HK SEMINAR, LECTURE 1, 2

Contents

Abstract. In lecture 1, we introduce the definition and basic examples of algebraic K3 surface. Then we study classical invariants about K3 surface X, including $\mathrm{Pic}(X)$, $\mathrm{NS}(X)$, $\mathrm{Num}(X)$, Chern numbers and Hodge numbers $h^{p,q}(X)$.

In lecture 2, we similarly define complex K3 surface and study these invariants described as above. Furthermore, we study the singular cohomology, for which we will introduce some lattice theory. Finally, we generalize our constructions and introduce the concept of hyperKähler manifolds.

1. Algebraic K3 surface

We mainly follow [Huy16, Chapter 1].

1.1. Basic definition.

Definition 1.1. Let k be a field, a K3 surface over k is a complete non-singular variety X of dimension 2 such that

$$\omega_X \cong \mathcal{O}_X \text{ and } \mathrm{H}^1(\mathrm{X}, \mathcal{O}_X) = 0.$$

Here $\omega_X = \wedge^2 \Omega_X$ is the canonical bundle.

Now $\Gamma(X, \omega) = \Gamma(X, \mathcal{O}) = k$ and any nonzero global section of ω gives an algebraic symplectic structure on X. (Recall that a symplectic structure on X is given by a closed 2-form which is non-degenerated everywhere.)

Local computation shows that the natural pairing

$$\Omega_X \times \Omega_X \to \omega_X \cong \mathcal{O}_X$$

is non-degenerated, which gives a non-canonical isomorphism $\Omega_X \cong \underline{\mathrm{Hom}}(\Omega_X, \mathcal{O}_X) = \mathcal{T}_X$, here $\mathcal{T}_X$ is the tangent bundle.

Remark 1.2. Any smooth complete surface is projective.

1.2. **Examples.**

Example 1.3. *A smooth quartic $\iota : X \hookrightarrow \mathbb{P}^3$ is* K3. *The exact sequence of graded $\mathbb{C}[x_0, \cdots, x_3]$-module*

$$0 \to k[x_0, \cdots, x_3] \cdot f \to k[x_0, \cdots, x_3] \to k[x_0, \cdots, x_3]/(f) \to 0$$

gives exact sequence of sheaves:

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \iota_* \mathcal{O}_X \to 0. \tag{1.1}$$

Note that

$$\mathrm{H}^i(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} k^{\binom{n+m}{n}}, & \text{if } i = 0 \text{ and } m \geq 0 \\ k^{\binom{-m-1}{n}}, & \text{if } i = n \text{ and } m < 0 \\ 0, & \text{else.} \end{cases}.$$

And $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(\mathbb{P}^3, \iota_* \mathcal{F})$ for $\mathcal{F}$ coherent, since ι is affine.

Then, the long exact sequence

$$\cdots \to \mathrm{H}^1(\mathbb{P}^3, \mathcal{O}) = 0 \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^2(\mathbb{P}^3, \mathcal{O}(-4)) = 0 \to \cdots$$

tells us $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

By [Har13, Proposition 8.20, Chapter 2], *we know that $\omega_X = \iota^*(\omega_{\mathbb{P}^3} \otimes \mathcal{L}(X)) = \iota^*(\mathcal{O}(-4) \otimes \mathcal{O}(4)) = \mathcal{O}_X$. So X is a* K3 *surface.*

We can describe a trivialization of ω_X as

$$\mathrm{Res} \frac{\sum_{i=0}^{3} (-1)^i x_i dx_0 \wedge \cdots \wedge dx_3}{f},$$

where f is the defining equation of X. For a detailed argument, see [Ree06, Lemma 7.20]. *(Maybe this only holds for $k = \mathbb{C}$, try to generalize* [Ree06, Lemma 7.20].*)*

Similarly, a smooth complete intersection of type $(d_1, \cdots, d_n)$ in $\mathbb{P}^{n+2}$ with $\sum_{i=1}^{n} d_i = n+3$ is a K3 *surface. Use the exact sequences 1.1 and 1.1 tensor with $\mathcal{O}(-d_i)$ inductively, one can show $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Direct computation shows that $\wedge^{n+1} J = \mathcal{O}(n+3)$ here J is the ideal sheaf of X and by* [Har13, Proposition 8.20, Chapter 2] *again, $\omega_X \cong \mathcal{O}_X$.*

We may assume $d_i > 1$ for $i = 1, \cdots, n$, then the only cases are $(2, 2, 2)$ in $\mathbb{P}^5$, $(2, 3)$ in $\mathbb{P}^4$ and (4) in $\mathbb{P}^3$.

Example 1.4 (Kummer surface)**.** *Let k be a field with $\mathrm{char}(k) \neq 2$.*

Consider an abelian surface A here Λ is a lattice. The natural involution $\iota : A \to A$, $x \mapsto -x$ has 16 two-torsion points as fixed points: $\frac{1}{2}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. Here we choose a basis of Λ as a of $\mathbb{C}^2$ and $\epsilon_i = 0$ or 1.

Blow-up A at these 16 points and we get $\widetilde{A}$ with natural morphism $p : \widetilde{A} \to A$. By [Har13, Lemma 7.15, Chapter 2]*, we know that ι naturally extend to $\widetilde{A}$. A local computation shows that fixed locus of ι on $\widetilde{A}$ is the union of these exceptional divisors. Let $X = \widetilde{A}/\iota$, we claim that X is a* K3 *surface.*

(1) We first prove X is complete and smooth. Since X admits a surjective morphism from a complete variety, X is also complete.

Denote the quotient morphism: $\widetilde{A} \to X$ by π. Then π is a local isomorphism (or an étale morphism) outside the exceptional divisors F_i of $\widetilde{A}$. So we only need to check the local ring of any point in $\pi(F_i)$ is a regular local ring.

After a left multiplication of A, we may assume that $F_i = F_0$ lying above $0 \in A$. Let R be the local ring at 0 and $\mathfrak{m}_0$ be its maximal ideal. The induced map $\iota^* : \mathfrak{m}_0/\mathfrak{m}_0^2 \to \mathfrak{m}_0/\mathfrak{m}_0^2$ is multiplied by -1. By A smooth, we can choose u, v generating $\mathfrak{m}_0$, then $\iota^* u = -u$, $\iota^* v = -v$ mod $\mathfrak{m}_0^2$. Now, considering $(u - \iota^* u, v - \iota^* v) \equiv (2u, 2v)$ mod $\mathfrak{m}_0^2$, we have $(u - \iota^* u, v - \iota^* v)$ generate $\mathfrak{m}_0$ by Nakayama's lemma. Replace them by (u, v) for simplicity, we may assume that $\iota^* u = -u$ and $\iota^* v = -v$.

Now the local ring of a point in $F_0 \subset \widetilde{A}$ can be written as the following:

$$R[x, y]/(ux - vy))(\frac{1}{y})_{deg=0} = R[\frac{x}{y}]/(v - u\frac{x}{y}).$$

So its maximal ideal is generated by $(u, \frac{x}{y})$ and $\frac{x}{y} = \frac{v}{u}$ is ι^*-invariant. So, the maximal ideal of ι^*-invariant ring (also the local ring of X) is generated by $(u^2, \frac{x}{y})$, which forces the invariant ring to be regular.

(2) Now we show $\omega_X \cong \mathcal{O}_X$. Since $du \wedge dv$ is a nonvanishing section of A ($\omega_A \cong \mathcal{O}_A$) and $du \wedge dv$ is ι^*-invariant. Now $p^*(du \wedge dv)$ is an ι^*-invariant section of $\tilde{A}$, which naturally descends to a 2-form ω' of ω_X with ω' support on these image of exceptional divisors. Take $x_0 \in F_0$, for example, we have:

$$p^*(du \wedge dv) = du \wedge d(u\frac{x}{y}) = udu \wedge d(\frac{x}{y}) = \frac{1}{2}d(u^2) \wedge d(\frac{x}{y}) = \pi^*(\omega').$$

From this, one can see $\omega' = d(u^2) \wedge d(\frac{x}{y})$ on image of these exceptional divisors. However, u^2 and $\frac{x}{y}$ is a local coordinate of X around $\pi(x_0)$, so ω' is nowhere vanishing, which gives a trivialization of ω_X.

(3) Finally, we show $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. We only need to show $\mathrm{H}^0(X, \Omega_X) = 0$ by Hodge symmetry. Suppose that there is a non-zero global section ω, then $\pi^*\omega$ is an ι^*-invariant 1-form on $\widetilde{A}$. Note that $\mathrm{H}^0(A, \omega)$ is also a birational invariant for surfaces (analog the argument of [Har13, Theorem 8.19, Chapter 2] *and note that pole of a non-zero 1-form must be a divisor). So $\pi^*\omega$ gives an ι^*-invariant 1-form on A, which is forced to be 0 because $\mathrm{H}^1(A, \Omega_A) = \mathrm{Span}_{\mathbb{C}}(du, dv)$ and $\iota^* du = -du$, $\iota^* dv = -dv$.*

The argument of the last paragraph uses the Hodge symmetry, which requires $\mathrm{char}(k) = 0$*. For a general argument for positive character, see* [Băd01, Theorem 10.6] *(classify minimal surfaces, forcing $h^{0,1} = 0$ holds).*

Example 1.5. *Consider the double covering $\pi : X \to \mathbb{P}^2$ branched along a smooth sextic curve C, then X is a* K3 *surface.*

Locally, one can write the equation of X as $y^2 - f = 0$, where f is a local equation of C. So X is smooth if and only if C is smooth by the Jacobian criterion.

The local computation also shows that $\pi_*\mathcal{O}_X$ is a locally free sheaf of $\mathcal{O}_{\mathbb{P}^2}$ of rank 2, locally the sections given by 1 and y. However, one can define the trace map from $\pi_*\mathcal{O}_X$ to $\mathcal{O}_{\mathbb{P}^2}$ as we did in number theory, which gives the splitting $\pi_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}$.

Now $\mathcal{L}$ generated locally by y, which should be of "degree 3". From this one can show that the transition maps have degree -3 and thus $\mathcal{L} \cong \mathcal{O}(-3)$. Note that the ramification divisor B of X is given by $y = 0$, which corresponds to the Cartiar divisor locally given by y. Note that $\mathcal{L}$ generated by y, so $\pi^*(\mathcal{L})$ is generated by y again, corresponding to the Cartier divisor y^{-1}. So $\pi^*(\mathcal{L}^{-1}) = \mathcal{L}(B)$ and the canonical bundle formula $\omega_X \cong \pi^*\omega_{\mathbb{P}^2} \otimes \mathcal{L}(B)$ (similar proof as [Har13, Proposition 2.3, Chapter 4]*, for a precise reference, see* [Iit82, Theorem 5.5, Page 202]*) reads that*

$$\omega_X \cong \pi^*(\mathcal{O}(-3) \otimes \mathcal{L}^{-1}) = \pi^*(\mathcal{O}(-3) \otimes \mathcal{O}(3)) = \mathcal{O}_X.$$

And

$$\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^1(\mathbb{P}^2, \pi_*\mathcal{O}_X) = \mathrm{H}^1(\mathbb{P}^2, \mathcal{O} \oplus \mathcal{O}(-3)) = 0.$$

For more details and general facts for coverings, see [BHPVdV03, P53-P56].

Example 1.6. *Let X be a generic surface of type (3,2), resp. (2,2,2), in $\mathbb{P}^2 \times \mathbb{P}^1$, resp. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X is a* K3 *surface.*

We have $\omega_X = (\omega_{\mathbb{P}^2} \boxtimes \omega_{\mathbb{P}^1}) \otimes \mathcal{L}(X)|_X = \mathcal{O}(-3,-2) \otimes \mathcal{O}(3,2)|_X = \mathcal{O}_X$.

And $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ follows from the following exact sequence and Künneth formula for cohomology of quasi-coherent sheaves:

$$0 \to \mathcal{O}(-3,-2) \to \mathcal{O} \to \mathcal{O}_X \to 0.$$

Example 1.7. *Consider the Plücker embedding $Gr(2,6) \hookrightarrow \mathbb{P}^{14}$. Let X be a generic intersection of $Gr := Gr(2,6)$ with $\mathbb{P}^8$. Use Bertini's theorem (*[Har13, Theorem 8.18, Chapter 2]*) iteratively, we know X is smooth.*

Use the fact that $\omega_{Gr(r,n)} = \mathcal{O}(-n)$ and [Har13, Theorem 8.20, Chapter 2]*, one can show that $\omega_X = \mathcal{O}_X$.*

We consider X over $\mathbb{C}$. Use weak Lefschetz theorem for the fundamental group ([Voi03, Theorem 1.23]*), one can show X is simply connected (Gr is simply connected, use fibration $P_{2,4} \to \mathrm{GL}(6, \mathbb{C}) \to Gr$). In particular, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ by Hodge decomposition theorem.*

Degree of X computes as following:

$$\begin{aligned} \deg(X) &= |X \cap \textit{generic 2 hyperplanes}| \\ &= |Gr \cap \textit{generic 8 hyperplnes}| \\ &= \deg(Gr) = 14. \end{aligned}$$

Example 1.8. *Recall that a Fano manifold F is a smooth projective variety over $\mathbb{C}$ such that ω_F^* is ample.*

It is a fact that F is always simply connected, [TSU88].

Assume that F is of co-index 3, that is, $\omega_Y = \mathcal{L}^{-r}$ with $\mathcal{L}$ very ample and $\dim(F) - r + 1 = 3$.

Take generic r hyperplanes $H_1, \cdots, H_r \in |\mathcal{L}|$ and consider $X := F \cap_{i=1}^r H_i$, we have

$$\mathcal{O}(\sum_{i=1}^{r} H_i) = \mathcal{L}^r = \omega_F^{-1}.$$

So $\omega_X = \mathcal{O}_X$ as before.

Since F is simply connected, X is simply connected by weak Lefschetz for the fundamental group. So X is a K3 *surface.*

Consider $\pi : F \to Gr(2,5)$ a double covering ramified along a quadratic (consider $Gr(2,5)$ a subvariety of $\mathbb{P}^9$ via Plücker embedding). Then by canonical bundle formular of ramified covering, F is a Fano manifold of coindex 3. So, F will give a family of K3 *surfaces.*

Actually,

$$\omega_F = \pi^*(\mathcal{O}(-5) \otimes \mathcal{O}(-1)) = \pi^*\mathcal{O}(-4) = \pi^*(\mathcal{O}(1))^{-4}.$$

And $\dim_{\mathbb{C}}(F) = \dim_{\mathbb{C}}(Gr(2,5)) = 6$ *with* $6 - 4 + 1 = 3$.

Consider a double covering $\pi : F \to \mathbb{P}^2 \times \mathbb{P}^2$ ramified alone a smooth (2,2) divisor. Use canonical bundle formula for ramified covering, we have F is Fano with $\omega_F = \pi^*\mathcal{O}(-2,-2) = \pi^*(\mathcal{O}(1,1))^{-2}$. So F gives K3 *surface X with*

$$\begin{aligned} \deg(X) &= (\pi^*\mathcal{O}(1,1)|_X)^2 \\ &= (\pi^*\mathcal{O}(1,1))^4 \\ &= \deg(\pi)\mathcal{O}(1,1)^4 \\ &= 2 \times 6 = 12. \end{aligned}$$

1.3. **Classical invariants.** Let X be an arbitrary non-singular complete surface over $\mathbb{C}$. We define the intersection form $(L_1.L_2)$ for line bundle L_1 and L_2 over X.

Definition 1.9. Following notations above, the following definitions for $(L_1.L_2)$ are equivalent:

1. Coefficient of monomial $n_1 \cdot n_2$ in the polynomial $\chi(X, L_1^{n_1} \otimes L_2^{n_2})$, see [Har06, §5, Chapter 1].
2. $(L_1.L_2) = \chi(X, \mathcal{O}_X) - \chi(X, L_1^*) - \chi(X, L_2^*) + \chi(X, L_1^* \otimes L_2^*)$, see [Mum66, Lecture 12].
3. $(L_1.L_2) = c_1(L_1) \cup c_1(L_2) \in \mathrm{H}^4(X, \mathbb{Z})$, cup product of corresponding chern classes (For $k = \mathbb{C}$ only).
4. Let $L_i = L(D_i)$, $i = 1, 2$, $(L_1.L_2) = \deg(D_1.D_2)$, where $D_i = c_1(L_i) \in \mathrm{CH}^1(X)$, see [Har13, Appendix A]
5. Axiomatic definition described in [Har13, Theorem 1.1, Chapter 5]

Visually, we have the following commutative diagram compatible with all kinds of multiplications defined above:

$$\begin{tikzcd}
 & D \arrow[r, mapsto] & {[D]} & \\
D \arrow[d, mapsto] & Cl(X) \arrow[r] \arrow[d, "\cong"] & CH^1(X) \arrow[r, hook] \arrow[d, "cl"'] & CH^*(X) \arrow[d] \\
L(D) & Pic(X) \arrow[r] & H^2(X,\mathbb{Z}) \arrow[r, hook] & H^{2*}(X,\mathbb{Z}) \\
 & L \arrow[r, mapsto] & c_1(L) &
\end{tikzcd}.$$

All of these definitions define the same symmetric bilinear form with the following basic properties:

1. If $L_1 = \mathcal{O}(C)$ for some integral curve $C \subset X$, then $(L_1.L_2) = \deg(L_2|_C)$.
2. If $L_i = \mathcal{O}(C_i)$ for two curves $C_i \subset X$, $i = 1, 2$, intersecting in only finitely many points $x_1, \dots, x_n$, then

$$(L_1.L_2) = \sum_{i=1}^{n} \dim_k(\mathcal{O}_{X,x_i}/(f_{1,x_i}, f_{2,x_i})),$$

   Here, f_{i,x_j} is the local equation of C_i near x_j.
3. If L_1 is ample and $L_2 = \mathcal{O}(C)$ for a curve $C \subset X$, then

$$(L_1.L_2) = \deg(L_1|_C) > 0.$$

   (Any ample line bundle on a curve has a positive degree. Since a sufficiently large tensor is very ample, which has a non-trivial section, thus corresponding to an effective divisor.)

Additionally, we have the following Riemann-Roch theorem:

Theorem 1.10. [Har13, Theorem 1.6, Chapter 5]

$$\chi(X, L) = \frac{(L.L \otimes \omega_X^*)}{2} + \chi(X, \mathcal{O}_X).$$

Definition 1.11. The Néron-Severi group of an algebraic surface X is the quotient

$$\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(\mathrm{X}),$$

where $\mathrm{Pic}^0(X)$ is the connected component of $\mathrm{Pic}(X)$, consisting line bundles algebraic equivalent to zero. (For a precise definition of algebraic equivalence, see [Har13, Example 9.8.5, Chapter 3 and exercise 1.7, Chapter 5])

Now we define invariants from quotients of $\mathrm{Pic}(X)$.

Definition 1.12. A line bundle L is numerically trivial if $(L, L') = 0$ for all line bundle L'.

The subgroup of all numerically trivial line bundles is denoted by $\mathrm{Pic}^\tau(X) < \mathrm{Pic}(X)$.

And the quotient is defined as:

$$\mathrm{Num}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X).$$

By [Har13, Exercise 1.7, Chapter 5], $\mathrm{Pic}^0(X) < \mathrm{Pic}^\tau(X)$. So, $\mathrm{Num}(X)$ is a quotient of $\mathrm{NS}(X)$.

Proposition 1.13. [LN59] *The Néron-Severi group* $\mathrm{NS}(X)$ *and its quotient* $\mathrm{Num}(X)$ *are finitely generated. The rank of* $\mathrm{NS}(X)$ *is called the Picard number* $\rho(X)$*.*

Clearly, $\mathrm{Num}(X)$ is torsion-free, so it is a free abelian group endowed with a non-degenerated, symmetric pairing:

$$(\ .\ ) : \mathrm{Num}(X) \times \mathrm{Num}(X) \to \mathbb{Z}.$$

The signature of above pairing is determined by the following theorem:

Theorem 1.14 (Hodge Index Theorem)**.** [Har13, Theorem 1.9, Chapter 5] *Let L_0 be an ample line bundle on the surface X, and suppose that L is a line bundle, L is not numerically equivalent to 0 and $(L.L_0) = 0$. Then $(L)^2 := (L.L) < 0$.*

Corollary 1.15. *The signature of the intersection form on* $\mathrm{Num}(X)$ *is* $(1, \rho(X) - 1)$*. Thus,* $(\ .\ )$ *on*

$$\mathrm{Num}(X)_{\mathbb{R}} := \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

can be diagonalized with entries $(1, -1, \cdots, -1)$*.*

Remark 1.16. The Hodge index theorem has the following immediate consequences:

The cone of all classes $L \in \mathrm{NS}(X)_{\mathbb{R}}$ with $L^2 > 0$ has two connected components. The positive cone $\mathcal{C}_X \subset \mathrm{NS}(X)_{\mathbb{R}}$ is defined as the connected component that contains an ample line bundle.

If L_1 and L_2 are line bundles such that $L_1^2 \geq 0$, then apply Hodge index theorem to $(L_1^2)L_2 - (L_1.L_2)L_1$ (clearly orthogonal to L_1), one get:

$$(L_1)^2(L_2)^2 \leq (L_1.L_2)^2.$$

We can also express the above inequality as the following intersection matrix is non-positive definite

$$\begin{pmatrix} (L_1)^2 & (L_1.L_2) \\ (L_1.L_2) & (L_2)^2 \end{pmatrix}.$$

Now we study the above invariants for K3 surfaces.

For a K3 surface X, $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. And we have $\mathrm{H}^0(X, \mathcal{O}_X) \cong \mathbb{C}$ by completeness of X. By Serre's duality, $\mathrm{H}^2(X, \mathcal{O}_X) \cong (\mathrm{H}^0(X, \omega_X))^* = \mathbb{C}$. So we have

$$\chi(X, \mathcal{O}_X) = 2$$

Proposition 1.17. *The algebraic fundamental group $\pi_1^{\acute{e}t}(X)$ (which classifies irreducible étale coverings of X) is trivial.*

Proof. If $\widetilde{X} \to X$ is an irreducible étale covering of degree d. Then $\widetilde{X}$ is smooth complete surface and it is easy to see that $\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = d\chi(X, \mathcal{O}_X) = 2d$ [BHPVdV03, Lemma 17.1, Chapter 1]. The covering is unramified and thus the canonical bundle formula shows that $\omega_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$. Now $\mathrm{H}^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong \mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = k$, which forces that $\mathrm{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ and $d = 1$. $\square$

By 1.3, the Riemann-Roch formula reads that:

$$\chi(X, L) = \frac{(L)^2}{2} + 2.$$

Proposition 1.18. *For a* K3 *surface X the natural surjections are isomorphisms:*

$$\mathrm{Pic}(X) \twoheadrightarrow \mathrm{NS}(X) \twoheadrightarrow \mathrm{Num}(X).$$

Moreover, the intersection pairing (.) *on* $\mathrm{Pic}(X)$ *is even, non-degenerated, and of signature* $(1, \rho(X) - 1)$.

Proof. Suppose L is non-trivial, but $(L.L') = 0$ for an ample line bundle L'. By 1.3, L cannot correspond to an effective divisor. Then $\mathrm{H}^0(X, L) = 0$ by [Har13, Proposition 7.7, Chapter 2]. Similarly, $(L^*, L') = 0$ implies $\mathrm{H}^0(X, L^*) = 0$. Then $\mathrm{H}^2(X, L) = 0$ by Serre's duality. So $0 \geq \chi(X, L) = \frac{(L)^2}{2} + 2$ and then $(L)^2 < 0$, which shows that $\mathrm{Pic}(X) \to \mathrm{Num}(X)$ is injective. Finally, the Riemann-Roch formula $(L)^2 = 2\chi(X, L) - 4$ shows that the pair is even.

$\square$

We shall next compute the Chern number $c_2(X)$ and the Hodge numbers

$$h^{p,q}(X) := \dim_k \mathrm{H}^q(X, \Omega_X^p).$$

We first state the Hirzebruch- Riemann-Roch formula, which is useful to compute invariants mentioned above.

Theorem 1.19 (Hirzebruch-Riemann-Roch)**.** [Har13, Theorem 4.1, Appendix A] *For a rank r locally free sheaf $\mathcal{E}$ over a non-singular projective variety X, we have:*

$$\chi(\mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E}).\mathrm{td}(\mathcal{T}_X))_n,$$

where $(\quad)_n$ *means the degree n part of* $\mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ *and $\mathcal{T}_X$ is the tangent bundle over X.*

For surface X and a locally free sheaf $\mathcal{E}$ of rank r,

$$\mathrm{ch}(\mathcal{E}) = r + c_1(\mathcal{E}) + \frac{1}{2}(c_1^2(\mathcal{E}) - 2c_2(\mathcal{E}))$$

and

$$\mathrm{td}(\mathcal{E}) = 1 + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}(c_1^2(\mathcal{E}) + c_2(\mathcal{E})).$$

When X is K3, $c_1(X) = c_1(\mathcal{T}_X) = c_1(\Omega_X) = c_1(\omega_X) = c_1(\mathcal{O}_X) = 0$. So, $\mathrm{td}(\mathcal{T}_X) = 1 + \frac{1}{12}c_2(X)$. And the Hirzebruch-Riemann-Roch formula reads that:

$$\chi(X, \mathcal{E}) = \frac{1}{2}(c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})) + \frac{\mathrm{rank}(\mathcal{E})}{12}c_2(X). \tag{1.2}$$

Apply 1.2 for $\mathcal{E} = \mathcal{O}_X$, then we have $2 = \frac{c_2(X)}{12}$, that is,

$$c_2(X) = 24.$$

Now apply 1.2 for $\mathcal{E} = \Omega_X \cong \mathcal{T}_X$, note that $\mathrm{H}^2(X, \Omega_X) \cong (\mathrm{H}^0(X, \mathcal{T}_X \otimes \omega_X))^* \cong (\mathrm{H}^0(X, \Omega_X))^*$, we get:

$$2h^{1,0} - h^{1,1} = \frac{1}{2}(-2c_2(X)) + \frac{1}{6}(c_2(X)) = -20.$$

Now we assume $\mathrm{char}(k) = 0$. By Hodge symmetry, we have $h^{1,0} = h^{0,1} = 0$, so $h^{1,1} = 20$. And $h^{0,2} = h^{2,0} = 1$. Use Serre's duality, we get all Hodge numbers. For an argument for arbitrary character, see [Huy16, Chapter 9]. (For general algebraic surface over positive character field k, $h^{0,1} = h^{1,0}$ no longer holds, see [Ser56, Proposition 16].)

2. Complex K3 surface

2.1. Definition and examples.

Definition 2.1. A complex K3 surface is a compact complex manifold X of dimension 2 such that $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

If a complex K3 surface X is an analytic closed subvariety of some projective space, by Serre's GAGA principle [Ser56], X can be seen as an algebraic K3 surface and it is equivalent to study its analytic coherent sheaves with algebraic coherent sheaves and the corresponding cohomology.

However, the following example shows that there exist complex K3 surfaces which are not algebraic.

We first introduce the following two theorems:

Theorem 2.2. [BHPVdV03, Theorem 6.2, Chapter 4] *A compact complex surface X is projective if and only if there exists on X a line bundle L with $c_1(L) > 0$.*

Theorem 2.3. [Ree06, Theorem 7.31] *Let X be a Kähler manifold and $\widetilde{X}$ is a blow-up of X along $Z \subset X$ with Z is of codimension r, then we have the following isomorphism of Hodge structures.*

$$\mathrm{H}^k(X, \mathbb{Z}) \oplus (\oplus_{i=0}^{r-2}\mathrm{H}^{k-2i-2}(Z, \mathbb{Z})(i+1)) \to \mathrm{H}^k(\widetilde{X}, \mathbb{Z}).$$

Here $*(i+1)$ means shift weight of Hodge structure by $(i+1, i+1)$.

Example 2.4. *Let $A = \mathbb{C}^2/\Lambda$ be a 2-dimensional complex torus and $\iota : A \to A$ the natural involution. Blow-up at the fixed points to get $\widetilde{A}$ and extend ι to $\widetilde{A}$. Quotient $\widetilde{A}$ by ι to get a surface X. A similar proof as example 1.4 shows that X is a complex* K3 *surface.*

We claim that X is algebraic if and only if A is algebraic. Concerning that a generic 2-dimensional complex torus is not algebraic, we get examples of non-algebraic K3 *surfaces.*

If A is algebraic, then X is algebraic.

If A is not algebraic, then A/ι cannot be algebraic because a pull back of numerically positive line bundle via a finite quotient is still numerically positive and use Theorem 2.2. Similarly, if $\widetilde{A}$ is not algebraic, then X cannot be algebraic. Therefore, it is remained to show that A is not algebraic implies that $\widetilde{A}$ is not algebraic.

Let $\mathcal{P}_Y = \{t \in \mathrm{H}^2(Y, \mathbb{Z})| \ t^2 > 0\}$ be the positive cone for an arbitrary complex surface Y. We have Y is algebraic if and only if $\mathcal{P}_Y \cap \mathrm{H}^{(1,1)}(Y) \neq \phi$. This follows from Lefschetz (1,1)-class theorem and theorem 2.2.

We apply the theorem 2.2 to A and $\widetilde{A}$. So, we only need to show $\mathcal{P}_A \cap \mathrm{H}^2(A, \mathbb{Z}) = \phi$ implies $\mathcal{P}_{\widetilde{A}} \cap \mathrm{H}^2(\widetilde{A}, \mathbb{Z}) = \phi$. This follows from theorem 2.3, which describes the cohomology of blow-up.

Actually, we apply theorem 2.3 for $X = A$, $Z = \{16$ points fixed by $\iota\}$ and $k = 2$. We know that

$$\mathrm{H}^2(\widetilde{A}, \mathbb{Z}) = \mathrm{H}^2(A, \mathbb{Z}) \oplus (\oplus_{i=1}^{16} \mathrm{H}^0(\mathrm{pt}^i, \mathbb{Z})(1)),$$

and thus

$$\mathrm{H}^2(\widetilde{A}, \mathbb{Z}) \cap \mathrm{H}^{1,1}(\widetilde{A}) = (\mathrm{H}^{1,1}(A) \cap \mathrm{H}^2(A, \mathbb{Z})) \oplus (\oplus_{i=1}^{16} \mathrm{H}^0(\mathrm{pt}^i, \mathbb{Z})(1)).$$

If there is a positive $(1,1)$-class in $\mathrm{H}^2(\widetilde{A}, \mathbb{Z})$, then it can be written as a $(1,1)$-class in $\mathrm{H}^2(A, \mathbb{Z})$ direct sum with some elements in $\mathrm{H}^0(\mathrm{pt}^i, \mathbb{Z})(1))$. However, generator of each $\mathrm{H}^0(\mathrm{pt}^i, \mathbb{Z})(1))$ is the class of the corresponding exceptional divisor, which have self-intersection number -1, forcing $(1,1)$-class in $\mathrm{H}^2(A, \mathbb{Z})$ positive, a contradiction!

2.2. **Classical invariants.** Note that definition 1.9, and two Riemann-Roch theorems 1.10, 1.19 still works for complex surfaces. However, the Hodge index theorem fails since the conception ample line bundle does not make sense for a general complex surface.

From the exponential sequence [Har13, §5, Appendix B]

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^\times \to 0,$$

we get the long exact sequence:

$$\begin{aligned} 0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp}{\twoheadrightarrow} \mathbb{C}^\times \to \mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{H}^1(X, \mathcal{O}_X) = 0 \\ \to \mathrm{Pic}(X) \overset{c_1}{\to} \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X) = \mathbb{C} \to \cdots \end{aligned}$$

So $\mathrm{H}^1(X, \mathbb{Z}) = 0$ and $\mathrm{H}^3(X, \mathbb{Z}) = 0$ up to torsion. And $\mathrm{H}^0(X, \mathbb{Z}) = \mathrm{H}^4(X, \mathbb{Z}) = \mathbb{Z}$. We will study $\mathrm{H}^2(X, \mathbb{Z})$ later.

Next we show $\mathrm{Pic}(X)$ is torsion free. Take a torsion line bundle L, then $(nL, nL) = 0$ for some $n \in \mathbb{Z} \setminus \{0\}$, which implies $(L)^2 = 0$. So $\chi(X, L) = 2$ by Theorem 1.10. So either $\mathrm{H}^0(X, L)$ or $\mathrm{H}^0(X, L^*)$ is nontrivial (Serre's duality). Take a nonzero section $s \in \mathrm{H}^0(X, L)$, then $s^n \in \mathrm{H}^0(X, L^n)$ is a nonzero constant. However, the zero locus of s and s^n coincides. So s gives a trivialization of L.

From $0 \to \mathrm{Pic}(X) \overset{c_1}{\to} \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X) = \mathbb{C} \to \cdots$, we know that $\mathrm{H}^2(X, \mathbb{Z})$ is torsion-free.

By the universal coefficient theory and Poincaré duality, we know that torsion part of $\mathrm{H}^3(X,\mathbb{Z})$ coincide with that of $\mathrm{H}^2(X,\mathbb{Z})$, so $\mathrm{H}^3(X,\mathbb{Z})=0$.

Moreover, the injection $\mathrm{Pic}(X)\hookrightarrow \mathrm{H}^2(X,\mathbb{Z})$ implies that $\mathrm{Pic}^0(X)=0$ and thus

$$\mathrm{Pic}(X)\cong \mathrm{NS}(X).$$

Remark 2.5. There is no conception of an ample line bundle for a general complex surface, so the proof of isomorphisms $\mathrm{Pic}(X)\cong \mathrm{Num}(X)$ 1.18 and the proof of the Hodge index theorem 1.14 no longer hold.

Actually, it can happen that $\mathrm{Pic}^\tau(X)$ is non-trivial so $\mathrm{Pic}(X)$ does not isomorphic to $\mathrm{Num}(X)$.

However, Lefschetz $(1,1)$-class theorem (which we will introduce later) and Hodge index theorem for $\mathrm{H}^{1,1}(X)_{\mathrm{prim}}$ (see [Ree06, §6.3.2]) ensure that the intersection form on $\mathrm{Pic}(X)$ has at most 1 positive eigenvalue.

The computations for $c_2(X)$ and $h^{p,q}(X)$ in §1.3 still hold, so we get:

$$c_2(X)=24,$$

and

$$h^{2,0}=h^{0,2}=1 \text{ and } h^{1,1}=20.$$

And the Hodge spectrum sequence degenerates for K3 surfaces [BHPVdV03, §2, Chapter 4], so the Hodge decomposition holds for complex K3 surfaces.

Combining above arguments, we have

$$\mathrm{H}^i(X,\mathbb{Z})=\begin{cases}\mathbb{Z} & i=0,4\\ \mathbb{Z}^{22} & i=2\\ 0 & \text{else.}\end{cases}$$

Remark 2.6. Another way to get $b_2(X):=\mathrm{rank}(\mathrm{H}^2(X,\mathbb{Z}))$ is to note that the Euler characteristic number $e(X)$ equals the top Chern number $c_2(X)=24$ [BHPVdV03, §4, Chapter 1].

As for $\rho(X)=\mathrm{rank}(\mathrm{Pic}(X))$, we have the following Lefschetz $(1,1)$-class theorem:

Theorem 2.7. [GH14, P 163] *Let X be a compact Kähler manifold, consider the following natural map:*

$$\begin{array}{ccc}\mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{H}^2(X,\mathbb{Z})\\ L & \mapsto & c_1(L).\end{array}$$

Then the image of above map is exactly $\mathrm{H}^{1,1}(X)\cap \mathrm{H}^2(X,\mathbb{Z})$.

Use the fact that every K3 surface is Kähler [Siu83] and note that $\mathrm{Pic}(X)\xrightarrow{c_1}\mathrm{H}^2$ is injective, we get:

$$\mathrm{Pic}(X) \cong \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{Z}),$$

and hence

$$\rho(X) \leq h^{1,1} = 20.$$

In fact, every Picard number between 0 and 20 is realized by some comlex K3 surfaces [Huy16, Chapter 17, Chapter 18].

2.3. **Basic lattice theory.** The intersection pairing on $\mathrm{H}^2(X, \mathbb{Z})$ gives it a lattice structure. To make it clear, we need to introduce some lattice theory. We follow [Huy16, §0, Chapter 14]. For more details, one can refer [Nik80].

Definition 2.8. A lattice Λ is a free $\mathbb{Z}-$module of finite rank together with a symmetric bilinear form:

$$(\ ,\ ) : \Lambda \times \Lambda \to \mathbb{Z}.$$

A lattice Λ is called even if

$$(x)^2 := (x.x) \in 2\mathbb{Z}$$

for all $x \in \Lambda$, otherwise Λ is called odd.

The determinant of the intersection matrix with respect to an arbitrary $\mathbb{Z}$-basis of Λ is called the discriminant, $\mathrm{disc}\Lambda$.

A lattice Λ can be extend to a $\mathbb{R}-$linear space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with extended $\mathbb{R}-$bilinear symmetric form $(\ .\ )$. The signature (n_+, n_-) of such $\mathbb{R}-$bilinear form is called the signature of Λ. And $\tau(\Lambda) := n_+ - n_-$ is called the index of Λ.

The lattice Λ is called definite if either $n_+ = 0$ or $n_- = 0$. Otherwise, Λ is indefinite.

One defines an injection of finite index:

$$\begin{aligned} i_\Lambda : \Lambda &\hookrightarrow \Lambda^* := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \\ x &\mapsto (x.\ ). \end{aligned}$$

Alternatively, if Λ^* is viewed as the subset of all $x \in \Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $(x.\Lambda) \subset \mathbb{Z}$, then i_Λ is identified with the natural inclusion. The cokernel of i_Λ is called the discriminant group:

$$A_\Lambda := \Lambda^*/\Lambda$$

of Λ, which is a finite group of order $|\mathrm{disc}\Lambda|$.

Definition 2.9. A lattice Λ is called unimodular if i_Λ is an isomorphism $\Lambda \stackrel{\cong}{\to} \Lambda^*$ or, equivalently, if A_Λ is trivial or, still equivalently, if $\mathrm{disc}\Lambda = \pm 1$.

The minimal number of generators of the finite group A_Λ is denoted $l(A_\Lambda) = l(\Lambda))$.

The pairing $(\ .\ )$ on Λ induces a $\mathbb{Q}-$valued pairing on Λ^* and hence a pairing $A_\Lambda \to \mathbb{Q}/\mathbb{Z}$. If the lattice is even, then the $\mathbb{Q}-$valued quadratic form on Λ^* yields

$$q_\Lambda : A_\Lambda \to \mathbb{Q}/2\mathbb{Z}.$$

A finite abelian group A with a quadratic form $q : A \to \mathbb{Q}/2\mathbb{Z}$ is called a finite quadratic form.

For two lattice Λ_1 and Λ_2, the direct sum $\Lambda_1 \oplus \Lambda_2$ shall always denote the orthogonal direct sum with $(x_1 + x_2.y_1 + y_2)_{\Lambda_1\oplus\Lambda_2} = (x_1.y_1)_{\Lambda_1} + (x_2.y_2)_{\Lambda_2}$. One can show that:

$$(A_{\Lambda_1\oplus\Lambda_2}, q_{\Lambda_1\oplus\Lambda_2}) \cong (A_{\Lambda_1}, q_{\Lambda_1}) \oplus (A_{\Lambda_2}, q_{\Lambda_2}).$$

A morphism between two lattices $\Lambda_1 \to \Lambda_2$ is by definition a linear map that respects the quadratic forms.

An injective morphism $\Lambda_1 \hookrightarrow \Lambda$ is called a primitive embedding if its cokernel is torsion free.

Two even lattices Λ_1, Λ_2 are called orthogonal if there exists a primitive embedding $\Lambda_1 \hookrightarrow \Lambda$ into an even unimodular lattice with $\Lambda_2 \cong \Lambda_1^\perp$.

Example 2.10. *1. By $\langle 1\rangle$, $\mathbb{Z}$, or I_1 one denote the lattice of rank one with intersection matrix 1. The direct sum $\langle 1\rangle^{\oplus n}$ is often denoted I_n.*

2. The hyperbolic plane U is the rank 2 lattice determined by the following intersection matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with basis denoted by $\{e, f\}$.

If $\mathrm{U} \hookrightarrow \Lambda$ is an arbitrary embedding, then $\Lambda \cong \mathrm{U} \oplus \mathrm{U}^\perp$. Indeed, for any $\alpha \in \Lambda$, $\alpha = (e.\alpha)f + (f.\alpha)e + \beta$ and $\beta \in \mathrm{U}^\perp$.

3. The E_8-lattice is given by the intersection matrix

$$E_8 := \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix},$$

and is an even, unimodular, positive definite ($n_- = 0$) of rank eight with $\mathrm{disc}(E_8) =$ 1 *lattice.*

4. For any given lattice Λ the twist $\Lambda(m)$ is obtained by changing the intersection form $(\ .\ )$ of Λ by the integer m, i.e. $\Lambda = \Lambda(m)$ as $\mathbb{Z}-$module but

$$(\ .\ )_{\Lambda(m)} := m \cdot (\ .\ )_\Lambda\ .$$

We have the following proposition which determines the lattice structure of $\mathrm{H}^2(X, \mathbb{Z})$.

Proposition 2.11. [Huy16, Corollary 1.3, Chapter 14] *Let Λ be an indefinite, unimodular even lattice of signature (n_+, n_-).*

Then the index $\tau = n_+ - n_-$ of Λ satisfies $\tau \equiv 0$ (mod 8), and

$$\Lambda \cong E_8^{\oplus \frac{\tau}{8}} \oplus \mathrm{U}^{\oplus n_-} \text{ or } \Lambda \cong E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus \mathrm{U}^{\oplus n_+}.$$

We want to apply Proposition 2.11 to $\mathrm{H}^2(X, \mathbb{Z})$.

First note that the Poincaré duality gives a perfect pairing:

$$\mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \to \mathbb{Z},$$

given by the usual intersection product.

Combining that $\mathrm{H}^2(X, \mathbb{Z})$ is torsion-free, $\mathrm{H}^2(X, \mathbb{Z})$ equipped with the intersection product is a unimodular lattice.

According to Wu's formula ([MS74]), the intersection product of a compact differentiable fourfold M is even if its second Stiefel-Whitney class $w_2(M)$ is trivial. Moreover, $w_2(M) \equiv c_1(M)$ mod(2) for any almost complex structure X on M. Hence, its intersection form is even.

The signature of the intersection pairing can be computed by the Thom-Hirzebruch index theorem which in dimension two says that the index is

$$\frac{p_1(X)}{3} = \frac{c_1^2(X) - 2c_2(X)}{3} = \frac{-2 \times 24}{3} = -16.$$

Combining $b_2(X) = \mathrm{rank}(\mathrm{H}^2(X, \mathbb{Z})) = 22$, the signature is therefore $(3, 19)$.

Apply Proposition 2.11, we get

Proposition 2.12. *The integral cohomology $\mathrm{H}^2(X, \mathbb{Z})$ of a complex* K3 *surface X endowed with the intersection form $(\ .\ )$ is abstractly isomorphic to the lattice*

$$E_8(-1) \oplus E_8(-1) \oplus \mathrm{U} \oplus \mathrm{U} \oplus \mathrm{U}.$$

3. HyperKähler manifolds

3.1. **Hilbert schemes and hyperKähler manifolds.** In this section, we generalize the concept K3 surfaces to hyperKähler manifolds. We also generalize the construction of example 1.4, which turns out give a family of hyperKähler manifolds.

We first introduce some notations, for detailed definitions, see [Nak99, Chapter 1]:

Let S be an smooth complete algebraic (compact complex) surface over $\mathbb{C}$.

Let $S^{(n)} := S^n/S_n$ be the n-th symmetric product of S, where S_n is the n-th permutation group.

Let $S^{[n]}$ be the n-th Hilbert scheme (Douady space) of S.

For A an abelian surface (complex torus of dimension 2), we define $A^{(n)}$ and $A^{[n]}$ similarly.

There is a natural morphism:

$$\begin{aligned} \pi : S^{[n]} &\to S^{(n)} \\ \xi &\mapsto \sum_{x \in S} \dim_k(\mathcal{O}_{\xi,x}/m_x)x. \end{aligned}$$

Theorem 3.1 ([Gro60],[Fog68]). *Follow the above notations, $S^{[n]}$ is a smooth projective variety of dimension 2n.*

And the natural morphism π is a resolution of singularity.

If S is symplectic, that is, $\omega_S \cong \mathcal{O}_S$, then $\sum_{i=1}^{n} p_i^*\omega$ over S^n will induce a non-vanishing closed holomorphic 2-form on $S^{[n]}$, where $p_i : S^n \to S$ is the projection to i-th coordinate and ω is a trivialization of ω_S. For more details, see [Bea83, Proposition 5]. In conclusion, we have:

Proposition 3.2. *If S is symplectic, then $S^{[n]}$ is symplectic.*

And we have the following fact:

Proposition 3.3. [Bea83, Lemma 1]

$$\pi_1(S^{[n]}) = \pi_1(S^{(n)}) = (\pi_1(S))^{ab}.$$

Proposition 3.4. [Bea83, Lemma 2] *Let S be a compact surface and $r \geq 2$.*

If $\mathrm{H}^1(S, \mathbb{C}) = 0$, then there is an isomorphism of Hodge structures:

$$\mathrm{H}^2(S^{[n]}, \mathbb{C}) \cong \mathrm{H}^2(S, \mathbb{C}) \oplus \mathbb{C}(1),$$

where $*(1)$ means shift weight of Hodge structure by (1).

Definition 3.5. A hyperKähler manifold M is a compact simply connected Kähler manifold such that $\mathrm{H}^{2,0}(M)$ is generated by a closed 2-form that is everywhere nondegenerate.

We want to show that, when S is a K3 surface, $S^{[n]}$ is a hyperKähler manifold.

It is known that a K3 surface is simply connected ([Huy16, Remark 3.6 (i), Chapter 1]). By Proposition 3.3, $S^{[n]}$ is simply connected.

Moreover, applying Proposition 3.4, we know that $\mathrm{H}^{2,0}(S^{[n]}) = \mathrm{H}^{2,0}(S)$ with a unique generator ω up to scalar. And by Proposition 3.2, ω is nondegenerated everywhere.

It remains to show that $S^{[n]}$ is Kähler.

Note that for an arbitrary algebraic K3 surface S, $S^{[n]}$ is also algebraic, so $S^{[n]}$ is Kähler.

In general, it is a fact that all K3 surfaces are deformation equivalent to each other, see [Huy16, Theorem 7.1, Chapter 7]. From construction of $S^{[n]}$, we can see that such $S^{[n]}$ are deformation equivalent to each other. Now we can deduce the conclusion by the following propostion:

Proposition 3.6. [GHJ12, Proposition 22.2] *Let X be a compact Kähler manifold and let $\mathcal{X} \to S$ be any deformation of X.*

For $t \in S$ close to $0 \in S$, the fiber is a compact Kähler manifold.

In conclusion, we have the following theorem:

Theorem 3.7. *For S a complex* K3 *surface, $S^{[n]}$ is a hyperKähler manifold of dimension 2n.*

3.2. **Generalized Kummer varieties.** Now we come back to the case A being a 2-dimensional complex torus.

Note that $A^{[n+1]}$ is symplectic and Kähler by the same reason. However, $A^{[n+1]}$ cannot give hyperKähler manifolds since $\pi_1(A^{[n+1]}) = \pi_1(A) = \mathbb{Z}^4$.

Consider the following construction:

Let $\Sigma : A^{n+1} \to A$ be the summation morphism, which is S_{n+1}-invariant. So it descends to

$$\Sigma : A^{(n+1)} \to A.$$

Composite $\pi : A^{[n+1]} \to A^{(n+1)}$, we get the following morphism:

$$s : A^{[n+1]} \to A.$$

Definition 3.8. We define the generalized Kummer varieties K^nA to be $s^{-1}(0)$ for $0 \in A$.

We claim that K^nA is a hyperKähler manifold of dimension $2n$.

We first prove it is smooth.

Note that the following diagram commutes.

$$\begin{tikzcd}
(a,\xi) \arrow[r, mapsto] & (a+\xi) \\
A \times A^{[n+1]} \arrow[r] \arrow[d, "id\times s"'] & A^{[n+1]} \arrow[d, "s"] \\
A \times A \arrow[r] & A \\
(a,b) \arrow[r, mapsto] & (n+1)a+b
\end{tikzcd}$$

So A acts on all fibers of s transitively, so all fibers are isomorphic. Note that both $A^{[n]}$ and A are smooth, so there must exist a smooth fiber, which implies that all fibers are smooth. In particular, K^nA is smooth.

Moreover, we have the following fiber bundle:

$$K^nA \to A^{[n+1]} \xrightarrow{s} A,$$

which induces the long exact sequence of homotopy groups:

$$\cdots \pi_2(A) = 0 \to \pi_1(K^nA) \to \pi_1(A^{[n]}) \xrightarrow{\cong} \pi_1(A) \to \cdots .$$

So, $\pi_1(K^nA) = 0$.

Use the same deformation technique, we know that K^nA is Kähler.

And a local computation shows that pulling back of holomorphic symplectic form on $A^{[n+1]}$ gives a holomorphic symplectic form on K^nA.

Moreover, we have the following description of $\mathrm{H}^2(K^nA, \mathbb{C})$:

Proposition 3.9. [Bea83, Proposition 8] *There is an isomorphism of Hodge structures:*

$$\mathrm{H}^2(K^nA, \mathbb{C}) \cong \mathrm{H}^2(A, \mathbb{C}) \oplus \mathbb{C}(1).$$

In conclusion, we have the following theorem:

Theorem 3.10. *For a 2-dimensional complex torus A, the generalized Kummer manifold K^nA is a hyperKähler manifold of dimension $2n$.*

In particular, K^1A is holomorphic symplectic simply connected surface. So K^1A is a K3 surface.

Then, we focus on identifying K^1A with the construction in example 1.4.

Let Δ be the diagonal of $A^{(2)}$, it is a fact that $\pi : A^{[2]} \to A^{(2)}$ is a blow-up of $A^{(2)}$ along Δ, see [Bea83, §6].

We can also embed A/ι into $A^{(2)}$ as follows (recall that ι is the natural involution given by multiplying -1).

$$\begin{aligned} A/\iota &\hookrightarrow A^{(2)} \\ \overline{a} &\mapsto \overline{(a, -a)}. \end{aligned}$$

Note that the embedding is well defined and identifies the 16 fixed points by $A/\iota \cap \Delta$.

Moreover, we have the following Cartesian diagram:

$$\begin{tikzcd}
K^1A \arrow[r] \arrow[d, dashed] \arrow[dd, bend right, two heads] & A^{[2]} \arrow[d, "\text{blow-up}"'] \arrow[dd, bend left, two heads] \\
A/\iota \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{(2)} \arrow[d, two heads] \\
0 \arrow[r] & A
\end{tikzcd}$$

So, we have a natural map $K^1A \to A/\iota$.

And the outside square is Cartesian by definition. Then we obtain that the top square is Cartesian again.

Now it remains to show $K^1A \to A$ is a blow-up at the 16-torsion points. Again, we only need to check $0 \in A/\iota$.

Let R be the function ring of A^2 around $(0, 0)$ with $(0, 0)$ corresponding to $\mathfrak{m} = (u, w, v, x)$ and $\mathfrak{p} := (u - v, w - x)$ corresponding to the diagonal locus Δ. The S_2-action permutes u, v and w, x respectively.

So R^{S_2} is the function ring of $A^{(2)}$ at around $(0, 0)$ with the maximal ideal $\mathfrak{m}^{S_2}$ corresponding to (0,0) and $\mathfrak{p}^{S_2} = ((u-v)^2, (w-x)^2, (u-v)(w-x))$ corresponding to the diagonal locus Δ. And $\mathfrak{m}^{S_2}$ is generated by $(u + v, w + x, (u - v)^2, (w - x)^2, (u - v)(w - x))$.

Then the sheaf of ring around exceptional divisor of $A^{[2]}$ over $(0,0)$ can be written as the following:

$$
\begin{aligned}
S &= R^{S_2} \oplus \mathfrak{p}^{S_2} \oplus (\mathfrak{p}^{S_2})^2 \oplus \cdots \\
\mathfrak{n} &= \mathfrak{p}^{S_2} \oplus (\mathfrak{p}^{S_2})^2 \oplus (\mathfrak{p}^{S_2})^3 \oplus \cdots .
\end{aligned}
$$

Let $I = (u+v, w+x)$,

Now, the sheaf of ring around the pre-image of (0,0) of $K^1A = 0 \times_A A^{[2]}$ is induced by:

$$\overline{S} := (R^{S_2}/I) \oplus J \oplus J^2 \oplus J^3 \oplus \cdots \tag{3.1}$$

$$\mathfrak{n} := J \oplus J^2 \oplus J^3 \oplus \cdots , \tag{3.2}$$

where $\mathfrak{m}^{S_2} = \mathfrak{p}^{S_2} + I$ and $J := \mathfrak{p}^{S_2}/(I \cdot \mathfrak{p}^{S_2}) = (\mathfrak{p}^{S_2} + I)/I$, is generated by $((u-v)^2, (w-x)^2, (u-v)(w-x))$.

However, the morphism

$$
\begin{aligned}
A/\iota &\to A^{(2)} \\
\overline{a} &\mapsto \overline{(a, -a)}
\end{aligned}
$$

corresponds to $R^{S_2} \to R^{S_2}/I$ around 0. So function ring of A/ι around 0 can be identifies as R^{S_2}/I with maximal ideal $\mathfrak{m}^{S_2}$ corresponding to $(0,0)$

Clearly, the ring in 3.1 is a blow-up of R^{S_2}/I and the conclusion follows.

In conclusion, we obtain the following result:

Proposition 3.11. *For a 2-dimensional complex torus A, the construction X in example 1.4 and K^1A are isomorphic to each other.*

Remark 3.12. Note that X in example 1.4 and K^1A are both K3 surfaces which are birational equivalent to A/ι. While a birational equivalence between K3 surfaces can be extended to an isomorphism, see [BHPVdV03, Theorem 1.1, Chapter 6].

The claim of last paragraph is related to that K3 surface S minimal. Actually, we have $\chi(C) = 1 + \frac{1}{2}(C.C + C.K) = 1 + \frac{1}{2}C.C = 0$ for arbitrary rational curve $C \subset S$. So $C.C = -2$ for any rational curve $C \subset S$. Then we apply [Har13, Theorem 5.7, Chapter 5].

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