What Is a Fréchet Derivative?

Nicholas J. Higham[*]

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Let U and V be Banach spaces (complete normed vector spaces). The Fréchet derivative of a function $f : U \to V$ at $X \in U$ is a linear mapping $L : U \to V$ such that

$$f(X+E) - f(X) - L(X,E) = o(\|E\|)$$

for all $E \in U$. The notation $L(X,E)$ should be read as "the Fréchet derivative of f at X in the direction E". The Fréchet derivative may not exist, but if it does exist then it is unique. When $U = V = \mathbb{R}$, the Fréchet derivative is just the usual derivative of a scalar function: $L(x,e) = f'(x)e$.

As a simple example, consider $U = V = \mathbb{R}^{n\times n}$ and $f(X) = X^2$. From the expansion

$$f(X+E) - f(X) = XE + EX + E^2$$

we deduce that $L(X,E) = XE+EX$, the first order part of the expansion. If X commutes with E then $L_X(E) = 2XE = 2EX$.

More generally, it can be shown that if f has the power series expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$ with radius of convergence r then for $X, E \in \mathbb{R}^{n\times n}$ with $\|X\| < r$, the Fréchet derivative is

$$L(X,E) = \sum_{i=1}^{\infty} a_i \sum_{j=1}^{i} X^{j-1} E X^{i-j}.$$

An explicit formula for the Fréchet derivative of the matrix exponential, $f(A) = \mathrm{e}^A$, is

$$L(A,E) = \int_0^1 \mathrm{e}^{A(1-s)} E \mathrm{e}^{As}\, ds.$$

Like the scalar derivative, the Fréchet derivative satisfies sum and product rules: if g and h are Fréchet differentiable at A then

$$\begin{aligned} f = \alpha g + \beta h \;&\Rightarrow\; L_f(A,E) = \alpha L_g(A,E) + \beta L_h(A,E),\\ f = gh \;&\Rightarrow\; L_f(A,E) = L_g(A,E)h(A) + g(A)L_h(A,E). \end{aligned}$$

A key requirement of the definition of Fréchet derivative is that $L(X,E)$ must satisfy the defining equation for all E. This is what makes the Fréchet derivative different from the Gâteaux derivative (or directional derivative), which is the mapping $G : U \to V$ given by

$$G(X,E) = \lim_{t\to 0} \frac{f(X+tE) - f(X)}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(X+tE).$$

[*]Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK (nick.higham@manchester.ac.uk).

Here, the limit only needs to exist in the particular direction E. If the Fréchet derivative exists at X then it is equal to the Gâteaux derivative, but the converse is not true.

A natural definition of condition number of f is

$$\operatorname{cond}(f,X) = \lim_{\epsilon\to 0} \sup_{\|\Delta X\|\le \epsilon\|X\|} \frac{\|f(X+\Delta X)-f(X)\|}{\epsilon\|f(X)\|},$$

and it can be shown that cond is given in terms of the Fréchet derivative by

$$\operatorname{cond}(f,X) = \frac{\|L(X)\|\|X\|}{\|f(X)\|},$$

where

$$\|L(X)\| = \sup_{Z\ne 0} \frac{\|L(X,Z)\|}{\|Z\|}.$$

For matrix functions, the Fréchet derivative has a number of interesting properties, one of which is that the eigenvalues of $L(X)$ are the divided differences

$$f[\lambda_i,\lambda_j] = \begin{cases} \dfrac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}, & \lambda_i \ne \lambda_j, \\ f'(\lambda_i), & \lambda_i = \lambda_j, \end{cases}$$

for $1 \le i,j \le n$, where the λ_i are the eigenvalues of X. We can check this formula in the case $F(X) = X^2$. Let (λ, u) be an eigenpair of X and (μ, v) an eigenpair of X^T, so that $Xu = \lambda u$ and $X^T v = \mu v$, and let $E = uv^T$. Then

$$L(X,E) = XE + EX = Xuv^T + uv^T X = (\lambda+\mu)uv^T.$$

So uv^T is an eigenvector of $L(X)$ with eigenvalue $\lambda+\mu$. But $f[\lambda,\mu] = (\lambda^2-\mu^2)/(\lambda-\mu) = \lambda+\mu$ (whether or not λ and μ are distinct).

References

This is a minimal set of references, which contain further useful references within.

- Kendall Atkinson and Weimin Han, Theoretical Numerical Analysis: A Functional Analysis Framework, Springer-Verlag, New York, 2009. (Section 5.3).
- Nicholas J. Higham, Functions of Matrices: Theory and Computation, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008. (Chapter 3).
- James Ortega and Werner Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2000. (Section 3.1).

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