CSE 250B: Homework 3 Solutions

1. *Checking convexity/concavity.*

   (a) $f(x) = e^{ax}$ is convex.
   Proof: The second partial derivative $H(x) = f''(x) = a^2 e^{ax} \geq 0$

   (b) $f(x) = |x|$ is convex.
   Proof: $\forall a, b \in \mathbb{R}$ and $\theta \in (0, 1)$,

   $$f(\theta a + (1-\theta)b) = |\theta a + (1-\theta)b| \leq |\theta a| + |(1-\theta)b| = \theta|a| + (1-\theta)|b| = \theta f(a) + (1-\theta)f(b)$$

   (c) $f(x) = \ln x$ is concave.
   Proof: $-f(x) = -\ln x$ is convex because the second derivative

   $$H(x) = -f''(x) = \frac{1}{x^2} \geq 0$$

   (d) $f(x) = x^a \quad (x > 0)$. Here we only consider $x > 0$ because $f(x)$ doesn't always have definition when x is negative. $f(x)$ is convex when $a \geq 1$ and $a \leq 0$, and is concave when $0 < a < 1$.
   Proof: The second derivative

   $$H(x) = a(a-1)x^{a-2}$$

   When $0 < a < 1$, $H(x) < 0$, which means the second derivative of $-f(x)$ is positive, so in this case $f(x)$ is concave. When $a \geq 1$ or $a \leq 0$, $H(x) \geq 0$, so in this case $f(x)$ is convex.

2. *Showing convexity.*

   (a) The Hessian of $f(x) = x^T M x$ is $H(x) = 2M$. Since M is positive semidefinite, so is $2M$; so f is convex.

   (b) The Hessian of $f(x) = e^{u \cdot x}$ is

   $$H(x) = e^{u \cdot x} u u^T,$$

   which can also be written as vv^T, where $v = (e^{u \cdot x}/2)u$. Thus $H(x)$ is P.S.D. and so $f(x)$ is convex.

   (c) Since $f(x) = \max(f_1(x), \ldots, f_k(x))$, where the individual f_i are all convex, we have that for all $x_1, x_2 \in \mathbb{R}$ and $t \in (0, 1)$,

   $$\begin{aligned}
   & f(tx_1 + (1-t)x_2) \\
   = \quad & \max\left(f_1(tx_1 + (1-t)x_2), f_2(tx_1 + (1-t)x_2), \ldots, f_k(tx_1 + (1-t)x_2)\right)) \\
   \leq \quad & \max\left(tf_1(x_1) + (1-t)f_1(x_2), tf_2(x_1) + (1-t)f_2(x_2), \ldots, tf_k(x_1) + (1-t)f_k(x_2)\right) \\
   \leq \quad & t\max\left(f_1(x_1), f_2(x_1), \ldots, f_k(x_1)\right) + (1-t)\max\left(f_1(x_2), f_2(x_2), \ldots, f_k(x_2)\right) \\
   = \quad & tf(x_1) + (1-t)f(x_2)
   \end{aligned}$$

   Therefore, $f(x)$ is convex.

3. *Entropy.* The negation of the entropy, $N(p) = -H(p)$, has Hessian with entries

   $$\frac{\partial N}{\partial p_i \partial p_j} = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{p_i \ln 2} & \text{if } i = j \end{cases}$$

This is a diagonal matrix with positive values on the diagonal. Thus the Hessian is P.S.D., whereupon N is convex and H is concave.

4. *Regression problem.*

   (a) Let

   $$X = \begin{pmatrix} \leftarrow & x^{(1)} & \rightarrow \\ \leftarrow & x^{(2)} & \rightarrow \\ \leftarrow & \dots & \rightarrow \\ \leftarrow & x^{(n)} & \rightarrow \end{pmatrix}$$

   Then we can write the Hessian as

   $$H(w) = 2\sum_{i=1}^{n} x^{(i)} \left(x^{(i)}\right)^T + 2\lambda I = 2X^T X + 2\lambda I$$

   (b) For all $z \in \mathbb{R}^d$

   $$z^T H z = z^T(2X^T X + 2\lambda I)z = 2(z^T X^T X z + \lambda z^T I z) = 2||Xz||^2 + 2\lambda||z||^2 \geq 0$$

   Therefore, $H(w)$ is P.S.D, which means $L(w)$ is convex.

5. *Convex sets.*

   (a) The circle is not a convex set: for any two points on the circle, the line joining them does not lie on the circle.

   (b) The ball is convex.

   (c) Hyperplanes are convex.

   (d) k-sparse points are not convex: lines joining two such points can be upto $(2k)$-sparse.

   (e) The set of positive semidefinite matrices is closed under addition and multiplication by positive scalars; therefore it is convex.

6. *Norms.*

   (a) We can check that ℓ_1 is a norm by going through the definition, one property at a time:

      i. $\|x\|_1 = \sum_{i=1}^d |x_i| \geq 0$.

      ii. If $x = 0$, then $\|x\|_1 = 0$. If $\exists i, x_i \neq 0$, then $\|x\|_1 \geq |x_i| > 0$. Therefore, $\|x\|_1 = 0$ if and only if $x = 0$.

      iii. For any real-valued t, we have $\|tx\|_1 = \sum_{i=1}^d |tx_i| = |t| \sum_{i=1}^d |x_i| = |t|\,\|x\|_1$

      iv. $\|x+y\|_1 = \sum_{i=1}^d |x_i + y_i| \leq \sum_{i=1}^d |x_i| + |y_i| = \sum_{i=1}^d |x_i| + \sum_{i=1}^d |y_i| = \|x\|_1 + \|y\|_1$

   (b) Invoking homogeneity and the triangle inequality, we have that for any norm f,

   $$f(\theta x + (1-\theta)y) \leq f(\theta x) + f((1-\theta)y) = |\theta| f(x) + |1-\theta| f(y) = \theta f(x) + (1-\theta) f(y).$$

   Thus any norm is a convex function.

   (c) Various inequalities relating $\|x\|_1$, $\|x\|$, and $\|x\|_\infty$:

      i. $\|x\|_1 = \sqrt{(\sum_{i=1}^d |x_i|)^2} = \sqrt{\sum_{i=1}^d \sum_{j=1}^d |x_i||x_j|} \geq \sqrt{\sum_{i=1}^d x_i^2} = \|x\|$.
      $\|x\| = \sqrt{\sum_{i=1}^d x_i^2} \geq \sqrt{\max_i x_i^2} = \max_i |x_i| = \|x\|_\infty$

      ii. Let vector $a = (|x_1|, |x_2|, \dots, |x_d|), b = (1, 1, \dots, 1)_d$
      $\|x\|_1 = \sum_{i=1}^d |x_i| = |a \cdot b| \leq \|a\|\,\|b\| = \sqrt{\sum_{i=1}^d x_i^2}\sqrt{\sum_{i=1}^d 1^2} = \|x\| \cdot \sqrt{d}$.
      $\|x\| = \sqrt{\sum_{i=1}^d x_i^2} \leq \sqrt{d \cdot \max_i x_i^2} = \|x\|_\infty \cdot \sqrt{d}$.
      Therefore, $\|x\|_1 \leq \|x\| \cdot \sqrt{d} \leq \|x\|_\infty \cdot d$.

(d) The unit ball $\{x : x^T A x \leq 1\}$ is an ellipsoid.

7. *A lower bound for the perceptron.* Pick any $\gamma > 0$. Consider the following data set in $\mathbb{R}^d$, where $d = 1/\gamma^2$:

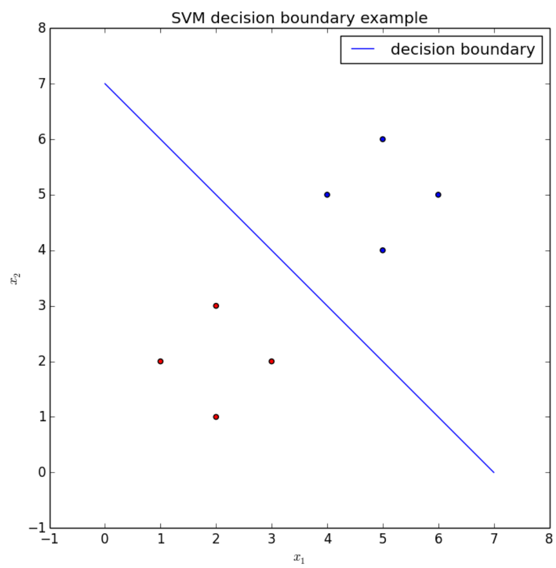
   - There are d points, each corresponding to one coordinate direction: $e_1, e_2, \ldots, e_d$, where e_i is the vector with all zeros except for a 1 at position i.
   - All points have label $+1$.

   These points are correctly classified by the vector $w^* = (\gamma, \gamma, \ldots, \gamma)$, which has unit length and has margin $\min_i(w^* \cdot e_i) = \gamma$.

   Now suppose the perceptron algorithm is run on this data set, and that it produces a linear separator w. If perceptron does not update on e_i, then $w_i = 0$ and w will not correctly classify e_i. Therefore, there must be at least one update for every data point: a total of $1/\gamma^2$ updates.

8. *Small SVM example.*

   (a)



   (b) The margin is $\sqrt{2}$.

   (c) w lies in the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and has length $1/\sqrt{2}$ (since the margin is $\sqrt{2}$); therefore, $w = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. We know that the point $x_o = (4, 3)$ lies on the decision boundary; solving $w \cdot x_o + b = 0$ yields $b = -7/2$.

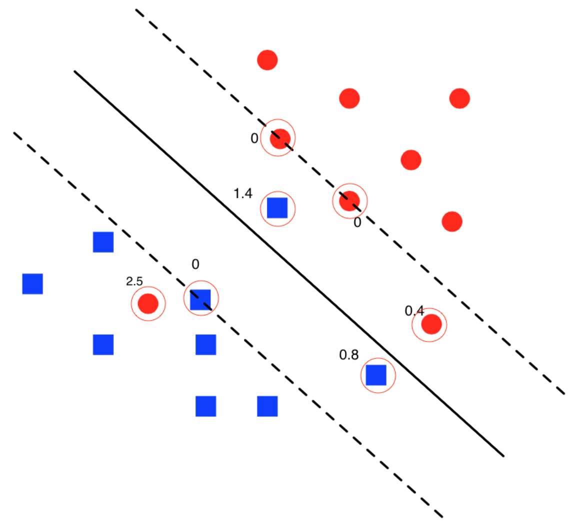
9. *Support vectors.* The margin decreases if the factor C is increased.

10. Here is a linear program, over variables $x \in \mathbb{R}^n$ and $v \in \mathbb{R}$:

$$\min v$$

$$-b_i + \sum_{j=1}^{n} a_{ij} x_j \leq v, \quad i = 1, 2, \ldots, m$$

$$b_i - \sum_{j=1}^{n} a_{ij} x_j \leq v, \quad i = 1, 2, \ldots, m$$



11. (a) Let K denote the intersection of halfspaces given by $w_1, w_2, \ldots \in \mathbb{R}^d$ and $b_1, b_2, \ldots \in \mathbb{R}$:

$$K = \bigcap_i \{x : w_i \cdot x \leq b_i\}.$$

For any $x, y \in K$ and $0 < \theta < 1$,

$$w_i \cdot (\theta x + (1-\theta)y) = \theta w_i \cdot x + (1-\theta) w_i \cdot y \leq \theta b_i + (1-\theta) b_i = b_i, \quad \text{for } i = 1, 2, \ldots$$

Therefore, $\theta x + (1-\theta)y \in K$; and K is a convex set.

(b) The unit ball in $\mathbb{R}^d$ can be written as

$$\bigcap_{\|w\|=1} \{x : w \cdot x \leq 1\}.$$

12. P_1 and P_2 are polyhedra that are intersections of finitely many halfspaces. Let the halfspaces for P_1 be given by $u_1, \ldots, u_m \in \mathbb{R}^d$ and $b_1, \ldots, b_m \in \mathbb{R}$:

$$P_1 = \bigcap_{i=1}^m \{x : u_i \cdot x \leq b_i\}.$$

Likewise, let P_2 be given by $v_1, \ldots, v_n \in \mathbb{R}^d$ and $c_1, \ldots, c_n \in \mathbb{R}$:

$$P_2 = \bigcap_{i=1}^n \{x : v_i \cdot x \leq c_i\}.$$

We wish to find the point $x_1 \in P_1$ and $x_2 \in P_2$ that are closest to one another. Let us write $z = x_1 - x_2$. Here is the optimization problem:

$$\begin{gathered} \min \ \|z\|^2 \\ u_i \cdot x_1 \leq b_i, \quad i = 1, 2, \ldots, m \\ v_i \cdot x_2 \leq c_i, \quad i = 1, 2, \ldots, n \\ z = x_1 - x_2 \end{gathered}$$

The constraints are all linear, and the objective function is convex, so this is a convex optimization problem.