

What Is the Jordan Canonical Form?

Nicholas J. Higham*

February 28, 2022

How close can similarity transformations take a matrix towards diagonal form? The answer is given by the Jordan canonical form, which achieves the largest possible number of off-diagonal zero entries (Brualdi, Pei, and Zhan, 2008).

Theorem 1. (Jordan canonical form) Any matrix $A \in \mathbb{C}^{n \times n}$ can be expressed as

$$A = ZJZ^{-1}, \quad J = \text{diag}(J_1, J_2, \dots, J_p),$$
$$J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{C}^{m_k \times m_k},$$

where Z is nonsingular and $m_1 + m_2 + \dots + m_p = n$. The matrix J is unique up to the ordering of the blocks J_k .

The matrix J is (up to reordering of the diagonal blocks) the *Jordan canonical form* of A (or the Jordan form, for short).

The bidiagonal matrices J_k are called *Jordan blocks*. Clearly, the eigenvalues of J_k are λ_k repeated m_k times and J_k has a single eigenvector, $e_1 \in \mathbb{R}^{m_k}$. Two different Jordan blocks can have the same eigenvalues.

In total, J has p linearly independent eigenvectors, and the same is true of A .

The Jordan canonical form is an invaluable tool in matrix analysis, as it provides a concrete way to prove and understand many results. However, the Jordan form can not be reliably computed in finite precision arithmetic, so it is of little use computationally, except in special cases such as when A is Hermitian or normal.

*Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (nick.higham@manchester.ac.uk).

For a Jordan block $J_k = J_k(\lambda_k) \in \mathbb{C}^{m_k \times m_k}$ we have

$$J_k - \lambda_k I = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad (J_k - \lambda_k I)^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix},$$

$$\dots, \quad (J_k - \lambda_k I)^{m_k-1} = \begin{bmatrix} 0 & 0 & \dots & 1 \\ & 0 & \dots & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}, \quad (J_k - \lambda_k I)^{m_k} = 0. \quad (*)$$

The superdiagonal of ones moves up to the right with each increase in the index of the power, until it disappears off the top corner of the matrix.

It is easy to see that $(A - \lambda I_n)^j = Z(J - \lambda I_n)^j Z^{-1} = Z \text{diag}((J_k(\lambda_k) - \lambda I_{m_k})^j) Z^{-1}$, and so

$$\text{rank}((A - \lambda I_n)^j) = \sum_{k=1}^p \text{rank}((J_k(\lambda_k) - \lambda I_{m_k})^j).$$

For $\lambda = \lambda_k$, these quantities provide information about the size of the Jordan blocks associated with λ_k . To be specific, let

$$d_j = \text{rank}((A - \lambda_k I_n)^j), \quad j \geq 1, \quad d_0 = n$$

and

$$\omega_j = d_{j-1} - d_j, \quad j \geq 1.$$

By considering the equations (*) above, it can be shown that ω_j is the number of Jordan blocks of size at least j in which λ_k appears. Moreover, the number of Jordan blocks of size j is $\omega_j - \omega_{j+1} = d_{j-1} - 2d_j + d_{j+1}$. Therefore if we know the eigenvalues and the ranks of $(A - \lambda_k I_n)^j$ for each eigenvalue λ_k and appropriate j then we can determine the Jordan structure. As an important special case, if $\text{rank}(A - \lambda_k I_n) = n - 1$ then we know that λ_k appears in a single Jordan block. The sequence of ω_j is known as the *Weyr characteristic*, and it satisfies $\omega_1 \geq \omega_2 \geq \dots$.

As an example of a matrix for which we can easily deduce the Jordan form consider the nilpotent matrix $B = \begin{bmatrix} 0_r & I_r \\ 0_r & 0_r \end{bmatrix}$, for which $B^2 = 0$ and all the eigenvalues are zero. Since $\text{rank}(B) = r$, we have $d_0 = 2r$, $d_1 = r$, and $d_2 = 0$. Hence $\omega_1 = 0$ and $\omega_2 = r$, so there are r 2×2 Jordan blocks. (In fact, A can be permuted into Jordan form by a similarity transformation.)

Here is an example with A the 11×11 matrix `anymatrix('core/collatz',11)`.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have $\text{rank}(A) = 10$ and $\text{rank}(A - 2I) = 10$, so 0 and 2 are simple eigenvalues. All the other eigenvalues are 1 and they have the following d_j and ω_j values:

j	d_j	ω_j	$\omega_j - \omega_{j+1}$
0	11		
1	7	4	2
2	5	2	1
3	4	1	0
4	3	1	0
5	2	1	1
6	2	0	0

We conclude that the eigenvalue 1 occurs in one block of order 5, one block of order 2, and two blocks of order 1.

A matrix and its transpose have the same Jordan form. One way to see this is to note that $A = ZJZ^{-1}$ implies

$$A^T = Z^{-T} J^T Z^T = Z^{-T} P \cdot P J^T P \cdot P Z^T = (ZP)^{-T} J (ZP)^T,$$

where P is the identity matrix with its columns reversed. A consequence is that A and A^T are similar.

Real Jordan Form

A version of the Jordan form with Z and J real exists for $A \in \mathbb{R}^{n \times n}$. The main change is how complex eigenvalues are represented. Since the eigenvalues now occur in complex conjugate pairs λ and $\bar{\lambda}$, and each of the pair has the same Jordan structure (which follows from the fact that a matrix and its complex conjugate have the same rank), pairs of Jordan blocks corresponding to λ and $\bar{\lambda}$ are combined into a real block of twice the size. For example, Jordan blocks

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \bar{\lambda} & 1 \\ 0 & \bar{\lambda} \end{bmatrix} \in \mathbb{C}^{2 \times 2}$$

become

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ \hline 0 & 0 & a & b \\ 0 & 0 & -b & a \end{array} \right] \in \mathbb{R}^{4 \times 4}$$

in the real Jordan form, where $\lambda = a + ib$. Note that the eigenvalues of $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are $a \pm ib$.

Notes

Proofs of the Jordan canonical form and its real variant can be found in many textbooks. See also Brualdi (1987) and Fletcher and Sorensen (1983), who give proofs that go via the Schur decomposition.

References

- Richard A. Brualdi. The Jordan Canonical Form: An Old Proof. Amer. Math. Monthly, 94(3):257–267, 1987.
- Richard A. Brualdi, Pei Pei, and Xingzhi Zhan. An Extremal Sparsity Property of the Jordan Canonical Form. Linear Algebra Appl., 429(10):2367–2372, 2008.
- R. Fletcher and D. C. Sorensen. An Algorithmic Derivation of the Jordan Canonical Form. Amer. Math. Monthly, 90(1):12–16, 1983.

This article is part of the “What Is” series, available from <https://nhigham.com/index-of-what-is-articles/> and in PDF form from the GitHub repository <https://github.com/higham/what-is>.