Vector Calculus Coursework 2

December 7, 2020

Question 1

Part a

If the closed path (C) does not contain a singularity then the line integral around a closed path for a conservative field $\left(\vec{A}\right)$ is zero.

$$\oint_C \vec{A} \cdot \mathrm{d}\vec{l} = 0$$

Part b

$$\vec{\nabla} F = \frac{\partial F}{\partial x}\ \hat{\mathbf{i}} + \frac{\partial F}{\partial y}\ \hat{\mathbf{j}} + \frac{\partial F}{\partial z}\ \hat{\mathbf{k}}$$

$$\vec{\nabla} \times \left(\vec{\nabla} F\right) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix}$$

$$= \ \hat{\mathbf{i}} \left(\frac{\partial^2 F}{\partial y \partial z} - \frac{\partial^2 F}{\partial z \partial y}\right) + \hat{\mathbf{j}} \left(\frac{\partial^2 F}{\partial z \partial x} - \frac{\partial^2 F}{\partial x \partial z}\right) + \ \hat{\mathbf{k}} \left(\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial x}\right)$$

For any function $\quad \dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial^2 F}{\partial y \partial x}$

$$\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial x} = 0$$

$$\therefore \vec{\nabla} \times \left(\vec{\nabla} F\right) = \ \hat{\mathbf{i}}(0) + \ \hat{\mathbf{j}}(0) + \ \hat{\mathbf{k}}(0) = 0$$

Part c

The line integral of $\vec{\nabla}F$ around a closed path (C) is zero, this is the same as our conservative field, $\vec{A}$.

$$\oint_C \vec{\nabla}F \cdot \mathrm{d}\vec{l} = \oint_C \vec{A} \cdot \mathrm{d}\vec{l}$$

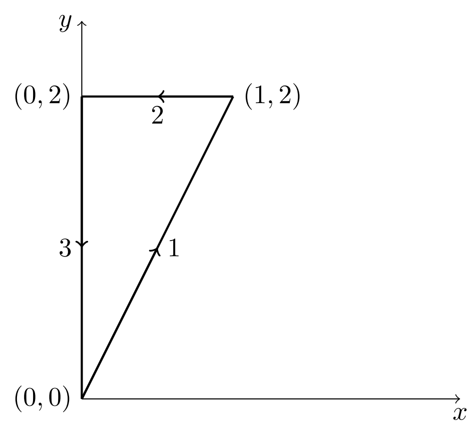
Therefore we can write our conservative field as a the gradient of a scalar field $\vec{\nabla}F = \vec{A}$. We previously showed that for any $\vec{\nabla}F$ the curl is zero therefore

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{\nabla}F\right) = 0$$

Therefore a conservative field must always have zero curl.

Question 2

Part a



$$\vec{A} = xy\ \hat{\mathbf{i}} + \ \hat{\mathbf{j}}$$
$$\mathrm{d}\vec{l} = \mathrm{d}x\ \hat{\mathbf{i}} + \mathrm{d}y\ \hat{\mathbf{j}}$$
$$\vec{A} \cdot \mathrm{d}\vec{l} = xydx + dy$$

Side 1

We can represent this side as a path of equation $y = 2x$, where the bounds for such a path are $x : 0 \to 1,\ y : 0 \to 2$

$$\int \vec{A} \cdot \mathrm{d}\vec{l} = \int_0^1 xy\mathrm{d}x + \int_0^2 \mathrm{d}y$$
$$y = 2x \therefore xy = 2x^2$$
$$\int_0^1 2x^2\mathrm{d}x + \int_0^2 \mathrm{d}y = \left[\frac{2}{3}x^3\right]_0^1 + 2$$
$$= \frac{2}{3} + 2 = \frac{8}{3}$$

Side 2

For this side $y = 2$ for the entire length, so we can make that substitution.

$$\int \vec{A} \cdot \mathrm{d}\vec{l} = \int_1^0 2x\mathrm{d}x + \int_2^2 \mathrm{d}y$$
$$= \left[x^2\right]_1^0 = -1$$

Side 3

$x = 0$ for this side, so this simplifies the integral a lot

$$\int \vec{A} \cdot \mathrm{d}\vec{l} = \int_2^0 \mathrm{d}y = -2$$

Summing all of the sides will gives us the final answer for our path integral

$$\oint \vec{A} \cdot \mathrm{d}\vec{l} = \frac{8}{3} - 1 - 2 = \frac{-1}{3}$$

Part b

$$
\begin{aligned}
\vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 1 & 0 \end{vmatrix} \\
&= \ \hat{\mathbf{k}}(\frac{\partial}{\partial x}1 - \frac{\partial}{\partial y}xy) \\
&= -\ \hat{\mathbf{k}}x
\end{aligned}
$$

Part c

$\vec{A}$ is non-conservative, neither its cross product nor its closed path integral are zero.

Question 3

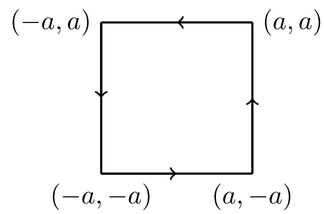
Part a

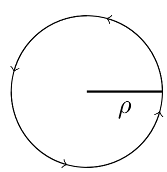
The magnetic field is symmetrical up and down in the z direction, so z does not matter. It is directed in *circles* (meaning it is pointed in the $\hat{\boldsymbol{\phi}}$ direction) around the wire, implying that for a certain magnitude (B) it is at a fixed radius (ρ). Given all of this we can state that $\vec{B}(\rho)\ \hat{\boldsymbol{\phi}}$, $\vec{B}$ is a function of ρ pointed in the $\hat{\boldsymbol{\phi}}$ direction.

$$
\begin{aligned}
\mathrm{d}\vec{l} &= \mathrm{d}\rho\ \hat{\boldsymbol{\rho}} + \rho\mathrm{d}\phi\ \hat{\boldsymbol{\phi}} + \mathrm{d}z\ \hat{\mathbf{z}} \\
\oint_C \vec{B} \cdot \mathrm{d}\vec{l} &= \int_0^{2\pi} B \cdot \rho\, \mathrm{d}\phi \\
&= 2\pi\rho B = \mu_0 I \\
B &= \frac{\mu_0 I}{2\pi\rho} \\
\text{We stated previously that} \quad \vec{B} &= B\ \hat{\boldsymbol{\phi}} \\
\vec{B} &= \frac{\mu_0 I}{2\pi\rho}\ \hat{\boldsymbol{\phi}}
\end{aligned}
$$

Part b

$$
\begin{aligned}
\rho &= \sqrt{x^2+y^2} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\ \hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}} \\
\vec{B} &= \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\left(-\sin\phi\ \hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}}\right) \\
\sin\psi &= \frac{y}{\rho} \\
\cos\psi &= \frac{x}{\rho} \\
\vec{B} &= \frac{(-y+x)\mu_0 I}{2\pi(x^2+y^2)} \\
\vec{B}\cdot \mathrm{d}\vec{l} &= \frac{-y\mu_0 I}{2\pi(x^2+y^2)}\mathrm{d}x + \frac{x\mu_0 I}{2\pi(x^2+y^2)}\mathrm{d}y \\
\text{let}\quad k &= \frac{\mu_0 I}{2\pi}
\end{aligned}
$$





Let's consider a side going from $(a,-a)$ to (a,a), this side has a constant x value, which means that the integral with respect to x is 0.

$$
\begin{aligned}
\int \vec{B}\cdot \mathrm{d}\vec{l} &= k\int_{-a}^{a}\frac{x}{x^2+y^2}\mathrm{d}y \\
x &= a \\
\int \vec{B}\cdot \mathrm{d}\vec{l} &= k\int_{-a}^{a}\frac{a}{a^2+y^2}\mathrm{d}y \\
\text{let}\quad y &= au \\
\int \vec{B}\cdot \mathrm{d}\vec{l} &= k\int_{-a}^{a}\frac{a}{a^2(1+u^2)}\mathrm{d}y \\
\frac{\mathrm{d}y}{\mathrm{d}u} &= a \\
\int \vec{B}\cdot \mathrm{d}\vec{l} &= k\int_{-a}^{a}\frac{a^2}{a^2(1+u^2)}\mathrm{d}u \\
&= k\int_{-a}^{a}\frac{1}{(1+u^2)}\mathrm{d}u \\
\text{using integral identity}\qquad &= k\left[\arctan\left(\frac{y}{a}\right)\right]_{-a}^{a} \\
&= k(\arctan(1)-\arctan(-1)) = k\frac{1}{2}\pi
\end{aligned}
$$

Now if we consider the side opposite to this one, going from $(-a,a)$ to $(-a,-a)$ we can see that x is still a constant but this time it's equal to $-a$, this is convinient since the bounds for this integral are reversed. It is known for any integral that

$$\int_a^b \mathrm{d}x = -\int_b^a \mathrm{d}x$$

Thus we can say that these sides have an equal line integral, $k\frac{1}{2}\pi$.

Now let's consider the top side, from (a,a) to $(-a,a)$, for this side $y=a$, and so any integral with respect to y is just 0.

$$\begin{aligned}
\int \vec{B}\cdot \mathrm{d}\vec{l} &= k\int_{a}^{-a} \frac{-y}{x^2+y^2}\mathrm{d}x \\
\text{let} \quad x &= au \\
\int \vec{B}\cdot \mathrm{d}\vec{l} &= k\int_{a}^{-a} \frac{-1}{(1+u^2)}\mathrm{d}u \\
&= k\int_{-a}^{a} \frac{1}{(1+u^2)}\mathrm{d}u \\
&= k\left[\arctan\left(\frac{x}{a}\right)\right]_{-a}^{a} \\
&= k(\arctan(1)-\arctan(-1)) = k\frac{1}{2}\pi
\end{aligned}$$

Using the same logic, the opposite side is the same except the bounds are reversed ($-a \to a$), and $y = -a$, these both "cancel" each other to give the same answer to the integral as previously. Summing all of these sides together gives us an answer of

$$\oint_C \vec{B}\cdot \mathrm{d}\vec{l} = k2\pi = \frac{\mu_0 I 2\pi}{2\pi} = \mu_0 I$$

Hence, this obeys Ampère's Law.

Part c

A field is conservative if $\vec{\nabla}\times\vec{B} = 0$.

$$\vec{\nabla}\times\vec{B} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \frac{\mu_0 I}{2\pi} & 0 \end{vmatrix} = 0$$

$\frac{\mu_0 I}{2\pi}$ is a constant (for a certain I) and so any derivative of it will always be zero, thus the field is conservative outside of the wire.