

# CMO: Existence and Characterization of Minimum

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Let  $S \subseteq \mathbb{R}^d$  and  $f : S \mapsto \mathbb{R}$ .

$x^*$  is a local minimum  $\iff \exists r > 0, \forall x \in N_r(x^*) \cap S, f(x^*) \leq f(x)$ .

We'll restrict our analysis in 2 ways:

- We'll only consider functions for which a global minimum exists. Here we'll discuss a sufficient condition for that.
- We'll only try to find a local minimum, since finding global minimum is difficult.

## 1 Necessary condition for local minimum of univariate function

**Theorem 1.** *If  $f : \mathbb{R} \mapsto \mathbb{R}$  is differentiable, then  $x^*$  is the local minimum of  $f \implies f'(x^*) = 0$ .*

*Proof.* Let

$$h(t) = \frac{f(t) - f(x^*)}{t - x^*}$$

Then  $f'(x^*) = \lim_{t \rightarrow x^*} h(t)$ .

Suppose  $x^*$  is a local minimum in  $(x - r, x + r)$ . Then for  $t \in (x - r, x)$ ,  $h(t) \leq 0$  and for  $t \in (x, x + r)$ ,  $h(t) \geq 0$ . Therefore, left derivative of  $f$  at  $x^*$  is non-positive and right derivative of  $f$  at  $x^*$  is non-negative. Since  $f$  is differentiable, left and right derivatives are equal. Therefore,  $f'(x^*) = 0$ .  $\square$

**Theorem 2.** *Let  $f$  be a  $C^2$  function and  $x^*$  be a local minimum. Then  $f''(x^*) \geq 0$ .*

*Proof.* Using Taylor series near  $x^*$ , we get

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + o((x - x^*)^2)$$

$$\implies 0 \leq f(x) - f(x^*) = \frac{1}{2}(x - x^*)^2 f''(x^*) + o((x - x^*)^2)$$

For this to hold true for all  $x$  near  $x^*$ ,  $f''(x^*) \geq 0$ .  $\square$

## 2 Characterization of functions which have a minimum

Consider a function from  $\mathbb{R}^d$  to  $\mathbb{R}$ . Global minimum exists iff  $f$  is lower-bounded.

**Definition 1.**

$$\lim_{\|x\| \rightarrow \infty} f(x) = \infty \iff \forall F > 0, \exists M > 0, \forall x \in \mathbb{R}^d, (\|x\| > M \implies f(x) \geq F)$$

If  $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$ , then  $f$  is called a **coercive** function.

**Theorem 3** (Weierstrass' theorem). *If a continuous function's domain is closed and bounded, the function has a global minimum and maximum.*

**Theorem 4.**

$$\lim_{\|x\| \rightarrow \infty} f(x) = \infty \wedge f \text{ is continuous} \implies f \text{ has global minimum}$$

*Proof.* Consider  $F = f(0)$ . Let  $S_1 = \{x : \|x\| > M\}$  and  $S_2 = \{x : \|x\| \leq M\}$ .

Since  $f$  is coercive,  $\forall x \in S_1, f(0) \leq f(x)$ . By Weierstrass' theorem, a global minimum exists in  $S_2$ . Let it be  $x^*$ . Therefore,  $f(x^*) \leq f(0)$ . Therefore,  $x^*$  is a global minimum of  $\mathbb{R}^d$ .  $\square$

## 3 Sufficient condition for local minimum of univariate function

**Theorem 5.**  $f'(x_0) = 0 \wedge f''(x_0) > 0 \implies x_0$  is local minimum.

*Proof.*

$$f(x) - f(x_0) = \frac{1}{2}(x - x^*)^2 f''(x^*) + o((x - x^*)^2)$$

In the neighborhood of  $x_0$ , the small-o term is negligible, so the  $f''(x^*)$  makes  $f(x) - f(x_0)$  positive. Therefore,  $x_0$  is a local minimum in that neighborhood.  $\square$

## 4 Necessary condition for local minimum of multivariate function

**Theorem 6.** *Let  $f : \mathbb{R}^d \mapsto \mathbb{R}$  be a differentiable function. Let  $x^*$  be a local minimum of  $f$ . Then  $\nabla_f(x^*) = 0$ .*

*Proof.* Let  $u \in \mathbb{R}^d$  and  $t \in \mathbb{R}$ .

$$x^* + tu \in N_r(x^*) \iff \|tu\| < r \iff |t| \leq \frac{r}{\|u\|} \iff t \in N_{\frac{r}{\|u\|}}(0)$$

Let  $g(t) = f(x^* + tu)$ .

$$\begin{aligned}
& x^* \text{ is local minimum of } f \\
& \Rightarrow \forall x \in N_r(x^*), f(x^*) \leq f(x) \\
& \Rightarrow \forall t \in N_{\frac{r}{\|u\|}}(0), f(x^*) \leq f(x^* + tu) \\
& \Rightarrow \forall t \in N_{\frac{r}{\|u\|}}(0), g(0) \leq g(t) \\
& \Rightarrow g \text{ has local minimum at } 0 \\
& \Rightarrow g'(0) = 0 \\
& \Rightarrow \nabla_f(x^*)^T u = 0
\end{aligned}$$

Since this is true for all  $u \in \mathbb{R}^d$ ,  $\nabla_f(x^*) = 0$ . □

**Theorem 7.** *Let  $f : \mathbb{R}^d \mapsto \mathbb{R}$  be a differentiable function. Let  $x^*$  be a local minimum of  $f$ . Then  $H_f(x^*)$  is positive semi-definite.*

*Proof.* Similar to above proof. Use the fact that if  $g$  has a local minimum at 0, then  $g''(0) \geq 0$ . □

## 5 Sufficient condition for local minimum of multivariate function

**Theorem 8.** *Let  $f : \mathbb{R}^d \mapsto \mathbb{R}$  be a differentiable function. Let  $\nabla_f(x_0) = 0$  and  $H_f(x_0)$  be positive definite. Then  $x_0$  is a local minimum of  $f$ .*

*Proof.* Proof follows directly from Taylor series. □