What Is a Diagonalizable Matrix?

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A matrix $A \in \mathbb{C}^{n\times n}$ is *diagonalizable* if there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that $X^{-1}AX$ is diagonal. In other words, a diagonalizable matrix is one that is similar to a diagonal matrix.

The condition $X^{-1}AX = D = \mathrm{diag}(\lambda_i)$ is equivalent to $AX = XD$ with X nonsingular, that is, $Ax_i = \lambda_i x_i$, $i = 1\colon n$, where $X = [x_1, x_2, \dots, x_n]$. Hence A is diagonalizable if and only if it has a complete set of linearly independent eigenvectors.

A Hermitian matrix is diagonalizable because the eigenvectors can be taken to be mutually orthogonal. The same is true for a normal matrix (one for which $A^*A = AA^*$). A matrix with distinct eigenvalues is also diagonalizable.

Theorem 1. *If $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues then it is diagonalizable.*

Proof. Let A have eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $x_1, x_2, \dots, x_n$. Suppose that $y = \sum_{i=1}^n \alpha_i x_i = 0$ for some $\alpha_1, \alpha_2, \dots, \alpha_n$. Then

$$\begin{aligned}
0 = (A-\lambda_2 I)\cdots(A-\lambda_n I)y &= \sum_{i=1}^n \alpha_i (A-\lambda_2 I)\cdots(A-\lambda_n I)x_i \\
&= \sum_{i=1}^n \alpha_i(\lambda_i-\lambda_2)\cdots(\lambda_i-\lambda_n)x_i = \alpha_1(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_n)x_1,
\end{aligned}$$

which implies $\alpha_1 = 0$ since $\lambda_1 \ne \lambda_j$ for $j \ge 2$ and $x_1 \ne 0$. Premultiplying $y = \sum_{i=2}^n \alpha_i x_i = 0$ by $\prod_{j=3}^n (A - \lambda_j I)$ shows, in the same way, that $\alpha_2 = 0$. Continuing in this way we find that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Therefore the x_i are linearly independent and hence A is diagonalizable.

A matrix can have repeated eigenvalues and be diagonalizable, as diagonal matrices with repeated diagonal entries show. What is needed for diagonalizability is that every k-times repeated eigenvalue has k linearly independent eigenvectors associated with it. Equivalently, the algebraic and geometric multiplicities of every eigenvalue must be equal, that is, the eigenvalues must all be semisimple. Another equivalent condition is that the degree of the minimal polynomial is equal to the number of distinct eigenvalues.

The simplest example of a matrix that is not diagonalizable is $\left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$. This matrix is a 2×2 Jordan block with the eigenvalue 0. Diagonalizability is easily understood in terms of the Jordan canonical form: A is diagonalizable if and only if all the Jordan blocks in its Jordan form are 1×1.

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Most matrices are diagonalizable, in the sense that the diagonalizable matrices are dense in $\mathbb{C}^{n\times n}$, that is, any matrix in $\mathbb{C}^{n\times n}$ is arbitrarily close to a diagonalizable matrix. This property is useful because it can be convenient to prove a result by first proving it for diagonalizable matrices and then arguing that by continuity the result holds for a general matrix.

Is a rank-1 matrix $A = xy^*$ diagonalizable, where $x, y \in \mathbb{C}^{n\times n}$ are nonzero? There are $n-1$ zero eigenvalues with eigenvectors any set of linearly independent vectors orthogonal to y. If $y^*x \neq 0$ then y^*x is the remaining eigenvalue, with eigenvector x, which is linearly independent of the eigenvectors for 0, and A is diagonalizable. If $y^*x = 0$ then all the eigenvalues of A are zero and so A cannot be diagonalizable, as the only diagonalizable matrix whose eigenvalues are all zero is the zero matrix. For the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ mentioned above, $x = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, so $y^*x = 0$, confirming that this matrix is not diagonalizable.

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