What Is a Stochastic a Matrix?

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A *stochastic matrix* is an $n \times n$ matrix with nonnegative entries and unit row sums. If $A \in \mathbb{R}^{n\times n}$ is stochastic then $Ae = e$, where $e = [1, 1, \dots, 1]^T$ is the vector of ones. This means that e is an eigenvector of A corresponding to the eigenvalue 1.

The identity matrix is stochastic, as is any permutation matrix. Here are some other examples of stochastic matrices:

$$A_n = n^{-1}ee^T, \quad \text{in particular} \quad A_3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \tag{1}$$

$$B_n = \frac{1}{n-1}(ee^T - I), \quad \text{in particular} \quad B_3 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \tag{2}$$

$$C_n = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \vdots & \vdots & \ddots & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}. \tag{3}$$

For any matrix A, the spectral radius $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ is bounded by $\rho(A) \le \|A\|$ for any norm. For a stochastic matrix, taking the ∞-norm (the maximum row sum of absolute values) gives $\rho(A) \le 1$, so since we know that 1 is an eigenvalue, $\rho(A) = 1$. It can be shown that 1 is a semisimple eigenvalue, that is, if there are k eigenvalues equal to 1 then there are k linearly independent eigenvectors corresponding to 1 (Meyer, 2000, p. 696).

A lower bound on the spectral radius can be obtained from Gershgorin's theorem. The ith Gershgorin disc is defined by $|\lambda - a_{ii}| \le \sum_{j\ne i} |a_{ij}| = 1 - a_{ii}$, which implies $|\lambda| \ge 2a_{ii} - 1$. Every eigenvalue λ lies in the union of the n discs and so must satisfy

$$2 \min_i a_{ii} - 1 \le |\lambda| \le 1.$$

The lower bound is positive if A is strictly diagonally dominant by rows.

If A and B are stochastic then AB is nonnegative and $ABe = Ae = e$, so AB is stochastic. In particular, any positive integer power of A is stochastic. Does A^k converge as $k \to \infty$? The answer is that it does, and the limit is stochastic, as long as 1 is the only

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eigenvalue of modulus 1, and this will be the case if all the elements of A are positive (by Perron's theorem). The simplest example of non-convergence is the stochastic matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which has eigenvalues 1 and -1. Since $A^2 = I$, all even powers are equal to I and all odd powers are equal to A. For the matrix (1), $A_n^k = A_n$ for all k, while for (2), $B_n^k \to A_n$ as $k \to \infty$. For (3), C_n^k converges to the matrix with 1 in every entry of the first column and zeros everywhere else.

Stochastic matrices arise in Markov chains. A transition matrix for a finite homogeneous Markov chain is a matrix whose (i, j) element is the probability of moving from state i to state j over a time step. It has nonnegative entries and the rows sum to 1, so it is a stochastic matrix. In applications including finance and healthcare, a transition matrix may be estimated for a certain time period, say one year, but a transition matrix for a shorter period, say one month, may be needed. If A is a transition matrix for a time period p then a stochastic pth root of A, which is a stochastic solution X of the equation $X^p = A$, is a transition matrix for a time period a factor p smaller. Therefore $p = 12$ (years to months) and $p = 7$ (weeks to days) are among the values of interest. Unfortunately, a stochastic pth root may not exist. For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

has no pth roots at all, let alone stochastic ones. Yet many stochastic matrices do have stochastic roots. The matrix (3) has a stochastic pth root for all p, as shown by Higham and Lin (2011), who give a detailed analysis of pth roots of stochastic matrices. For example, to four decimal places,

$$C_6^{1/7} = \begin{bmatrix} 1.000 & & & & & \\ 0.094 & 0.906 & & & & \\ 0.043 & 0.102 & 0.855 & & & \\ 0.027 & 0.050 & 0.103 & 0.820 & & \\ 0.019 & 0.032 & 0.052 & 0.103 & 0.795 & \\ 0.014 & 0.023 & 0.034 & 0.053 & 0.102 & 0.774 \end{bmatrix}.$$

A stochastic matrix is sometime called a *row-stochastic matrix* to distinguish it from a *column-stochastic matrix*, which is a nonnegative matrix for which $e^T A = e^T$ (so that A^T is row-stochastic). A matrix that is both row-stochastic and column-stochastic is called *doubly stochastic*. A permutation matrix is an example of a doubly stochastic matrix. If U is a unitary matrix then the matrix with $a_{ij} = |u_{ij}|^2$ is doubly stochastic. A magic square scaled by the magic sum is also doubly stochastic. For example,

```
>> M = magic(4), A = M/sum(M(1,:)) % A is doubly stochastic.
M =
    16     2     3    13
     5    11    10     8
     9     7     6    12
```

```
     4    14    15     1
A =
   4.7059e-01   5.8824e-02   8.8235e-02   3.8235e-01
   1.4706e-01   3.2353e-01   2.9412e-01   2.3529e-01
   2.6471e-01   2.0588e-01   1.7647e-01   3.5294e-01
   1.1765e-01   4.1176e-01   4.4118e-01   2.9412e-02
>> [sum(A) sum(A')]
ans =
     1     1     1     1     1     1     1     1
>> eig(A)'
ans =
   1.0000e+00   2.6307e-01  -2.6307e-01   8.5146e-18
```

References

- Nicholas J. Higham and Lijing Lin, On pth Roots of Stochastic Matrices, Linear Algebra Appl. 435, 448–463, 2011.
- Carl D. Meyer, Matrix Analysis and Applied Linear Algebra, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2000.

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