What Is a Toeplitz Matrix?

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$T \in \mathbb{C}^{n\times n}$ is a Toeplitz matrix if $t_{ij} = t_{i-j}$ for $2n-1$ parameters $t_{1-n}, \dots, t_{n-1}$. A Toeplitz matrix has constant diagonals. For $n = 4$:

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & t_{-3} \\ t_1 & t_0 & t_{-1} & t_{-2} \\ t_2 & t_1 & t_0 & t_{-1} \\ t_3 & t_2 & t_1 & t_0 \end{bmatrix}.$$

Toeplitz matrices arise in various problems, including analysis of time series, discretization of constant coefficient differential equations, and discretization of convolution equations $\int a(t-s)x(s)\,\mathrm{d}s = b(t)$.

Since a Toeplitz matrix depends on just $2n-1$ parameters it is reasonable to expect that a linear system $Tx = b$ can be solved in less than the $O(n^3)$ flops that would be required by LU factorization. Indeed methods are available that require only $O(n^2)$ flops; see Golub and Van Loan (2013) for details.

Upper triangular Toeplitz matrices can be written in the form

$$T = \sum_{j=1}^{n} t_{1-j} N^{j-1}, \quad N = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix},$$

where N is upper bidiagonal with a superdiagonal of ones and $N^n = 0$. It follows that the product of two upper triangular Toeplitz matrices is again upper triangular Toeplitz, upper triangular Toeplitz matrices commute, and T^{-1} is also an upper triangular Toeplitz matrix (assuming t_0 is nonzero, so that T is nonsingular).

Tridiagonal Toeplitz matrices arise frequently:

$$T(c,d,e) = \begin{bmatrix} d & e & & \\ c & d & \ddots & \\ & \ddots & \ddots & e \\ & & c & d \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

The eigenvalues of $T(c,d,e)$ are

$$d + 2(ce)^{1/2}\cos\left(\frac{k\pi}{n+1}\right), \quad k = 1:n.$$

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The Kac–Murdock–Szegö matrix is the symmetric Toeplitz matrix

$$A(\rho) = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & 1 & \rho & \dots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho \\ \rho^{n-1} & \rho^{n-2} & \dots & \rho & 1 \end{bmatrix}.$$

It has a number of interesting properties.

In MATLAB, a Toeplitz matrix can be constructed using `toeplitz(c,r)`, which produces the matrix with first column `c` and first row `r`. Example:

```
>> n = 5; A = toeplitz(1:n,[1 -2:-1:-n])
A =
     1    -2    -3    -4    -5
     2     1    -2    -3    -4
     3     2     1    -2    -3
     4     3     2     1    -2
     5     4     3     2     1
```

References

- Gene Golub and Charles F. Van Loan, Matrix Computations, fourth edition, Johns Hopkins University Press, Baltimore, MD, USA, 2013. Section 4.7.

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- What Is a Circulant Matrix? (2022)
- What Is a Tridiagonal Matrix? (2022)
- What Is the Kac–Murdock–Szegö Matrix? (2020)

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