What is the Polar Decomposition?

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A polar decomposition of $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ is a factorization $A = UH$, where $U \in \mathbb{C}^{m\times n}$ has orthonormal columns and $H \in \mathbb{C}^{n\times n}$ is Hermitian positive semidefinite. This decomposition is a generalization of the polar representation $z = r\mathrm{e}^{\mathrm{i}\theta}$ of a complex number, where H corresponds to $r \geq 0$ and U to $\mathrm{e}^{\mathrm{i}\theta}$. When A is real, H is symmetric positive semidefinite. When $m = n$, U is a square unitary matrix (orthogonal for real A).

We have $A^*A = H^*U^*UH = H^2$, so $H = (A^*A)^{1/2}$, which is the unique positive semidefinite square root of A^*A. When A has full rank, H is nonsingular and $U = AH^{-1}$ is unique, and in this case U can be expressed as

$$U = \frac{2}{\pi} A \int_0^\infty (t^2 I + A^*A)^{-1}\, \mathrm{d}t.$$

An example of a polar decomposition is

$$A = \begin{bmatrix} 4 & 0 \\ -5 & -3 \\ 2 & 6 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{2}{3} \end{bmatrix} \cdot \sqrt{2} \begin{bmatrix} \frac{9}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{9}{2} \end{bmatrix} \equiv UH.$$

For an example with a rank-deficient matrix consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

for which $A^*A = \mathrm{diag}(0, 1, 1)$ and so $H = \mathrm{diag}(0, 1, 1)$. The equation $A = UH$ then implies that

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \theta & 0 & 0 \end{bmatrix}, \quad |\theta| = 1,$$

so U is not unique.

The polar factor U has the important property that it is a closest matrix with orthonormal columns to A in any unitarily invariant norm. Hence the polar decomposition provides an optimal way to orthogonalize a matrix. This method of orthogonalization is used in various applications, including in quantum chemistry, where it is called Löwdin

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orthogonalization. Most often, though, orthogonalization is done through QR factorization, trading optimality for a faster computation.

An important application of the polar decomposition is to the orthogonal Procrustes problem[1]

$$\min\{\, \|A - BW\|_F : W \in \mathbb{C}^{n\times n},\ W^*W = I \,\},$$

where $A, B \in \mathbb{C}^{m\times n}$ and the norm is the Frobenius norm $\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2$. This problem, which arises in factor analysis and in multidimensional scaling, asks how closely a unitary transformation of B can reproduce A. Any solution is a unitary polar factor of B^*A, and there is a unique solution if B^*A is nonsingular. Another application of the polar decomposition is in 3D graphics transformations. Here, the matrices are 3×3 and the polar decomposition can be computed by exploiting a relationship with quaternions.

For a square nonsingular matrix A, the unitary polar factor U can be computed by a Newton iteration:

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-*}), \quad X_0 = A.$$

The iterates X_k converge quadratically to U. This is just one of many iterations for computing U and much work has been done on the efficient implementation of these iterations.

If $A = P\Sigma Q^*$ is a singular value decomposition (SVD), where $P \in \mathbb{C}^{m\times n}$ has orthonormal columns, $Q \in \mathbb{C}^{n\times n}$ is unitary, and Σ is square and diagonal with nonnegative diagonal elements, then

$$A = PQ^* \cdot Q\Sigma Q^* \equiv UH,$$

where U has orthonormal columns and H is Hermitian positive semidefinite. So a polar decomposition can be constructed from an SVD. The converse is true: if $A = UH$ is a polar decomposition and $H = Q\Sigma Q^*$ is a spectral decomposition (Q unitary, D diagonal) then $A = (UQ)\Sigma Q^* \equiv P\Sigma Q^*$ is an SVD. This latter relation is the basis of a method for computing the SVD that first computes the polar decomposition by a matrix iteration then computes the eigensystem of H, and which is extremely fast on distributed-memory manycore computers.

The nonuniqueness of the polar decomposition for rank deficient A, and the lack of a satisfactory definition of a polar decomposition for $m < n$, are overcome in the *canonical polar decomposition*, defined for any m and n. Here, $A = UH$ with U a partial isometry, H is Hermitian positive semidefinite, and $U^*U = HH^+$. The superscript "+" denotes the Moore–Penrose pseudoinverse and a partial isometry can be characterized as a matrix U for which $U^+ = U^*$.

Generalizations of the (canonical) polar decomposition have been investigated in which the properties of U and H are defined with respect to a general, possibly indefinite, scalar product.

References

This is a minimal set of references, which contain further useful references within.

- Nicholas J. Higham, Functions of Matrices: Theory and Computation, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008. (Chapter 8.)

[1] Procrustes: an ancient Greek robber who tied his victims to an iron bed, stretching their legs if too short for it, and lopping them if too long.

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