What Is the Schur Complement of a Matrix?

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For an $n \times n$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad (1)$$

with nonsingular $(1,1)$ block A_{11} the *Schur complement* is $A_{22} - A_{21}A_{11}^{-1}A_{12}$. It is denoted by A/A_{11}. The block with respect to which the Schur complement is taken need not be the $(1,1)$ block. Assuming that A_{22} is nonsingular, the Schur complement of A_{22} in A is $A_{11} - A_{12}A_{22}^{-1}A_{21}$.

More generally, for an index vector $\alpha = [i_1, i_2, \dots, i_k]$, where $1 \le i_1 < i_2 < \cdots < i_p \le n$, the Schur complement of $A(\alpha,\alpha)$ in A is $A(\alpha^c,\alpha^c) - A(\alpha^c,\alpha)A(\alpha,\alpha)^{-1}A(\alpha,\alpha^c)$, where α^c is the complement of α (the vector of indices not in α). Most often, it is the Schur complement of A_{11} in A that is of interest, and the general case can be reduced to it by row and column permutations that move $A(\alpha,\alpha)$ to the $(1,1)$ block of A.

Schur Complements in Gaussian Elimination

The reduced submatrices in Gaussian elimination are Schur complements. Write an $n \times n$ matrix A with nonzero $(1,1)$ element α as

$$A = \begin{bmatrix} \alpha & a^* \\ b & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b/\alpha & I_{n-1} \end{bmatrix} \begin{bmatrix} \alpha & a^* \\ 0 & C - ba^*/\alpha \end{bmatrix}.$$

This factorization represents the first step of Gaussian elimination.

After k stages of Gaussian elimination we have computed the following factorization, in which the $(1,1)$ blocks are $k \times k$:

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ 0 & S \end{bmatrix},$$

where the first matrix on the left is the product of the elementary transformations that reduce the first k columns to upper triangular form. Equating $(2,1)$ and $(2,2)$ blocks in this equation gives $L_{21}A_{11} + A_{21} = 0$ and $L_{21}A_{12} + A_{22} = S$. Hence $S = A_{22} + L_{21}A_{12} = A_{22} + (-A_{21}A_{11}^{-1})A_{12}$, which is the Schur complement of A_{11} in A.

The next step of the elimination, which zeros out the first column of S below the diagonal, succeeds as long as the $(1,1)$ element of S is nonzero (or if the whole first column of S is zero, in which case there is nothing to do, and A is singular in this case).

For a number of matrix structures, such as

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- Hermitian (or real symmetric) positive definite matrices,
- totally positive matrices,
- matrices diagonally dominant by rows or columns,
- M-matrices,

one can show that the Schur complement inherits the structure. For these four structures the $(1,1)$ element of the matrix is nonzero, so the success of Gaussian elimination (or equivalently, the existence of an LU factorization) is guaranteed. In the first three cases the preservation of structure is also the basis for a proof of the numerical stability of Gaussian elimination.

Inverse of a Block Matrix

The Schur complement arises in formulas for the inverse of a block 2×2 matrix. If A_{11} is nonsingular, we can write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}, \quad S = A_{22} - A_{21}A_{11}^{-1}A_{12}. \qquad (2)$$

If S is nonsingular then inverting gives

$$\begin{aligned} A^{-1} &= \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}. \end{aligned}$$

So S^{-1} is the $(2,2)$ block of A^{-1}.

One can obtain an analogous formula for A^{-1} in terms of the Schur complement of A_{22} in A, in which the inverse of the Schur complement is the $(1,1)$ block of A^{-1}. More generally, we have $(A/A(\alpha,\alpha))^{-1} = A^{-1}(\alpha^c,\alpha^c)$.

Test for Positive Definiteness

For Hermitian matrices the Schur complement provides a test for positive definiteness. Suppose

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

with A_{11} positive definite. The factorization

$$A = \begin{bmatrix} I & 0 \\ A_{12}^*A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}, \quad S = A_{22} - A_{12}^*A_{11}^{-1}A_{12},$$

shows that A is congruent to the block diagonal matrix $\mathrm{diag}(A_{11}, S)$, and since congruences preserve inertia (the number of positive, zero, and negative eigenvalues), A is positive definite if and only if $\mathrm{diag}(A_{11}, S)$ is positive definite, that is, if and only if S is positive definite. The same equivalence holds with "definite" replace by "semidefinite" (with A_{11} still required to be positive definite).

Generalized Schur Complement

For A in (1) with a possibly singular $(1,1)$ block A_{11} we can define the *generalized Schur complement* $A_{22} - A_{21}A_{11}^{+}A_{12}$, where "+" denotes the Moore-Penrose pseudo-inverse. The generalized Schur complement is useful in the context of Hermitian positive semidefinite matrices, as the following result shows.

Theorem 1. *If*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \in \mathbb{C}^{n\times n}$$

is Hermitian then A is positive semidefinite if and only if $\mathrm{range}(A_{12}) \subseteq \mathrm{range}(A_{11})$ *and A_{11} and the generalized Schur complement $S = A_{22} - A_{12}^* A_{11}^{+} A_{12}$ are both positive semidefinite.*

If A_{11} is positive definite then the range condition is trivially satisfied.

Notes and References

The term "Schur complement" was coined by Haynsworth in 1968, thereby focusing attention on this important form of matrix and spurring many subsequent papers that explore its properties and applications. The name "Schur" was chosen because a 1917 determinant lemma of Schur says that if A_{11} is nonsingular then

$$\det\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det(A_{11})\det(A/A_{11}),$$

which is obtained by taking determinants in (2).

For much more about the Schur complement see Zhang (2005) or Horn and Johnson (2013).

- Emilie V. Haynsworth, Determination of the Inertia of a Partitioned Matrix, Linear Algebra Appl. 1(1), 73–81, 1968.
- Roger A. Horn and Charles R. Johnson, Matrix Analysis, second edition, Cambridge University Press, 2013. My review of the second edition.
- Fuzhen Zhang, editor, The Schur Complement and Its Applications. Springer-Verlag, New York, 2005, xvi+295 pp.

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