What Is the Pascal Matrix?

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The Pascal matrix $P_n \in \mathbb{R}^{n\times n}$ is the symmetric matrix defined by

$$p_{ij} = \binom{i+j-2}{j-1} = \frac{(i+j-2)!}{(i-1)!(j-1)!}.$$

It contains the rows of Pascal's triangle along the anti-diagonals. For example:

$$P_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}.$$

In MATLAB, the matrix is `pascal(n)`.

The Pascal matrix is positive definite and has the Cholesky factorization

$$P_n = L_n L_n^T, \qquad (1)$$

where the rows of L_n are the rows of Pascal's triangle. For example,

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

From (1) we have $\det(P_n) = \det(L_n)^2 = 1$. Form this equation, or by inverting (1), it follows that P_n^{-1} has integer elements. Indeed the inverse is known to have (i,j) element

$$(-1)^{i-j} \sum_{k=0}^{n-i} \binom{i+k-1}{k} \binom{i+k-1}{i+k-j}, \quad i \ge j. \qquad (2)$$

The Cholesky factor L_n can be factorized as

$$L_n = M_{n-1} M_{n-2} \dots M_1, \qquad (3)$$

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where M_i is unit lower bidiagonal with the first $i-1$ entries along the subdiagonal of M_i zero and the rest equal to 1. For example,

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The factorization (3) shows that P_n is totally positive, that is, every minor (a determinant of a square submatrix) is positive. Indeed each bidiagonal factor M_i is totally nonnegative, that is, every minor is nonnegative, and the product of totally nonnegative matrices is totally nonnegative. Further results in the theory of totally positive matrices show that the product is actually totally positive.

The positive definiteness of P_n implies that the eigenvalues are real and positive. The total positivity, together with the fact that P_n is (trivially) irreducible, implies that the eigenvalues are distinct.

For a symmetric positive semidefinite matrix A with nonnegative entries, we define $A^{\circ t} = (a_{ij}^t)$, which is the matrix with every entry raised to the power $t \in \mathbb{R}$. We say that A is *infinitely divisible* if $A^{\circ t}$ is positive semidefinite for all nonnegative t. The Pascal matrix is infinitely divisible.

It is possible to show that

$$L_n^{-1} = DL_nD, \qquad (4)$$

where $D = \mathrm{diag}((-1)^i)$. In other words, $Y_n = L_nD$ is involutory, that is, $Y_n^2 = I$. It follows from $P_n = Y_nY_n^T$ that

$$P_n^{-1} = Y_n^{-T}Y_n^{-1} = Y_n^TY_n = Y_n^{-1}(Y_nY_n^T)Y_n = Y_n^{-1}P_nY_n,$$

so P_n and P_n^{-1} are similar and hence have the same eigenvalues. This means that the eigenvalues of P_n appear in reciprocal pairs and that the characteristic polynomial is palindromic. Here is an illustration in MATLAB:

```
>> P = pascal(5); evals = eig(P); [evals 1./evals], coeffs = charpoly(P)
ans =
   1.0835e-02   9.2290e+01
   1.8124e-01   5.5175e+00
   1.0000e+00   1.0000e+00
   5.5175e+00   1.8124e-01
   9.2290e+01   1.0835e-02
coeffs =
     1   -99   626  -626    99    -1
```

Now

$$p_{nn} \le \|P\|_2 \le (\|P\|_1\|P\|_\infty)^{1/2} = \binom{2n-1}{n} p_{nn},$$

where for the equality we used a binomial coefficient summation identity. The fact that P_n is positive definite with reciprocal eigenvalues implies that $\kappa_2(P) = \|P\|_2^2$. Hence, using Stirling's approximation ($n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$),

$$\kappa_2(P_n) \sim p_{nn}^2 \sim \left(\frac{(2n)!}{(n!)^2}\right)^2 \sim \frac{16^n}{n\pi}.$$

Thus P_n is exponentially ill conditioned as $n \to \infty$.

The matrix Y_n is obtained in MATLAB with `pascal(n,1)`; this is a lower triangular square root of the identity matrix. Turnbull (1927) noted that by rotating Y_n through 90 degrees one obtains a cube root of the identity matrix. This matrix is returned by `pascal(n,2)`. For example, with $n = 5$:

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -3 & -2 & -1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & -1 & -2 & -3 & -4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad X^3 = I.$$

The logarithm of L_n is explicitly known: it is the upper bidiagonal matrix

$$\log L_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & & & \ddots & n-1 \\ & & & & 0 \end{bmatrix}. \qquad (5)$$

Notes

For proofs of (2) and (4) see Cohen (1975) and Call and Velleman (1993). respectively. For (5), see Edelman and Strang (2004). The infinite divisibility of the Pascal matrix is infinitely is shown by Bhatia (2006). For the total positivity property see Fallat and Johnson (2011).

References

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