

What Is Backward Error?

Nicholas J. Higham*

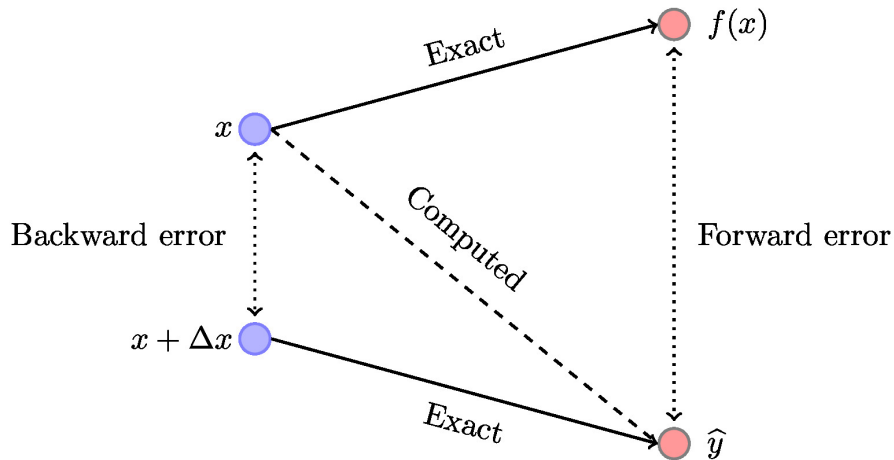
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Backward error is a measure of error associated with an approximate solution to a problem. Whereas the forward error is the distance between the approximate and true solutions, the backward error is how much the data must be perturbed to produce the approximate solution.

For a function f from \mathbb{R}^n to \mathbb{R}^n and an approximation y to $f(x)$, the backward error in y is the smallest Δx such that $y = f(x + \Delta x)$, for some appropriate measure of size. There can be many Δx satisfying this equation, so the backward error is the solution to a minimization problem. Using a vector norm and measuring perturbations in a relative sense, we can define the backward error in y as

$$\eta(y) = \min\{ \epsilon : y = f(x + \Delta x), \|\Delta x\| \leq \epsilon \|x\| \}.$$

In the following figure the solid lines denote exact mappings and the dashed line shows the mapping that was actually computed.



Usually, but not always, the errors in question are rounding errors, but backward error can also be a useful way to characterize truncation errors (for example in deriving algorithms based on Padé approximation for computing matrix functions).

As an example, for the inner product $u^T v$ of two vectors the backward error of an approximation w can be defined as

$$\eta(w) = \min\{ \epsilon : w = (u + \Delta u)^T v, \|\Delta u\|_2 \leq \epsilon \|u\|_2 \},$$

*Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (nick.higham@manchester.ac.uk).

where $\|u\|_2 = (u^T u)^{1/2}$. It can be shown that

$$\eta(w) = \frac{|w - u^T v|}{\|u\|_2 \|v\|_2}.$$

The definition of $\eta(w)$ is clearly unsymmetric in that u is perturbed but v is not. If v is perturbed instead of u then the same formula is obtained. If both u and v are perturbed then the constraint in the definition of $\eta(w)$ becomes nonlinear in Δu and Δv and no explicit formula is available for $\eta(w)$.

For some problems a backward error may not exist. An example is computation of the outer product $A = uv^T$ of two n -vectors, which has rank 1. In floating-point arithmetic the computed matrix \hat{A} is unlikely to be of rank 1, so $\hat{A} = (u + \Delta u)(v + \Delta v)^T$ is not in general possible for any Δu and Δv . In this situation one can consider a mixed backward–forward error that perturbs \hat{A} as well as u and v .

Backward error analysis refers to rounding error analysis in which a bound is obtained for a suitably defined backward error. If the backward error can be shown to be small then y is the solution to a nearby problem. Indeed, if the backward error can be shown to be of the same size as any uncertainties in the data then y is as good a solution as can be expected.

Backward error analysis was developed and popularized by James Wilkinson in the 1950s and 1960s. He first used it in the context of computing zeros of polynomials, but the method’s biggest successes came when he applied it to linear algebra computations.

Backward error analysis has also been used in the context of the numerical solution of differential equations, where it is used in different forms known as *defect control* and *shadowing*.

The forward error of $y \approx f(x)$ is bounded in terms of the backward error $\eta(y)$ by

$$\frac{\|y - f(x)\|}{\|f\|} \leq \text{cond}(f, x) \eta(y) + O(\eta(y))^2,$$

in view of the definition of condition number. Consequently, we have the rule of thumb that

$$\text{forward error} \lesssim \text{condition number} \times \text{backward error}.$$

References

This is a minimal set of references, which contain further useful references within.

- Robert M. Corless and Nicolas Fillion, *A Graduate Introduction to Numerical Methods from the Viewpoint of Backward Error Analysis*, Springer, 2013.
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- Nicholas J. Higham, *Accuracy and Stability of Numerical Algorithms*, second edition, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2002.

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