What Is a Nilpotent Matrix?

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An $n \times n$ matrix A is *nilpotent* if $A^k = 0$ for some positive integer k. A nonzero nilpotent matrix must have both positive and negative entries in order for cancellation to take place in the matrix powers. The smallest k for which $A^k = 0$ is called the *index of nilpotency*. The index does not exceed n, as we will see below.

Here are some examples of nilpotent matrices.

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1^2 = 0, \\
A_2 &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2^3 = 0, \\
A_3 &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_3^2 = 0, \\
A_4 &= \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 1 & 1 & 2 \end{bmatrix}, \quad A_4^2 = 0.
\end{aligned}
$$

Matrix A_1 is the 2×2 instance of the upper bidiagonal $p \times p$ matrix

$$
N = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \qquad (1)
$$

for which

$$
N^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & 0 & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}, \quad \dots, \quad N^{p-1} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ & 0 & \ddots & & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}
$$

and $N^p = 0$. The superdiagonal of ones moves up to the right with each increase in the index of the power until it disappears off the top right corner of the matrix.

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Matrix A_4 has rank 1 and was constructed using a general formula: if $A = xy^T$ with $y^Tx = 0$ then $A^2 = xy^Txy^T = (y^Tx)xy^T = 0$. We simply took orthogonal vectors $x = [2, -4, 1]^T$ and $y = [1, 1, 2]^T$.

If A is nilpotent then every eigenvalue is zero, since $Ax = \lambda x$ with $x \neq 0$ implies $0 = A^n x = \lambda^n x$ or $\lambda = 0$. Consequently, the trace and determinant of a nilpotent matrix are both zero.

If A is nilpotent and Hermitian or symmetric, or more generally normal ($A^*A = AA^*$), then $A = 0$, since such a matrix has a spectral decomposition $A = Q\mathrm{diag}(\lambda_i)Q^*$ and the matrix $\mathrm{diag}(\lambda_i)$ is zero. It is only for nonnormal matrices that nilpotency is a nontrivial property, and the best way to understand it is with the Jordan canonical form (JCF). The JCF of a matrix with only zero eigenvalues has the form $A = XJX^{-1}$, where $J = \mathrm{diag}(J_{m_1}, J_{m_2}, \dots, J_{m_p})$, where J_{m_i} is of the form (1) and hence $J_{m_i}^{m_i} = 0$. It follows that the index of nilpotency is $k = \max\{\, m_i : i = 1\colon p \,\} \le n$.

What is the rank of an $n \times n$ nilpotent matrix A? The minimum possible rank is 0, attained for the zero matrix. The maximum possible rank is $n - 1$, attained when the JCF of A has just one Jordan block of size n. Any rank between 0 and $n - 1$ is possible: rank j is attained when there is a Jordan block of size $j+1$ and all other blocks are 1×1.

Finally, while a nilpotent matrix is obviously not invertible, like every matrix it has a Moore–Penrose pseudoinverse. The pseudoinverse of a Jordan block with eigenvalue zero is just the transpose of the block: $N^+ = N^T$ for N in (1).

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