Recall that a free module is a module with a basis: a subset that is both spanning and linearly independent over the base ring.

> Statement: Let R be a commutative ring and M a free R-module. Dimension of M is well-defined, that is all basis of M have the same cardinality.

The strategy is to reduce the problem to dimension of vector spaces. Let I be a maximal ideal of R, and let $k = R/I$. The R-module $N = M/IM$ is actually a vector space over k.
Now let $(x_j)_{j \in J}$ be a basis of M: - Clearly $x_j + IM$ generates M. - Let $\sum_{j \in J} \overline{\lambda_j} \overline{x_j} = 0$ with $\lambda_i \in R, \overline{\lambda_i} \in k$ and only finitely many λ_j's nonzero. It translates to $\sum_{j \in J} \lambda_j x_j \in IM$, meaning we have some $\mu_i \in I$ such that $\sum \lambda_j x_j = \sum \mu_j x_j$. By the basis property, we must have $\lambda_j = \mu_j$, up to a permutation, and $\lambda_j \in I$ means precisely $\overline{\lambda_j} = 0$, that is $(\overline{x_j})_{j \in J}$ is linearly. Since $(x_j)_{j \in J}$ corresponds bijectively to a basis of the k-vector space N and dimension of a vector space is well-defined, we are done.