Subject: PDE

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1 Theory

Here is the equation, representing the heat conduction forward model for a one-dimensional frozen soil temperature field, accounting for ice-water phase change:

$$C(T,x,t)\frac{\partial T(x,t)}{\partial t} + L\frac{\partial \theta(T,x)}{\partial t} = \frac{\partial}{\partial x}\left[\lambda \frac{\partial T(x,t)}{\partial x}\right]$$

Boundary and initial conditions:

1. Initial condition:

$$T(x,0) = T_0(x), \quad x \in [0,l]$$

2. Upper boundary condition:

$$\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = q_1(t), \quad T(0,t) = T_s(t)$$

3. Lower boundary condition:

$$\lambda \frac{\partial T}{\partial x}\bigg|_{x=l} = q_2(t), \quad T(l,t) = T_g(t)$$

Where: C: Volumetric heat capacity of the soil (J/(m$^3 \cdot$ K));
λ: Thermal conductivity of frozen soil (W/(m $\cdot$ K));
L: Latent heat of phase change per unit volume (3.34×10^5 KJ/m^3);
$\theta(T,x)$: Unfrozen water content, depending on temperature and spatial position;
$T_0(x)$: Initial temperature distribution;
$q_1(t)$, $q_2(t)$: Heat flux at the boundaries;
$T_s(t)$, $T_g(t)$: Temperatures at the boundaries.
The unfrozen water content $\theta(T,x)$ is expressed as:

$$\theta(T,x) = \eta(x)\phi(T,x)$$

Where: $\eta(x)$: Porosity of the soil; $\phi(T,x)$: Fraction of pore water content.

The fraction of pore water content $\phi(T,x)$ is defined as:

$$\phi(T,x) = \begin{cases} 1, & T \geq T_\nabla \\ |T_\nabla|^b |T(x,t)|^{-b}, & T < T_\nabla \end{cases}$$

Where:
b: Coefficients related to the unfrozen water content;
$T_\nabla(x)$: Freezing temperature of the soil at position x;
$T(x,t)$: Temperature.

The volumetric heat capacity C and thermal conductivity λ are expressed as a weighted combination of frozen and thawed states:

$$C = \phi C_t + (1-\phi)C_f, \quad \lambda = \lambda_t^{\phi}\lambda_f^{1-\phi}$$

Where: ϕ: Fraction of unfrozen water in the soil; C_f, C_t: Volumetric heat capacities of frozen and thawed states, respectively; λ_f, λ_t: Thermal conductivities of frozen and thawed states, respectively.

The effective thermal properties of saturated soil, including C_f, C_t, λ_f, and λ_t, are calculated as weighted averages of the components (soil particles, ice, and water):

$$C_f = (1-\eta)C_s + \eta C_i, \quad C_t = (1-\eta)C_s + \eta C_l$$

$$\lambda_f = \lambda_s^{1-\eta}\lambda_i^{\eta}, \quad \lambda_t = \lambda_s^{1-\eta}\lambda_l^{\eta}$$

Where: η: Porosity of the soil;
C_s, λ_s: Volumetric heat capacity and thermal conductivity of soil particles;
C_i, λ_i: Volumetric heat capacity and thermal conductivity of ice;
C_l, λ_l: Volumetric heat capacity and thermal conductivity of water.

The relationship between the thermal properties in the thawed and frozen states is expressed as:

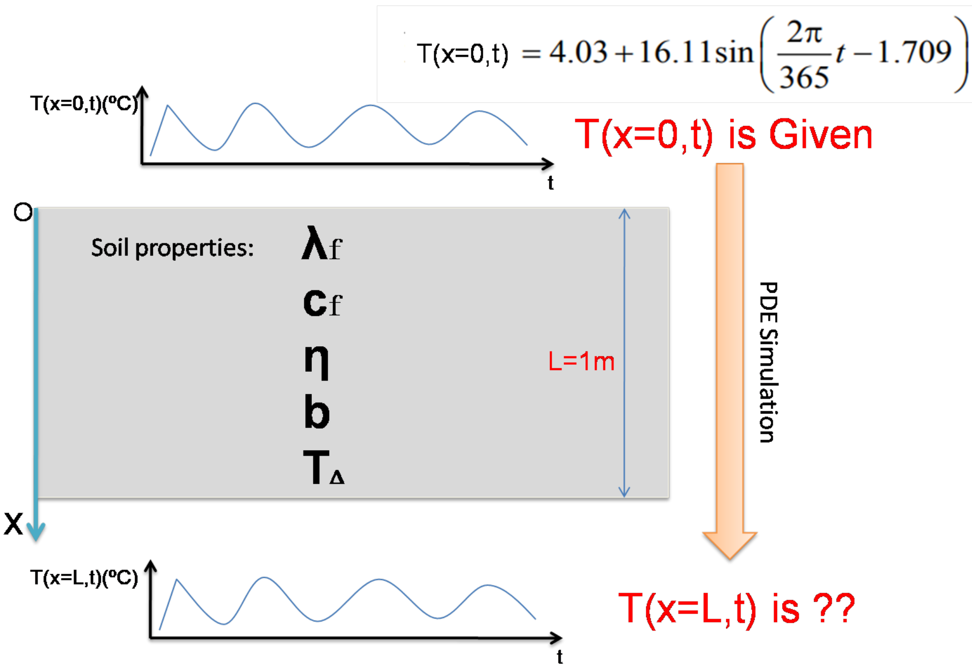
$$C_t = C_f + \eta(C_l - C_i), \quad \lambda_t = \lambda_f(\frac{\lambda_l}{\lambda_i})^{\eta}$$

Where:
C_t, λ_t: Volumetric heat capacity and thermal conductivity in the thawed state;
C_f, λ_f: Volumetric heat capacity and thermal conductivity in the frozen state;
ϕ: Fraction of unfrozen water in the soil;
C_l, λ_l: Volumetric heat capacity and thermal conductivity of water;
C_i, λ_i: Volumetric heat capacity and thermal conductivity of ice.

Parameter	**Value Range or Specific Value**	**Unit**
C_i (Heat capacity of ice)	1.672	KJ/(m$^3 \cdot$ K)
C_l (Heat capacity of water)	4.18	KJ/(m$^3 \cdot$ K)
λ_i (Thermal conductivity of ice)	$2.210 \sim 2.326$	W/(m $\cdot$ K)
λ_l (Thermal conductivity of water)	$0.465 \sim 0.582$	W/(m $\cdot$ K)
L (Latent heat of phase change per unit volume)	3.34×10^5	KJ/m^3

Table 1: Values of thermal properties of ice and water.

There are 5 soil thermal properties: (λ_f, C_f, η, b, T_∇)



Numerical setup:

λ_f /(W/(m·K))	C_f /(kJ/(m^3·K))	η	b	T_∇ /℃
2.0	1 600	0.09	0.70	−0.03

Figure 1: Schematic diagram