What Is a Vector Norm?

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September 22, 2021

A vector norm measures the size, or length, of a vector. For complex n-vectors, a vector norm is a function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ satisfying

- $\|x\| \ge 0$ with equality if and only if $x = 0$, nonnegativity,
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$ $x \in \mathbb{C}^n$, homogeneity,
- $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in \mathbb{C}^n$, the triangle inequality.

An important class of norms is the Hölder p-norms

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad p \ge 1. \tag{1}$$

The ∞-norm is interpreted as $\lim_{p\to\infty} \|x\|_p$ and is given by

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

Other important special cases are

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^n |x_i|, && \text{“Manhattan” or “taxi cab” norm,} \\ \|x\|_2 &= \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} = (x^* x)^{1/2}, && \text{Euclidean length.} \end{aligned}$$

A useful concept is that of the *dual norm* associated with a given vector norm, which is defined by

$$\|y\|^D = \max_{x\ne 0} \frac{\operatorname{Re} x^* y}{\|x\|}.$$

The maximum is attained because the definition is equivalent to $\|y\|^D = \max\{ \operatorname{Re} x^* y : \|x\| = 1 \}$, in which we are maximizing a continuous function over a compact set. Importantly, the dual of the dual norm is the original norm (Horn and Johnson, 2013, Thm. 5.5.9(c)).

We can rewrite the definition of dual norm, using the homogeneity of vector norms, as

$$\|y\|^D = \max_{|c|=1} \|cy\|^D = \max_{|c|=1} \max_{x\ne 0} \frac{\operatorname{Re} x^*(cy)}{\|x\|} = \max_{x\ne 0} \max_{|c|=1} \frac{\operatorname{Re} c(x^* y)}{\|x\|} = \max_{x\ne 0} \frac{|x^* y|}{\|x\|}.$$

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Hence we have the attainable inequality

$$|x^*y| \le \|x\| \, \|y\|^D, \qquad (2)$$

which is the generalized Hölder inequality.

The dual of the p-norm is the q-norm, where $p^{-1} + q^{-1} = 1$, so for the p-norms the inequality (2) becomes the (standard) Hölder inequality,

$$|x^*y| \le \|x\|_p \, \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

An important special case is the Cauchy–Schwarz inequality,

$$|x^*y| \le \|x\|_2 \, \|y\|_2.$$

The notation $\|x\|_0$ is used to denote the number of nonzero entries in x, even though it is not a vector norm and is not obtained from (1) with $p = 0$. In portfolio optimization, if x_k specifies how much to invest in stock k then the inequality $\|x\|_0 \le k$ says "invest in at most k stocks".

In numerical linear algebra, vector norms play a crucial role in the definition of a subordinate matrix norm, as we will explain in the next post in this series.

All norms on $\mathbb{C}^n$ are equivalent in the sense that for any two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ there exist positive constants ν_1 and ν_2 such that

$$\nu_1 \|x\|_\alpha \le \|x\|_\beta \le \nu_2 \|x\|_\alpha \quad \text{for all } x \in \mathbb{C}^n.$$

For example, it is easy to see that

$$\begin{aligned} \|x\|_2 &\le \|x\|_1 \le \sqrt{n}\|x\|_2, \\ \|x\|_\infty &\le \|x\|_2 \le \sqrt{n}\|x\|_\infty, \\ \|x\|_\infty &\le \|x\|_1 \le n\|x\|_\infty. \end{aligned}$$

The 2-norm is invariant under unitary transformations: if $Q^*Q = I$, then $\|Qx\|^2 = x^*Q^*Qx = x^*x = \|x\|_2^2$.

Care must be taken to avoid overflow and (damaging) underflow when evaluating a vector p-norm in floating-point arithmetic for $p \ne 1, \infty$. One can simply use the formula $\|x\|_p = \|(x/\|x\|_\infty)\|_p \|x\|_\infty$, but this requires two passes over the data (the first to evaluate $\|x\|_\infty$). For more efficient one-pass algorithms for the 2-norm see Higham (2002, Sec. 21.8) and Harayama et al. (2021).

References

This is a minimal set of references, which contain further useful references within.

- Takeyuki Harayama, Shuhei Kudo, Daichi Mukunoki, Toshiyuki Imamura, and Daisuke Takahashi, A Rapid Euclidean Norm Calculation Algorithm that Reduces Overflow and Underflow, in Computational Science and Its Applications–ICCSA 2021, O. Gervasi et al., eds, 95–110, Springer, 2021.
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Note: This article was revised October 4, 2021 to change the definition of dual norm to use Re.

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