What Is the Trace of a Matrix?

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The trace of an $n \times n$ matrix is the sum of its diagonal elements: $\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}$. The trace is linear, that is, $\mathrm{trace}(A+B) = \mathrm{trace}(A)+\mathrm{trace}(B)$, and $\mathrm{trace}(A) = \mathrm{trace}(A^T)$.

A key fact is that the trace is also the sum of the eigenvalues. The proof is by considering the characteristic polynomial $p(t) = \det(tI-A) = t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$. The roots of p are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, so p can be factorized

$$p(t) = (t-\lambda_1)(t-\lambda_2)\ldots(t-\lambda_n),$$

and so $a_{n-1} = -(\lambda_{11}+\lambda_{22}+\cdots+\lambda_{nn})$. The Laplace expansion of $\det(tI-A)$ shows that the coefficient of t^{n-1} is $-(a_{11}+a_{22}+\cdots+a_{nn})$. Equating these two expressions for a_{n-1} gives

$$\mathrm{trace}(A) = \sum_{i=1}^{n} \lambda_i. \qquad (1)$$

A consequence of (1) is that any transformation that preserves the eigenvalues preserves the trace. Therefore the trace is unchanged under similarity transformations: $\mathrm{trace}(X^{-1}AX) = \mathrm{trace}(A)$ for any nonsingular X.

An an example of how the trace can be useful, suppose A is a symmetric and orthogonal $n \times n$ matrix, so that its eigenvalues are ± 1. If there are p eigenvalues 1 and q eigenvalues -1 then $\mathrm{trace}(A) = p - q$ and $n = p + q$. Therefore $p = (n + \mathrm{trace}(A))/2$ and $q = (n - \mathrm{trace}(A))/2$.

Another important property is that for an $m \times n$ matrix A and an $n \times m$ matrix B,

$$\mathrm{trace}(AB) = \mathrm{trace}(BA) \qquad (2)$$

(despite the fact that $AB \neq BA$ in general). The proof is simple:

$$\begin{aligned}\mathrm{trace}(AB) &= \sum_{i=1}^{m}(AB)_{ii} = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki}a_{ik}\\ &= \sum_{k=1}^{n}(BA)_{kk} = \mathrm{trace}(BA).\end{aligned}$$

This simple fact can have non-obvious consequences. For example, consider the equation $AX - XA = I$ in $n \times n$ matrices. Taking the trace gives $0 = \mathrm{trace}(AX) - \mathrm{trace}(XA) =$

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$\mathrm{trace}(AX - XA) = \mathrm{trace}(I) = n$, which is a contradiction. Therefore the equation has no solution.

The relation (2) gives $\mathrm{trace}(ABC) = \mathrm{trace}((AB)C) = \mathrm{trace}(C(AB)) = \mathrm{trace}(CAB)$ for $n \times n$ matrices A, B, and C, that is,

$$\mathrm{trace}(ABC) = \mathrm{trace}(CAB). \qquad (3)$$

So we can cyclically permute terms in a matrix product without changing the trace.

As an example of the use of (2) and (3), if x and y are n-vectors then $\mathrm{trace}(xy^T) = \mathrm{trace}(y^Tx) = y^Tx$. If A is an $n \times n$ matrix then $\mathrm{trace}(xy^TA)$ can be evaluated without forming the matrix xy^TA since, by (3), $\mathrm{trace}(xy^TA) = \mathrm{trace}(y^TAx) = y^TAx$.

The trace is useful in calculations with the Frobenius norm of an $m \times n$ matrix:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} = \bigl(\mathrm{trace}(A^*A)\bigr)^{1/2},$$

where $*$ denotes the conjugate transpose. For example, we can generalize the formula $|x + \mathrm{i}y|^2 = x^2 + y^2$ for a complex number to an $m \times n$ matrix A by splitting A into its Hermitian and skew-Hermitian parts:

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) \equiv B + C,$$

where $B = B^*$ and $C = -C^*$. Then

$$\begin{aligned} \|A\|_F^2 = \|B + C\|_F^2 &= \mathrm{trace}((B + C)^*(B + C)) \\ &= \mathrm{trace}(B^*B + C^*C) + \mathrm{trace}(B^*C + C^*B) \\ &= \mathrm{trace}(B^*B + C^*C) + \mathrm{trace}(BC - CB) \\ &= \mathrm{trace}(B^*B + C^*C) \\ &= \|B\|_F^2 + \|C\|_F^2. \end{aligned}$$

If a matrix is not explicitly known but we can compute matrix–vector products with it then the trace can be estimated by

$$\mathrm{trace}(A) \approx x^TAx,$$

where the vector x has elements independently drawn from the standard normal distribution with mean 0 and variance 1. The expectation of this estimate is

$$\begin{aligned} E\bigl(x^TAx\bigr) &= E\bigl(\mathrm{trace}(x^TAx)\bigr) = E\bigl(\mathrm{trace}(Axx^T)\bigr) = \mathrm{trace}\bigl(E(Axx^T)\bigr) \\ &= \mathrm{trace}\bigl(AE(xx^T)\bigr) = \mathrm{trace}(A), \end{aligned}$$

since $E(x_ix_j) = 0$ for $i \neq j$ and $E(x_i^2) = 1$ for all i, so $E(xx^T) = I$. This stochastic estimate, which is due to Hutchinson, is therefore unbiased.

References

- Haim Avron and Sivan Toledo, Randomized Algorithms for Estimating the Trace of an Implicit Symmetric Positive Semi-definite Matrix, J. ACM 58, 8:1-8:34, 2011.

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