What Is a Circulant Matrix?

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An $n \times n$ circulant matrix is defined by n parameters, the elements in the first row, and each subsequent row is a cyclic shift forward of the one above:

$$C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ c_n & c_1 & \dots & \vdots \\ \vdots & \ddots & \ddots & c_2 \\ c_2 & \dots & c_n & c_1 \end{bmatrix}.$$

Circulant matrices have the important property that they are diagonalized by the discrete Fourier transform matrix

$$F_n = \Big(\exp(-2\pi\mathrm{i}(r-1)(s-1)/n)\Big)_{r,s=1}^{n},$$

which satisfies $F_n^* F_n = nI$, so that $n^{-1/2}F_n$ is a unitary matrix. (F_n is a Vandermonde matrix with points the roots of unity.) Specifically,

$$F_n C F_n^{-1} = D = \mathrm{diag}(d_i). \qquad (1)$$

Hence circulant matrices are normal ($C^*C = CC^*$). Moreover, the eigenvalues are given by $d = F_n C e_1$,

Note that one particular eigenpair is immediate, since $Ce = \big(\sum_{i=1}^n c_i\big)e$.

The factorization (1) enables a circulant linear system to be solved in $O(n \log n)$ flops, since multiplication by F_n can be done using the fast Fourier transform.

A particular circulant matrix is the (up) shift matrix K_n, the 4×4 version of which is

$$K_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is not hard to see that

$$C = c_1 I + c_2 K_n + c_3 K_n^2 + \dots + c_n K_n^{n-1}.$$

Since powers of K_n commute, it follows that any two circulant matrices commute (this is also clear from (1)). Furthermore, the sum and product of two circulant matrices is a circulant matrix and the inverse of a nonsingular circulant matrix is a circulant matrix.

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One important use of circulant matrices is to act as preconditioners for Toeplitz linear systems. Several methods have been proposed for constructing a circulant matrix from a Toeplitz matrix, including copying the central diagonals and wrapping them around and finding the nearest circulant matrix to the Toeplitz matrix. See Chan and Ng (1996) or Chan and Jin (2017) for a summary of work on circulant preconditioners for Toeplitz systems.

An interesting circulant matrix is `anymatrix('core/circul_binom',n)` in the Anymatrix collection, which is the $n \times n$ circulant matrix whose first row has ith element $\binom{n}{,i-1}$. The eigenvalues are $(1+w^i)^n - 1$, $i = 1:n$, where $w = \exp(2\pi \mathrm{i}/n)$. The matrix is singular when n is a multiple of 6, in which case the null space has dimension 2. Example:

```
>> n = 6; C = anymatrix('core/circul_binom',n), svd(C)
C =
     1     6    15    20    15     6
     6     1     6    15    20    15
    15     6     1     6    15    20
    20    15     6     1     6    15
    15    20    15     6     1     6
     6    15    20    15     6     1
ans =
   6.3000e+01
   2.8000e+01
   2.8000e+01
   1.0000e+00
   2.0244e-15
   7.6607e-16
```

Notes

A classic reference on circulant matrices is Davis (1994).

References

- Raymond Chan and Michael Ng, Conjugate Gradient Methods for Toeplitz Systems, SIAM Rev. 38(3), 427–482, 1996.
- Raymond Chan and Xiao-Qing Jin, An Introduction to Iterative Toeplitz Solvers, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2007.
- Philip Davis, Circulant Matrices, Second edition, Chelsea, New York, 1994.
- Nicholas J. Higham and Mantas Mikaitis, Anymatrix: An Extendable MATLAB Matrix Collection, Numer. Algorithms, 90:3, 1175-1196, 2021.

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