Maths Semester II

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1 First Order Differential Equations(ODEs)

1.1 Separable ODEs

A separable ODE is one in which the functions of both x and y can be separated.

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$
$$\frac{dy}{dx} f(y) = g(x)$$

This can then be rearranged and integrated so that

$$\int f(y)dy = \int g(x)dx$$

Example

$$\begin{aligned} \frac{dy}{dx} &= e^{x+y} = e^x e^y \\ \int e^{-y} dy &= \int e^x dx \\ -e^{-y} &= e^x + c \\ e^x + e^{-y} &= c \end{aligned}$$

c is just a constant so its sign does not matter at this point.
We can then solve explicitly for y.

$$\begin{aligned} e^{-y} &= c - e^x \\ y &= -\ln(c - e^x) \end{aligned}$$

1.2 Boundary and Initial Conditions

The general solution to a 1st Order DE always contains one undefined constant of integration, like c. A boundary (for x) or initial (for t) condition is given, typcially y(x) at a given x value.

Example: T goes from 90°c to 70°c in 10 minutes(t), the room temperature is 20°c(T_0), find T after 20 minutes.

$$\begin{aligned}
\frac{dT}{dt} &= -\alpha(T - T_0) \\
\int \frac{dT}{T - T_0} &= -\alpha \int dt \\
\ln(T - T_0) &= -\alpha t + c \\
T - T_0 &= e^{-\alpha t + c} = Ae^{-\alpha t} \\
A &= e^c
\end{aligned}$$

We can now use our initial conditions to solve for T(20)

$$\begin{aligned}
&\text{When } t = 0 \\
T - T_0 &= 90 - 20 = 70 \\
A &= 70 \\
&\text{When } t = 10 \\
T &= T_0 + Ae^{-\alpha t} \\
70 &= 20 + 70e^{-10\alpha} \\
e^{-10\alpha} &= \frac{70 - 20}{70} = \frac{5}{7} \\
&\text{When } t = 20 \\
T &= 20 + 70e^{-20\alpha} \\
(e^{-10\alpha})^2 &= e^{-20\alpha} \\
T &= 20 + 70\left(\frac{5}{7}\right)^2 = 55.7^\circ c
\end{aligned}$$

1.3 Homoegeneous ODEs

Replace y with yt and x with xt, if all the ts cancel then the ODE is homogeneous.

$$\begin{aligned}
\frac{dy}{dx} &= f\left(\frac{y}{x}\right) \\
\text{let } v &= \frac{y}{x} \text{ so } y = vx \\
\frac{dy}{dx} &= \frac{d}{dx}vx
\end{aligned}$$

v is a function of x so we must use the product rule.

$$\begin{aligned}
\frac{dy}{dx} &= v + x\frac{dv}{dx} = f(v) \\
\frac{dv}{dx} &= \frac{f(v) - v}{x}
\end{aligned}$$

This is now a separable function, and can be solved as before.

Example: $y(1) = 1$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x^2}(x - y) = \frac{y}{x} - \left(\frac{y}{x}\right)^2 \\
\text{let } v &= \frac{y}{x} \text{ so } y = vx \\
\frac{dy}{dx} &= v + x\frac{dv}{dx} = v - v^2 \qquad \text{separable} \\
\int \frac{-dv}{v^2} &= \int \frac{dx}{x} \\
\frac{1}{v} &= \ln x + c = \frac{x}{y} \\
y &= \frac{x}{\ln x + c} \\
y(1) &= 1 \\
1 &= \frac{1}{\ln(1) + c} \\
c &= 1
\end{aligned}$$

1.4 Linear ODEs

Linear ODEs always take the form of

$$\frac{dy}{dx} + y \cdot P(x) = Q(x)$$

When $Q(x) = 0$ this becomes a separable ODE and can be solved as before such that

$$\begin{aligned} dy \cdot \frac{1}{y} &= -P(x)dx \\ y &= Ae^{\int -P(x)dx} \quad \text{where } A = e^c \end{aligned}$$

When $P(x) = 0$ this becomes a simple separable ODE and can be solved by integration.

Recall the product rule $\frac{d}{dx}(u \cdot v) = u'v + uv'$
This is strikingly close to our Linear ODE, except we are missing a factor, the integration factor, $\mu(x)$. We want μ to be such that $\mu' = P(x)\mu$, allow us to do some algebraic manipulation.

$$\begin{aligned} \mu' &= P(x)\mu \\ P(x) &= \frac{\mu'}{\mu} = \frac{d}{dx}\ln(\mu) \\ \int P(x)dx &= \ln(\mu) \\ \mu &= e^{\int P(x)dx} \end{aligned}$$

We can then multiply by our integrating factor, then apply the chain rule, however this is best shown with an example.

$$\begin{aligned} \cos(x)\frac{dy}{dx} + \sin(x)y &= 1 \\ \frac{dy}{dx} + \tan(x)y &= \sec(x) \quad \text{Linear ODE} \\ \mu &= e^{\int \tan(x)dx} \\ \int \tan(x)dx &= \ln(sec(x)) \\ \mu &= \sec(x) \quad \text{Multiply ODE by } \mu \\ \frac{dy}{dx}\sec(x) + \sec(x)\tan(x)y &= \sec^2(x) \end{aligned}$$

Since $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$ we should now be able to see how the LHS of our equation can be withdrawn into its chain rule form.

$$\begin{aligned}
\frac{d}{dx}\sec(x) \cdot y &= \frac{dy}{dx}\sec(x) + \sec(x)\tan(x)y \\
\frac{d}{dx}\sec(x) \cdot y &= sec^2(x) \\
\sec(x) \cdot y &= \int \sec^2(x) = \tan(x) + c \\
y &= \sin(x) + c \cdot \cos(x)
\end{aligned}$$

1.5 Exact ODEs

When a variable in a multi variable function $f(x, y)$ has variables that rely on another e.g. $y = y(t), x = x(t)$ then the derivative of that function is known as the total derivative and is given as

$$\frac{d}{dt}f(x, y) = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$$

This can also occur for when one variable is the function of another e.g. $x = x, y = y(x)$, the total derivative of such is

$$\frac{d}{dx}f(x, y) = \frac{\delta f}{\delta x} + \frac{\delta f}{\delta x}\frac{dy}{dx} \tag{1}$$

Exact ODEs take the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Where M and N must be separated by a plus. As you can see, this is in the same form as a the derivative of the function from equation (1), thus we can assume that

$$\frac{\delta f}{\delta x} = M \quad \frac{\delta f}{\delta x} = N$$

In order to verify this assumption we can test an 'exactness condition'

$$\begin{aligned}
\frac{\delta^2 f}{\delta x \delta y} &= \frac{\delta^2 f}{\delta y \delta x} \\
\frac{\delta M}{\delta x} &= \frac{\delta N}{\delta y}
\end{aligned}$$

We must now find a function to satisfy these conditions, however this is best left to an example.

$$
\begin{aligned}
x + y^2 + 2xy\frac{dy}{dx} &= 0 \\
M = x + y^2 \quad N &= 2xy \\
\frac{\delta M}{\delta y} &= 2y \\
\frac{\delta N}{\delta x} &= 2y
\end{aligned}
$$

This satisfies the 'exactness condition', and so we can begin to find the function that solves the ODE.

$$
\begin{aligned}
M &= \frac{\delta f}{\delta x} \\
\int M dx = f &= \frac{1}{2}x^2 + xy^2 + g(y) \\
N &= \frac{\delta f}{\delta y} \\
\int N dy = f &= xy^2 + h(x)
\end{aligned}
$$

For $\int M dx = \int N dy$ to be true their constants must satisfy each other, thus $g(y) = 0, h(x) = \frac{1}{2}x^2$

$$
f(x, y) = \frac{1}{2}x^2 + xy^2 + c = 0
$$

2 Second Order ODEs

2.1 Introduction

They take the form of

$$m(x)\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$
$$r(x) = 0 \quad \text{Homogeneous}$$
$$r(x) \neq 0 \quad \text{Inhomogeneous}$$

The general solution is

$$y(x) = Ay_1(x) + By_2(x)$$

Where y_1 & y_2 are two functions of x and A & B are integration constants. We normally assume we have constant coefficients, as in driven oscillators.

$$m\frac{d^2y}{dx^2} + b\frac{dy}{dx} + ky = F_0\cos(\omega t)$$

2.2 Homogeneous

Consider the second order homogeneous ODE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

We normally take an educated guess at a solution, e.g. $y = e^{mx}$, where m is some constant. Thus

$$\frac{dy}{dx} = me^{mx} \qquad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Subbing this back into our original homogeneous equation and then dividing by e^{mx} gives us

$$(m^2 + am + b) = 0 \quad \text{axuillary equation}$$
$$(m - m_1)(m - m_2) = 0 \quad m_1 \text{ \& } m_2 \text{ are roots}$$

There are three cases to consider, when m_1 & m_2 are

a. distinct and real

b. distance and complex

c. the same

2.2.1 Case a.

Where $m_1 \neq m_2$ and both m_1 & m_2 are real. From our educated guess we know

$$y_1(x) = e^{m_1 x} \qquad y_2(x) = e^{m_2 x}$$

So the general solution is

$$y(x) = Ae^{m_1 x} + Be^{m_2 x}$$

Example:

$$\begin{aligned}
3\frac{d^2y}{dx^2} + -5\frac{dy}{dx} + -2y &= 0 \\
3m^2 - 5m - 2 &= 0 \quad \text{auxillary equation} \\
(3m+1)(m-2) &= 0 \\
m_1 &= \frac{-1}{3} \\
m_2 &= 2 \\
y(x) &= Ae^{-x/3} + Be^{2x} \quad \text{general solution}
\end{aligned}$$

2.2.2 Case b.

Where $m_1 \neq m_2$ and both m_1 & m_2 are complex. We know that

$$y_1(x) = e^{m_1 x} \qquad y_2(x) = e^{m_2 x}$$

But now y_1 & y_2 are complex. This means we can rewrite the general solution using Euler's Formula.

$$\begin{aligned}
\text{let} \quad m_1 &= \mu + i\lambda \\
\text{let} \quad m_2 &= \mu - i\lambda
\end{aligned}$$

So the general solution is

$$\begin{aligned}
y(x) = Ae^{m_1 x} + Be^{m_2 x} &= Ae^{(\mu + i\lambda)x} + Be^{(\mu - i\lambda)x} \\
&= e^{\mu x}[Ae^{i\lambda x} + Be^{-i\lambda x}] \\
&= e^{\mu x}[A(\cos(\lambda x) + i\sin(\lambda x)) + B(\cos(\lambda x) - i\sin(\lambda x))] \\
&= e^{\mu x}[(A+B)(\cos(\lambda x) + (A-B)i\sin(\lambda x)) \\
&= e^{\mu x}[(C)(\cos(\lambda x) + (D)\sin(\lambda x))
\end{aligned}$$

The final general solution is

$$e^{\mu x}[(C)(\cos(\lambda x) + (D)\sin(\lambda x))$$

Where $C = A + B$ & $D = i(A - B)$ are both integration constants.

2.2.3 Case c.

Where $m_1 = m_2$. Thus

$$y_1 = e^{m_1 x}$$

So we need another solution. We will prove this in a later section.

$$\begin{aligned} y_2(x) &= xe^{m_1 x} \\ y(x) &= Ay_1(x) + By_2(x) \quad \text{general solution} \\ y(x) &= (A + Bx)e^{m_1 x} \end{aligned}$$

Example:

$$\begin{aligned} &\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \\ m^2 + 2m + 1 &= (m+1)^2 = 0 \quad \text{auxillary equation} \\ m &= -1 \\ y_1(x) &= e^{-x} \\ y_2(x) &= xe^{-x} \\ y(x) &= (A + Bx)e^{-x} \\ &\text{general solution} \end{aligned}$$

We can check that y_2 is a solution.

$$\begin{aligned} \frac{dy_2}{dx} &= e^{-x} - xe^{-x} = (1-x)e^{-x} \\ \frac{d^2y_2}{dx^2} &= -e^{-x} - e^{-x} + xe^{-x} = (x-2)e^{-x} \\ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 & \\ \text{original ODE} & \\ [(x-2) + 2(1-x) + x]e^{-x} = 0 & \\ 0 = 0 & \end{aligned}$$