lambda_k(A), \quad k = 1\colon n,$$
>
> *where $\lambda_n(X^*X) \le \theta_k \le \lambda_1(X^*X)$.*

The theorem shows that the further X is from being orthogonal the greater the potential change in the eigenvalues.

Finally, we note that everything here generalizes to complex Hermitian matrices by replacing transpose by conjugate transpose.

Related Blog Posts

- What Is a Symmetric Indefinite Matrix? (2022)
- What Is an Eigenvalue? (2022)

This article is part of the "What Is" series, available from `https://nhigham.com/category/what-is` and in PDF form from the GitHub repository `https://github.com/higham/what-is`.