What Is the Nearest Symmetric Matrix?

Nicholas J. Higham*

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What is the nearest symmetric matrix to a real, nonsymmetric square matrix A? This question arises whenever a symmetric matrix is approximated by formulas that do not respect the symmetry. For example, a finite difference approximation to a Hessian matrix can be nonsymmetric even though the Hessian is symmetric. In some cases, lack of symmetry is caused by rounding errors. The natural way to symmetrize A is to replace it by $(A+A^T)/2$. Is this the best choice?

As our criterion of optimality we take that $\|A-X\|$ is minimized over symmetric X for some suitable norm. Fan and Hoffman (1955) showed that $(A+A^T)/2$ is a solution in any unitarily invariant norm. A norm is unitarily invariant if $\|A\| = \|UAV\|$ for all unitary U and V. Such a norm depends only on the singular values of A, and hence $\|A\| = \|A^T\|$ since A and A^T have the same singular values. Examples of unitarily invariant norms are the 2-norm and the Frobenius norm.

The proof that $(A+A^T)/2$ is optimal is simple. For any symmetric Y,

$$\begin{aligned}\Big\|A - \frac{1}{2}(A+A^T)\Big\| &= \frac{1}{2}\|A-A^T\| = \frac{1}{2}\|A-Y+Y^T-A^T\| \\ &\le \frac{1}{2}\|A-Y\| + \frac{1}{2}\|(Y-A)^T\| \\ &= \|A-Y\|.\end{aligned}$$

Simple examples of a matrix and a nearest symmetric matrix are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that any A can be written

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) \equiv B + C,$$

where B and C are the symmetric part and the skew-symmetric part of A, respectively, so the nearest symmetric matrix to A is the symmetric part of A.

For the Frobenius norm, $(A+A^T)/2$ is the unique nearest symmetric matrix, which follows from the fact that $\|S+K\|_F^2 = \|S\|_F^2 + \|K\|_F^2$ for symmetric S and skew-symmetric

*Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (nick.higham@manchester.ac.uk).

K. For the 2-norm, however, the nearest symmetric matrix is not unique in general. An example of non-uniqueness is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} \theta & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix},$$

for which $\|A - X\|_2 = 0.5$, and $\|A - Y\|_2 = 0.5$ for any θ such that $|\theta| \le 0.5$.

Entirely analogous to the above is the nearest skew-symmetric matrix problem, for which the solution is the skew-symmetric part for any unitarily invariant norm. For complex matrices, these results generalize in the obvious way: $(A + A^*)/2$ is the nearest Hermitian matrix to A and $(A - A^*)/2$ is the nearest skew-Hermitian matrix to A in any unitarily invariant norm.

Reference

- Ky Fan and A. J. Hoffman, Some Metric Inequalities in the Space of Matrices, Proc. Amer. Math. Soc. 6, 111–116, 1955.

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