Vector Calculus

§0 Coördinate Systems

§0.1 Converting

§0.1.1 Polar coördinates

A point P in Cartesian coördinates (x, y) has polar coördinates (r, θ)

$$r = (x^2 + y^2)^{\frac{1}{2}} \qquad x = r\cos\theta$$
$$\theta = \arctan\left(\frac{1}{2}\right) \qquad y = r\sin\theta$$

The unit vectors in polar coördinates are converted to and from as such

$$\hat{\mathbf{r}} = \cos\theta\ \hat{\mathbf{i}} + \sin\theta\ \hat{\mathbf{j}} \qquad \hat{\mathbf{i}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}} \qquad \hat{\mathbf{j}} = \sin\theta\ \hat{\mathbf{r}} + \cos\theta\ \hat{\boldsymbol{\theta}}$$

The same applies to their components, for some arbitrary vector $\vec{\mathrm{v}}$

$$v_r = v_x\cos\theta + v_y\sin\theta \qquad v_x = v_r\cos\theta - v_\theta\sin\theta$$
$$v_\theta = -v_x\sin\theta + v_y\cos\theta \qquad v_y = v_r\sin\theta + v_\theta\cos\theta$$

§1 Integral Vector Calculus

§1.1 Line integrals

A line of path integral is the integral of a scalar or vector field along a specified path C from point A to point B, there are three general forms.

$$\vec{\mathrm{I}} = \int_C \mathrm{F}\vec{\mathrm{dl}} \qquad \vec{\mathrm{I}} = \int_C \vec{\mathrm{F}} \cdot \vec{\mathrm{dl}} \qquad \vec{\mathrm{I}} = \int_C \vec{\mathrm{F}} \times \vec{\mathrm{dl}}$$

Where $\vec{\mathrm{dl}}$ is the infitesimal line element, remember that it is different for each coördinate system. The direction of the path matters, if the path is reversed, as in from B to A, then $\vec{\mathrm{dl}}$ becomes $-\vec{\mathrm{dl}}$

If A and B are the same point then the integral is *closed*, and as such is written $\oint$. By convention the path taken for a closed integral is anti-clockwise (widdershins).

Example 1.1. Calculate the line integral $\int_C \vec{\mathrm{A}}\cdot\vec{\mathrm{dl}}$ where $\vec{\mathrm{A}} = (x+y)\,\hat{\mathbf{i}} + (y-x)\,\hat{\mathbf{j}}$ along the path $y = x^2$ from $(1,1)$ to $(2,4)$

The first step is to evaluate the dot product inside the integral. It is clear we should be using Cartesian coördinates and so $\vec{\mathrm{dl}} = \mathrm{d}x\ \hat{\mathbf{i}} + \mathrm{d}x\ \hat{\mathbf{j}} + \mathrm{d}z\ \hat{\mathbf{k}}$

$$\vec{\mathrm{A}} \cdot \vec{\mathrm{dl}} = (x+y)\mathrm{d}x + (y-x)\mathrm{d}y$$

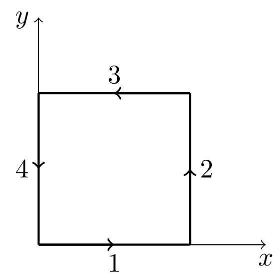
x and y are not independent along the path, i.e. a change in x produces a change in y. Whilst not always possible, it is much easier if we contract our integral down to one variable.

$$\begin{aligned} y &= x^2 \\ \frac{dy}{dx} &= 2x \\ \vec{\mathrm{A}} \cdot \vec{\mathrm{dl}} &= (x+x^2)\ \mathrm{d}x + (x^2-x)2x\ \mathrm{d}x \\ \vec{\mathrm{A}} \cdot \vec{\mathrm{dl}} &= 2x^3 - x^2 + x\ \mathrm{d}x \\ \int_C \vec{\mathrm{A}} \cdot \vec{\mathrm{dl}} &= \int_C 2x^3 - x^2 + x\ \mathrm{d}x \end{aligned}$$

Now we need to determine our limits, we are integrating with respect to x, and our path is from $(1,1)$ to $(2,4)$ to our limits are from 1 to 2. The order is very important so make sure to double check.

$$\begin{aligned} \int_1^2 2x^3 - x^2 + x\ \mathrm{d}x &= \left[\frac{2x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\right]_1^2 \\ &= \frac{22}{3} - \frac{2}{3} = \frac{20}{3} \end{aligned}$$

Example 1.2. Calculate the line integral $\oint_C \vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}}$ where $\vec{\mathrm{F}} = x^2 y\ \hat{\mathbf{i}} + 3xy\ \hat{\mathbf{j}}$ around a unit square with vertices at $(0,0),\ (1,0),\ (1,1),\ (0,1)$.



For this problem we need to break it down into pieces, namely each side. It does not matter which order we evaluate our sides, only that we evaluate them anti-clockwise, i.e. from (0,0) to (0,1).

Side 1: (0,0) to (0,1), $y = 0,\ x : 0 \to 1,\ \vec{\mathrm{F}} = 0\ \hat{\mathbf{i}} + 0\ \hat{\mathbf{j}}$ so $\int \vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}} = 0$

Side 2: (0,1) to (1,1), $x = 0,\ y : 0 \to 1,\ \vec{\mathrm{F}} = y\ \hat{\mathbf{i}} + 3y\ \hat{\mathbf{j}}$
Now we need to find $\mathrm{d}\vec{\mathrm{l}}$, since $\vec{\mathrm{F}}$ only points in the $\hat{\mathbf{j}}$ direction on this side we can say $\mathrm{d}\vec{\mathrm{l}} = \mathrm{d}y\ \hat{\mathbf{j}}$.

$$
\begin{aligned}
\vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}} &= 3y\ \mathrm{d}y \\
\int_0^1 \vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}} &= \int_0^1 3y\ \mathrm{d}y \\
&= \left[\frac{3y^2}{2}\right]_0^1 = \frac{3}{2}
\end{aligned}
$$

Side 3: (1,1) to (0,1), $y = 0,\ x : 1 \to 0,\ \vec{\mathrm{F}} = x^2\ \hat{\mathbf{i}} + 3x\ \hat{\mathbf{j}}$
$\mathrm{d}\vec{\mathrm{l}}$ only moves in the x direction so $\mathrm{d}\vec{\mathrm{l}} = \mathrm{d}x\ \hat{\mathbf{i}}$.

$$
\begin{aligned}
\vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}} &= x^2 \mathrm{d}x \\
\int_1^0 x^2 \mathrm{d}x &= \left[\frac{x^3}{3}\right]_1^0 = \frac{-1}{3}
\end{aligned}
$$

Ensure the limits are in the right order!!

Side 4: (0,1) to (0,0), $x = 0,\ y : 1 \to 0,\ \vec{\mathrm{F}} = 0\ \hat{\mathbf{i}} + 0\ \hat{\mathbf{j}}$ so $\int \vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}} = 0$

So $\oint \vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{l}} = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}$