CMO 1: Preliminaries

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1 Central problem and algorithm template

Central Problem of the course 'Computational Methods of Optimization': Given an objective function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and a constraint set $S \subseteq \mathbb{R}^d$, find $x^* = \operatorname{argmin}_{x \in S} f(x)$ and $f^* = f(x^*)$.

Example: for $\min_{x \in \mathbb{R}} (x - t)^2$, $x^* = t$ and $f^* = 0$.

All algorithms we develop to find x^* will follow this template:

```
Pick x ∈ S.
while x is not optimal do
    Pick another x ∈ S such that f(x) decreases.
end while
return x
```

2 Metric space

For any set S (we'll usually consider $S = \mathbb{R}^d$), $D : S \times S \mapsto \mathbb{R}$ is a distance function iff all of the following are true:

- $D(x, y) = 0 \iff x = y$.
- $D(x, y) \geq 0$.
- Symmetry: $D(x, y) = D(y, x)$.

- Triangle inequality: $D(x, y) + D(y, z) \geq D(x, z)$.

Theorem 1. *$D(x, y) = \|x - y\|$ is a distance function. Here*

$$\|x\| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{d} x_i^2}$$

Theorem 2. *$D(x, y) = \sum_{i=1}^{d} |x_i - y_i|$ is a distance function.*

3 Neighborhood function and Open sets

Definition 1. *For $r > 0$ and $x \in \mathbb{R}^d$, $N_r(x) = \{z : D(x, z) < r\}$ is called a neighborhood of x of radius r.*

Definition 2. *$x \in \mathbb{R}^d$ is an interior point of S iff $\exists r > 0, N_r(x) \subseteq S$.*

Definition 3. *Let $x, y \in \mathbb{R}$.*

- $(x, y) = \{z : x < z < y\}$.
- $(x, y] = \{z : x < z \leq y\}$.
- $[x, y) = \{z : x \leq z < y\}$.
- $[x, y] = \{z : x \leq z \leq y\}$.

Definition 4. *S is an open set iff $\forall x \in S$, x is an interior point of S.*

Example 1. *$(0, 1)$ is an open set but $[0, 1)$ is not.*

Definition 5. *$x \in \mathbb{R}^d$ is a limit point of S iff $N_r(x) \cap S \neq \phi$.*

Example 2. *$0, \frac{1}{2}, 1$ are 3 of the limit points of $(0, 1]$.*

Definition 6. *Closure of a set S is the set of all limit points of S.*

Definition 7. *A set S is closed iff all limit points of S lie in S.*

Example 3. *$[0, 1]$ is a closed set.*

4 Limit and Bounds

Definition 8. *Let $[x_i]_{i \in \mathbb{N}}$ be an infinite sequence where $x \in \mathbb{R}^d$. Then*

$$\lim_{i \to \infty} x_i = x \iff \forall \epsilon > 0, \exists n, \forall i \geq n, \|x - x_i\| < \epsilon$$

Definition 9. *$S \subseteq \mathbb{R}^d$ is a bounded set iff $\exists M, \forall x \in S, \|x\| \leq M$.*

Definition 10. *For $x_i \in \mathbb{R}$, M is an upper bound of $[x_i]_{i \in \mathbb{N}}$ iff $\forall i, x_i \leq M$. A sequence with an upper bound is called an upper-bounded sequence.*

Definition 11. *g is a least upper bound (LUB) (of $[x_i]_{i \in \mathbb{N}}$) iff g is an upper bound and for every upper bound h, $g \leq h$.*

Example 4. *For $x_i = 1 - \frac{1}{i}$, LUB is 1.*

Theorem 3. *A monotonic bounded sequence has a limit.*

5 Continuity

Definition 12.

$$\lim_{x \to p} f(x) = q \iff \forall \epsilon > 0, \exists \delta > 0, \forall x \in N_\delta(p), f(x) \in N_\epsilon(q)$$

Definition 13. *f is continuous at $x \iff \lim_{x \to p} f(x) = f(p)$. f is continuous over $S \iff f$ is continuous at all points $x \in S$.*

Theorem 4. *Let $S \subseteq \mathbb{R}^d$ be closed and bounded. Let $f(S) = \{f(x) : x \in S\}$. Let f be continuous over S. Then $f(S)$ is closed and bounded.*

For optimization problems, x^* is guaranteed to exist iff f is continuous and S is closed and bounded. Henceforth, we will assume S to be closed and bounded and assume functions to be continuous.

6 Asymptotics

$$a(x) \in o(b(x)) \iff \lim_{x \to x_0} \left| \frac{a(x)}{b(x)} \right| = 0$$

For example, at $x = 0$, $x^3 \in o(x^2)$.

If f is continuous at $x = p$, $f(x) = f(p) + o(1)$.