Exercise 1

The identity $(1+x)\sum_{k=0}^{n}(-x)^n = 1 \pm x^{n+1}$ gives an inverse for n large enough. If a is a unit, $a^{-1}x$ is still nilpotent. We just showed that $1 + a^{-1}x$ must be a unit, and it follows $a + x$ is a unit.

Exercise 2

i) Suppose $f = a_0 + a_1x + \ldots + a_nx^n$ is invertible and nonconstant and let $g = b_0 + b_1x + \ldots + b_mx^m$ be an inverse of f, i.e $fg = 1$. That means in particular $f(0)g(0) = a_0g(0) = 1$ and a_0 is a unit. We clearly have $a_nb_m = 0$. We also have $a_nb_{m-1} + a_{n-1}b_m = 0$ so $a_n^2b_{m-1} + a_{n-1}a_nb_m = a_n^2b_{m-1} = 0$. Iterating this process we get $a_n^{r+1}b_{m-r} = 0$. Setting for $r = m$ gives $a_n^{m+1} = 0$ as b_0 is a unit.

Exercise 1 gives the other implication.

ii) If all the a_i are nilpotent, f is a sum of nilpotents, so it is itself nilpotent.

In the other direction, since f is nilpotent, $\frac{f(x)-f(0)}{x}$ is also nilpotent, because sum of nilpotents is nilpotent and x is regular in $A[x]$. Iterating, we get the result.

iii) Pick g of minimal degree such that $fg = 0$. Since $a_nb_m = 0$, $\deg a_ng < \deg g$ and $a_ngf = 0$ so $a_ng = 0$ by minimality. It follows that $f - a_nx^n$ is still annihilated by g, and thus $a_{n-1}b_m = 0$, and then by the same argument $a_{n-1}g = 0$. Iterating, we get $a_kg = 0$ for $0 \leq k \leq n$. This means that $b_m \cdot f = 0$.

iv) Let's denote by (f) the ideal generated by the coefficients of f, by abuse of notation. We clearly have $(fg) \subset (f)$ and $(fg) \subset (g)$, which implies that if fg is primitive, f and g are. Let's now suppose f, g are primitive and fg is not, and let I be a prime ideal containing (fg). Clearly, f and g are not zero mod I, otherwise 1 would be zero mod I, but fg is, contradicting that I is prime.

Exercise 3

Induction (too lazy to do it properly)

Exercise 4

Let f be in the Jacobson radical. We will show f is nilpotent. For any $g \in A[X]$, $1 - gf$ is a unit. Setting $g = x$, we get that $1 - xf$ is a unit. By exercise 2, this mean all of the coefficients of f are nilpotent, that is, f is nilpotent.

Exercise 5

i) If f is a unit, with inverse g, $f(0)g(0) = 1$ and a_0 is a unit.

In the other direction, assuming a_0 is a unit we construct the coefficients (b_n) of an inverse $g = \sum_n b_n x^n$ inductively. Set $b_0 = \frac{1}{a_0}$. Assuming $b_0, \dots, b_n$ are already defined, let $b_{n+1} = -a_0^{-1}(b_n a_1 + \ldots + b_0 a_{n+1})$. It is now routine to check that $fg = 1$

ii) We argue similarly to exercise 2. If f is nilpotent, $f(0)$ is nilpotent, and then $f - f(0)$ is nilpotent and finally $\frac{f-f(0)}{x}$ is nilpotent.

As for the converse, let $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ and let $a_n = (p\delta_{i,n})_{i\in\mathbb{N}^*}$. Clearly, each a_n is nilpotent. The nth coefficient of f^k is $[n]f^k = \sum_{i_1+\ldots+i_k=n} a_{i_1} \ldots a_{i_k}$. For it to be zero, we must have in particular that the nth coordinate (of the nth coefficient…) is zero. It is equal to kp, since any term containing a_n as a factor must have the kth other factor to be a_0, and there is then k choice for the position of the a_n coefficient. For n large enough, $kp \neq 0 \pmod{p}^n$, and thus $f^k \neq 0$.

iii) If f is in the Jacobson radical of $A[[X]]$, then for any constant b, $1 - fb$ is a unit and thus $1 - f(0)b$ is a unit, meaning a_0 is in the Jacobson radical of A.

We showed in i) that $1 - fg$ is a unit iff $1 - f(0)g(0)$ is a unit. If $f(0)$ is in the Jacobson radical, then $1 - f(0)g(0)$ is always invertible and we're done.

iv) Let $\pi : A[[X]] \to A$ be the quotient map and $\mathfrak{m}$ be a maximal ideal of $A[[X]]$. As π is surjective, $\mathfrak{m}^c = \pi(\mathfrak{m})$ is an ideal, and the image of a maximal ideal is maximal. We clearly have $\mathfrak{m} \subset (\mathfrak{m}^c, x)$, and since $(\mathfrak{m}^c, x) \subsetneq A[[X]$, maximality gives $\mathfrak{m} = (\mathfrak{m}^c, x)$.

v) Let $\mathfrak{p}$ be a prime ideal of A. Then $I = (\mathfrak{p}, x)$ is a prime ideal of $A[[X]]$, with contraction $\mathfrak{p}$. Indeed, $A[[X]]/\mathfrak{p} \simeq A/\mathfrak{p}$.

Exercise 6

Assume $\mathfrak{n} \subsetneq \mathfrak{r}$. Then there is some idempotent $e \in \mathfrak{r}$, meaning that $1 - e$ is both invertible and a zero divisor, as $e(1-e) = 0$, absurd.

Exercise 7

Let $\mathfrak{p}$ be a prime ideal, and $\pi : A \to A/\mathfrak{p}$ the quotient map. For any $x \in A/\mathfrak{p}$, there is some $n \in \mathbb{N}$ such that $x^n = x \iff x(x^{n-1} - 1) = 0$. For any $x \neq 0$, $x^{n-1} = 1$, meaning x is invertible, and $A/\mathfrak{p}$ is a field.

Exercise 8

Let $\{\mathfrak{p}_i \mid i \in I\}$ be a totally ordered set of prime ideals. We show that $\mathfrak{q} = \bigcap_i \mathfrak{p}_i$ is still a prime ideal. Indeed, it is an ideal and if $xy \in \mathfrak{q}$, and $x \notin \mathfrak{q}$, there is a $\mathfrak{p}_i$ such that $x \notin \mathfrak{p}_i$. It follows that for any $\mathfrak{p}_j$, the possibilities are: - $y \in \mathfrak{p}_j$ if

$\mathfrak{p}_i \subset \mathfrak{p}_j$ - Otherwise, $\mathfrak{p}_j \subset \mathfrak{p}_i$ and x or y must be in $\mathfrak{p}_j$, x can't be, so $y \in \mathfrak{p}_j$. In any case, we conclude that $y \in \mathfrak{q}$. Zorn then applies.

Exercise 9

If $\mathfrak{a} = r(\mathfrak{a})$ then since $r(\mathfrak{a})$ is the intersection of the prime ideals containing $\mathfrak{a}$, $\mathfrak{a}$ is indeed an intersection of prime ideals.

Now, assume $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$. For any i, we have $\mathfrak{a} \subset \mathfrak{p}_i$, meaning $r(\mathfrak{a}) \subset \cap_i \mathfrak{p}_i = \mathfrak{a}$ and the other inclusion always holds, that is $\mathfrak{a} = r(\mathfrak{a})$.

Exercise 10

Assume i), we show ii). Let $\mathfrak{p}$ be that prime ideal and let x not be a unit. Then x is contained in some prime ideal, which must be $\mathfrak{p}$. Then x is in the nilradical, that is x is nilpotent.

Assume ii), we show iii). Let $\pi : A \to A/\mathfrak{N}$ be the quotient map. For any $x \in A$, either x is nilpotent and $\pi(x) = 0$ or x is a unit, meaning $\pi(x)$ is invertible. As π is surjective, $A/\mathfrak{N}$ is a field.

Assume iii), we show i). Prime ideals of A (containing $\mathfrak{N}$, which is a trivial condition) and prime ideals of $A/\mathfrak{N}$ are in bijective correspondence. Since $A/\mathfrak{N}$ is a field, the only prime ideal of A is $\pi^{-1}(0) = \mathfrak{N}$.

Exercise 11

i) Since $2 \in A$, $2 = 2^2 = 4$, meaning $2x = 4x \iff 2x = 0$
ii) Let $\mathfrak{p}$ be a prime ideal. Since $A/\mathfrak{p}$ is integral, $x^2 = x$ gives $x(x-1) = 0$, meaning $x \in \{0; 1\}$. Since $A/\mathfrak{p} \neq 0$, we get $A/\mathfrak{p} \sim \mathbf{F}_2$
iii) Let $\mathfrak{q}$ be a finitely generated ideal of A, with set of generators $\{x_1, \dots, x_n\}$. Assume $n > 1$, then $x_1 \neq x_2$. Let $y = x_1 + x_2 + x_1x_2$. Computations give $x_1y = x_1$ and $x_2y = x_2$, that is $\mathfrak{q} = (x_1, x_2, \dots, x_n) = (y, x_3, \dots, x_n)$. Iterating this process, we get that $\mathfrak{q}$ is principal.

Exercise 12

A nontrivial idempotent $e \in A$ gives a nontrivial decomposition $A \simeq A/e \times A/(1-e)$. Indeed, (e) and $(1-e)$ are coprime ideals and CRT applies. A maximal ideal $\mathfrak{m}$ of A/e gives a maximal ideal $\mathfrak{m} \times A/(1-e)$ of A, and symetrically. This gives atleast two distinct maximal ideals, as the decomposition is nontrivial.

Exercise 13

Let K be a field, Σ the set of all its irreducible polynomials and A be the polynomial ring $K[x_f \mid f \in \Sigma]$. Let $\mathfrak{a} = (f(x_f))_{f\in\Sigma}$. For any $p \in \mathfrak{a}$, $p = \sum_{f\in\Sigma} g_f f(x_f)$ with finitely many $g_f \in A$ being non-zero. Setting each of these x_f to be a root of f in a finite extension of K gives that $p \neq 1$, and as p is arbitrary, $\mathfrak{a} = (1)$. We now let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$ and $K_1 = A/\mathfrak{m}$.

Repeating the construction, we get some increasing sequence of fields $K_1, K_2, \dots$ and let $L = \bigcup_{n=1}^{\infty} K_n$. The fact that any $f \in \Sigma$ of degree $n+1$ splits in K_n is clear by induction, and it follows that any $f \in \Sigma$ splits in L. Let $\overline{K}$ be the subset of L of algebraic elements over K. Any finite extension $F/\overline{K}$ is in L, as if $f \in \overline{K}[X]$ we must have $f \in K_n[X]$ for n large enough. Since F is a subset of L algebraic over K, it follows that $F = \overline{K}$ and $\overline{K}$ is algebraically closed.

Exercise 14

Wrong proof: See here

Correct proof: Increasing union of ideals is an ideal, so Zorn applies. Let $\mathfrak{a}$ be a maximal ideal in Σ and $xy \in \mathfrak{a}$. We have $(\mathfrak{a}, x)(\mathfrak{a}, y) \subset (\mathfrak{a}, xy) = \mathfrak{a}$. By maximality, if $x, y \notin \mathfrak{a}$, there is a non-zerodivisor in $(\mathfrak{a}, x)$ and $(\mathfrak{a}, y)$. Product of non-zerodivisor is not a zerodivisor, absurd.

Exercise 15

i) $V(E) = V(\mathfrak{a})$ is clear, as is $V(\mathfrak{a}) = V(r(\mathfrak{a})$
ii) Same
iii) Same
iv) The inclusion $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b}$, gives $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{ab})$ Now let $\mathfrak{ab} \subset \mathfrak{p}$ be a prime. For any $x \in \mathfrak{a} \cap \mathfrak{b}$, $x^2 \in \mathfrak{ab} \subset \mathfrak{p}$ giving $x \in \mathfrak{p}$, and the reverse inclusion. For any prime $\mathfrak{p}$,$\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$, meaning $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. The other inclusion is clear.

Exercise 16

- *Spec*($\mathbf{Z}$) is $\{0\}$ and the set of positive primes $\mathbf{P}$

- *Spec*($\mathbf{R}$) is a single point 0.
- Nonzero prime ideals of $\mathbf{C}[X]$ are in correspondence with irreductible monic polynomials over $\mathbf{C}$, that is of the form $(X - z), z \in \mathbf{C}$.
- Same thing except that we add polynomials of the form $x^2 + ax + b$ with $a^2 - 4b < 0$
- Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[X]$ and let $(p) = \mathfrak{p} \cap \mathbb{Z}$. As any retract of a prime ideal is prime, p must be a prime or 0.
  If $p = 0$, let $f \in \mathfrak{p}$ of minimal degree, f must be irreducible over $\mathbb{Q}$, and we write $f = \alpha f'$ with f' irreducible over $\mathbb{Z}$ and $\alpha \in \mathbb{Z}^*$ by Gauss' lemma. Since $\alpha \notin \mathfrak{p}$, $f' \in \mathfrak{p}$ and $\mathfrak{p} = (f')$. Otherwise, $\mathfrak{p}$ induces a prime ideal of $\mathbf{F}_p[X]$ through a quotient by (p) and it must be of the form (f) with f irreducible over $\mathbf{F}_p$, that is $\mathfrak{p} = (p, f)$. ### Exercise 17 We have the equality $V(\mathfrak{a}) = \bigcap_{a \in \mathfrak{a}} V(a)$ by exercise 15. It follows that the $X_a = X \setminus V(a)$ form a basis of open sets.

i) Exercise 15 gives $V(a) \cup V(b) = V(ab)$ giving $X_a \cap X_b = X_{ab}$
ii) $X_f = \varnothing$ means f is in every prime ideal, i.e f is in the nilradical, so f is nilpotent

iii) $X_f = X$ means that it is in no prime ideal. By Krull's theorem, $(f) = A$ and f is a unit

iv) $X_f = X_g$ means that $f \in \mathfrak{p} \iff g \in \mathfrak{p}$ for any prime $\mathfrak{p}$. It follows that intersecting over that property $r((f)) = r((g))$.

v) It is enough to consider a covering given by the basis of open sets: $X = \bigcup_{i \in I} X_{x_i}$. This equality gives $\bigcap_{i \in I} V((x_i)) = V((x_i)_{i \in I}) = \varnothing$, i.e $(x_i)_{i \in I} = (1)$, and there is a finite subset generating 1, giving a finite subcover.

vi) This time, a covering may be reduced to something of the form $X_f \subset \bigcup_{i \in I} X_{x_i}$. It implies $V((x_i)_{i \in I}) \subset V(f)$, and thus $f \in r((x_i)_{i \in I})$, meaning there is some $n \in \mathbb{N}$ such that $f^n \in ((x_i)_{i \in I})$, and there is finitely many x_i's generating f^n. Then f^n is in any prime ideal containing all these x_i, and thus f by primality.

vii) If U is a finite union of X_f, since a finite union of quasi-compact spaces is still quasi-compact, U is quasi-compact. Now assume U is quasi-compact and consider the family of open sets given by the set of all X_f such that $X_f \subset U$. As the X_f are a basis, this is an open-cover. The result follows.

Exercise 18

i) If $\{x\}$ is closed means there is some ideal $\mathfrak{a}$ such that $\{x\} = V(\mathfrak{a})$. We must have $\mathfrak{p}_x = \mathfrak{a}$, and if $\mathfrak{p}_x$ is not maximal then any larger ideal containing it contradicts the equality.

Reciprocally if $\mathfrak{p}_x$ is maximal, $V(\mathfrak{p}_x) = \{x\}$.

ii) Let $V(\mathfrak{a})$ be any closed set containing $\mathfrak{p}_x$. As $\mathfrak{p}_x \in V(\mathfrak{a})$, we must have $V(\mathfrak{p}_x) \subset V(\mathfrak{a})$.

iii) We just showed that $y \in \overline{\{x\}} \iff y \in V(\mathfrak{p}_x)$, which in turn gives $\mathfrak{p}_x \subset \mathfrak{p}_y$ by definition.

iv) Let $x \neq y$, and assume $y \in \overline{\{x\}}$, as otherwise we are done. Then by iii), $\mathfrak{p}_x \subsetneq \mathfrak{p}_y$ and $x \in X \setminus \overline{\{x\}}$.

Exercise 19

Let's denote by $\mathfrak{n}$ the nilradical.

Assume $\mathfrak{n}$ is prime, we show X is irreducible. It is enough to show that basic open set intersect. By hypothesis $\mathfrak{n} \in X$, and for basic open sets $X_f, X_g \neq \varnothing$, we must have $f, g \notin \mathfrak{n}$, that is $\mathfrak{n} \in X_f \cap X_g$

Now assume X is irreducible and let X_f, X_g be non-empty basic open sets (that is $f, g \notin \mathfrak{n}$). Since $X_f \cap X_g = X_{fg} \neq \varnothing$, we get $fg \notin \mathfrak{n}$.

Exercise 20

Let X be a top space. i) Let $Y \subset X$ be irreducible and $O \subset X$ be open. Then $O \cap Y$ is dense in Y, that is any closed set containing $O \cap Y$ contains Y, and thus contains $\overline{Y}$. It implies that $O \cap Y$ is still dense in $\overline{Y}$ and a fortiori $O \cap \overline{Y}$ is.

ii) Let Y_i be irreducible subspaces totally ordered subspace ordered by inclusion and $Y = \bigcup_i Y_i$. Let $O \subset X$ be open. For any i, $Y_i \cap O$ is dense in Y_i. That is $O \cap Y$ is an open subset of Y whose closure contains Y_i, for i arbitrary, i.e it contains Y. Zorn holds and yadi yada

iii) If they're not closed, by i) taking closure gives a strictly larger irreducible subspace. In an Hausdorff space, sets containing atleast two points cannot be irreducible, meaning irreducible components are singletons.

iv) Let $V(\mathfrak{a})$ be a maximal irreducible subset of X, with $\mathfrak{a}$ radical. Let $ab \in \mathfrak{a}$. Radicality gives $V(\mathfrak{a}, a) \cup V(\mathfrak{a}, b) = V(\mathfrak{a})$ and irreducibility implies one of these is equal to $V(\mathfrak{a})$, that is $\mathfrak{a}$ is prime.

Exercise 21

If $\mathfrak{q}$ is prime and $xy \in \phi^{-1}(\mathfrak{q})$ then $\phi(x)\phi(y) \in \mathfrak{q}$, meaning wlog $\phi(x) \in \mathfrak{q}$, that is $\phi^{-1}(\mathfrak{q})$ is prime.

i) Let $\mathfrak{p}$ be a prime ideal containing $

Exercise 22

Let $\mathfrak{p}$ be a prime ideal of a finite product $A = \prod_{i=1}^n A_i$ and let $e_1, \ldots, e_n$ be such that e_i is 0 except at the i-th component where it is equal to 1. Since $\mathfrak{p}$ is proper, some $e_i \notin \mathfrak{p}$, say e_1 wlog. Then, since $e_1 e_j = 0 \in \mathfrak{p}$ for any $j > 1$, we get $0 \times \prod_{i=2}^n A_i \subset \mathfrak{p}$. The ideal $\mathfrak{p}' = \pi_1(\mathfrak{p})$ as $\mathfrak{p}$ is prime and contains the kernel of π_1, and then $\mathfrak{p} = \pi_1(\mathfrak{p}) \times \prod_{i=2}^n A_i$. The description $\mathrm{Spec}(A) = \bigsqcup_{i=1}^n \mathrm{Spec}(A_i)$ follows.

i) => ii) Let $Spec(A) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ with $V(\mathfrak{a}), V(\mathfrak{b})$ disjoint and non-empty. Then $\mathfrak{a} \cap \mathfrak{b}$

ii) => iii) Elements of the form $(0, 1)$ and $(1, 0)$ in the product induce nontrivial idempotents in A

iii) => i) Let e be a nontrivial idempotent. We have $X = V(eA) \cup V((1 - e)A) = V(0)$, both being nontrivial closed sets.