

# CMO: Convex functions

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**Definition 1** (Convex set). Let  $S \subseteq \mathbb{R}^d$ ,  $S$  is convex iff

$$\forall u, v \in S, \forall \alpha \in (0, 1), (1 - \alpha)u + \alpha v \in S$$

**Definition 2** (Convex function). Let  $f : S \mapsto \mathbb{R}$ , where  $S$  is convex.  $f$  is convex iff

$$\forall \alpha \in (0, 1), \forall u, v \in S, f((1 - \alpha)u + \alpha v) \leq (1 - \alpha)f(u) + \alpha f(v)$$

**Theorem 1.**  $f$  is convex  $\iff (\forall u, v \in S, \nabla_f(u)^T(v - u) \leq f(v) - f(u))$

*Proof.*

$f$  is convex

$$\begin{aligned} &\Rightarrow f((1 - \alpha)u + \alpha v) \leq (1 - \alpha)f(u) + \alpha f(v) \\ &\Rightarrow f(u + \alpha(v - u)) \leq f(u) + \alpha(f(v) - f(u)) \\ &\Rightarrow \frac{1}{\alpha}(f(u + \alpha(v - u)) - f(u)) \leq f(v) - f(u) \end{aligned}$$

Let  $g(\alpha) = f(u + \alpha(v - u))$ .

$f$  is convex

$$\begin{aligned} &\Rightarrow \frac{g(\alpha) - g(0)}{\alpha} \leq g(1) - g(0) \\ &\Rightarrow \lim_{\alpha \rightarrow 0} \frac{g(\alpha) - g(0)}{\alpha} \leq \lim_{\alpha \rightarrow 0} (g(1) - g(0)) \\ &\Rightarrow g'(0) \leq g(1) - g(0) \\ &\Rightarrow \nabla_f(u)^T(v - u) \leq f(v) - f(u) \end{aligned}$$

Suppose  $\forall u, v \in S, \nabla_f(u)^T(v - u) \leq f(v) - f(u)$ .

For any arbitrarily chosen  $x_1$  and  $x_2$  ( $x_1 \neq x_2$ ), let  $x = (1 - \alpha)x_1 + \alpha x_2$ . Then  $x_1 - x = \alpha(x_1 - x_2)$  and  $x_2 - x = (1 - \alpha)(x_2 - x_1)$ .

Setting  $u = x$  and  $v = x_1$ , we get

$$\nabla_f(x)^T \alpha(x_1 - x_2) \leq f(x_1) - f(x)$$

Setting  $u = x$  and  $v = x_2$ , we get

$$-\nabla_f(x)^T (1 - \alpha)(x_1 - x_2) \leq f(x_2) - f(x)$$

Adding these equations with weights  $1 - \alpha$  and  $\alpha$ , we get

$$\begin{aligned}
& (1 - \alpha) \nabla_f(x)^T \alpha(x_1 - x_2) - \alpha \nabla_f(x)^T (1 - \alpha)(x_1 - x_2) \\
& \leq (1 - \alpha)(f(x_1) - f(x)) + \alpha(f(x_2) - f(x)) \\
& \Rightarrow 0 \leq (1 - \alpha)f(x_1) + \alpha f(x_2) - f(x) \\
& \Rightarrow f((1 - \alpha)x_1 + \alpha x_2) \leq (1 - \alpha)f(x_1) + \alpha f(x_2) \\
& \Rightarrow f \text{ is convex}
\end{aligned}$$

□

**Theorem 2.** *If  $f$  is convex, and  $x^*$  is a local minimum, then  $x^*$  is also a global minimum.*

*Proof.* For all  $x \in \mathbb{R}^d$ ,

$$0 = f(x^*)^T(x - x^*) \leq f(x) - f(x^*)$$

□

**Theorem 3** (Proof omitted). *Let  $f : \mathbb{R}^d \mapsto \mathbb{R}$  and  $f \in C^2$ . Then  $f$  is convex iff  $H_f$  is positive semi-definite.*