The Quantum World

§1 Fourier Analysis

5/10/20 — Week 2

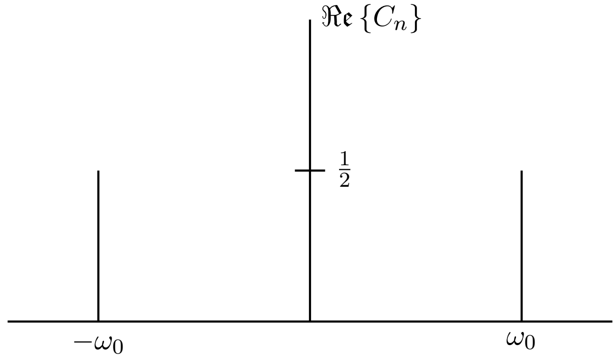
§1.1 Fourier Series

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 t}$$

Where ω_0 is the fundamental frequency, and $n\omega_0$ is the nth harmonic. The superposition of these waves gives us our final function. Do not panic about negative frequencies, it will all become clear later on.

$$\begin{aligned}
f(t) &= \cos(\omega_0 t) \\
&= \frac{1}{2}e^{i\omega_0 t} + \quad \frac{1}{2}e^{-i\omega_0 t} \\
n &= 1 \qquad\quad n = -1 \\
C_1 &= \frac{1}{2} \qquad C_{-1} = \frac{1}{2} \\
f(t) &= \frac{1}{2}e^{i\omega_0 t} + \quad \frac{1}{2}e^{-i\omega_0 t}
\end{aligned}$$

Plotting this as a fourier spectrum. We have ommited the imaginary part (because there isn't one), however if there were we would simply plot the modulus squared.



But how do we find C_n for a general case?

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-in\omega_0 t}\ \mathrm{d}t$$

Where T is the period. Just like frequency is the reciprocal of time, we can have reciprocal space. We have a *spacial frequency*, k, that is related to the inverse of the period of a wave in space (its wavelength). We call this spacial frequency the *wavenumber*. It is more the spacial equivalent of angular frequency.

$$k = \frac{2\pi}{\lambda}$$

From the de Brogile relationship it is obvious that the momentum of a quantum particle is inversely related to its wavelength

$$p = \frac{h}{\lambda}$$
$$p = \hbar k$$

We can see how momentum and position are inversely related (k is the inverse of position.)

§1.2 FOURIER TRANSFORMS Fourier series are only really useful for periodic functions. An aperiodic function is very useful for things like wavepackets and 1D particles in boxes which do have a repeating pattern along the x-axis.

We will have to use a Fourier transform, unlike a Fourier series, which is a discrete set of coefficients, c_n, for a discrete set of frequencies, a Fourier transform is a continuous function. As we can see below when we increase the seperation between the pulses, $T \to \infty$, more frequency components appear on the Fourier spectrum. This means that the separation between the components $\Delta\omega$ gets smaller until it approaches dω. This arises from the reciprocal nature of time and frequency, as the period increases it should be expected that frequency separtion is to reduce. The same is true of position and wavenumber.

Wide in time, narrow in frequency
Wide in position space, narrow in momentum space

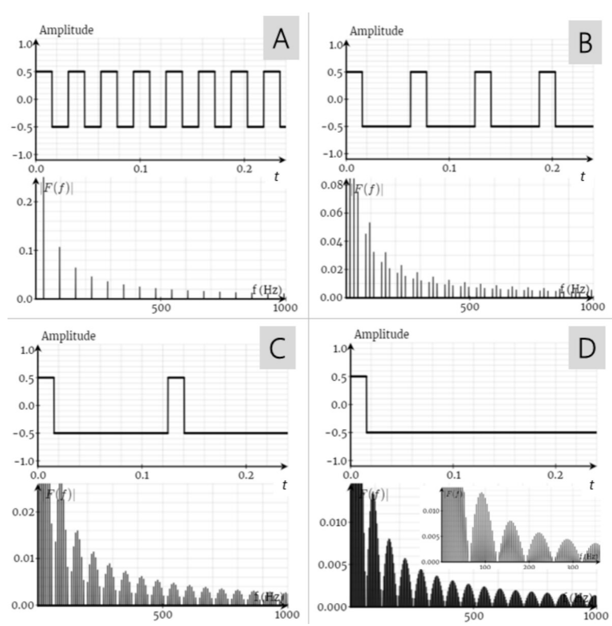


Figure 1: Increasing of T results in lowering of f separation

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$
$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-ikt}dt$$

As $T \to \infty$ the frequency separation dω becomes infitesimally small, so the Fourier transformation is the *integral above.*