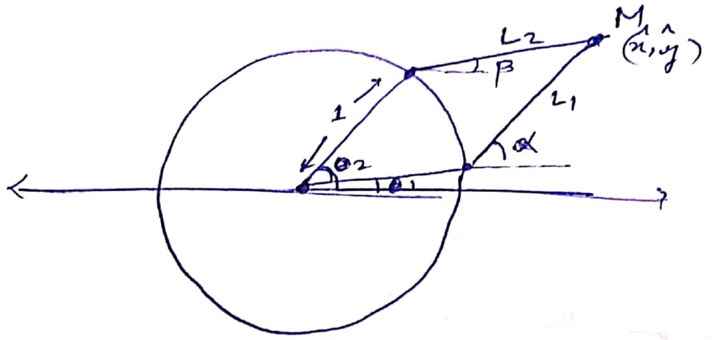


Let $M = (\hat{x}, \hat{y})$

L_1 : passes through $(\cos\theta_1, \sin\theta_2)$ has slope $\tan\alpha$

$$\therefore \quad y - \sin\theta_1 = \tan\alpha\,(x - \cos\theta_1) \longrightarrow (1)$$

L_2 : $P\,(\cos\theta_2, \sin\theta_2)$; $m = \tan\beta$

$$y - \sin\theta_2 = \tan\beta\,(x - \cos\theta_2) \longrightarrow (2)$$

Intersection pt. is

$$\Rightarrow \sin\theta_2 + \tan\beta\,(x - \cos\theta_2) - \sin\theta_1 = \tan\alpha\,(x - \cos\theta_1)$$

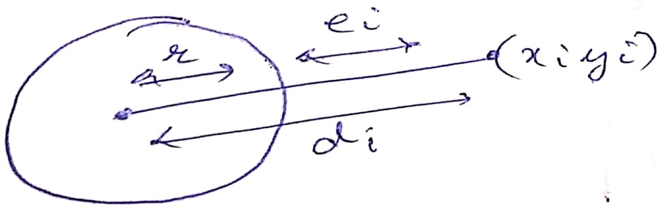
$$\Rightarrow \sin\theta_2 - \sin\theta_1 + x\,(\tan\beta - \tan\alpha) = \tan\beta\cos\theta_2 - \tan\alpha\cos\theta_1$$

$$\Rightarrow \boxed{\hat{x} = \frac{\tan\beta\cos\theta_2 - \tan\alpha\cos\theta_1 - \sin\theta_2 + \sin\theta_1}{\tan\beta - \tan\alpha}}$$

Substituting in (1)

$$\boxed{\hat{y} = \sin\theta_1 + \tan\alpha\,(\hat{x} - \cos\theta_1)}$$

Let $d_i^2 = \hat{x}_i^2 + \hat{y}_i^2$ for $i = 1, 2 .., 5$ data point (x_i, y_i)

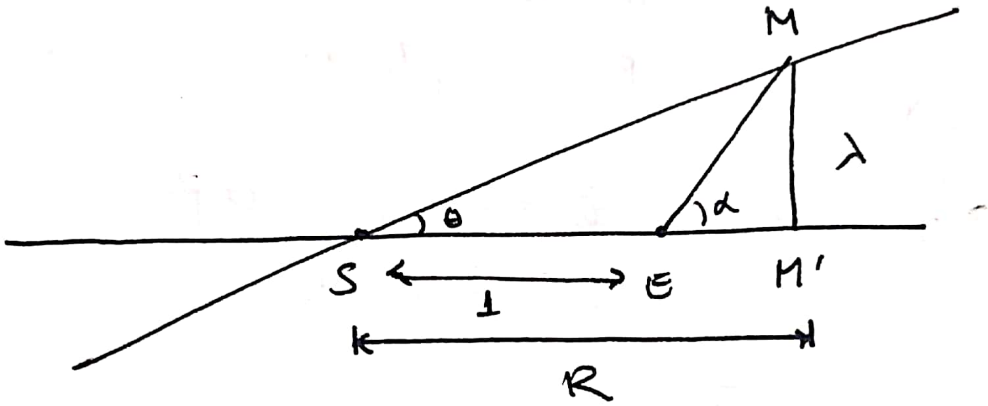


$$L = \sum_{i=1}^{r} e_i^2$$

$$= \sum_{i=1}^{5} (d_i - r)^2$$

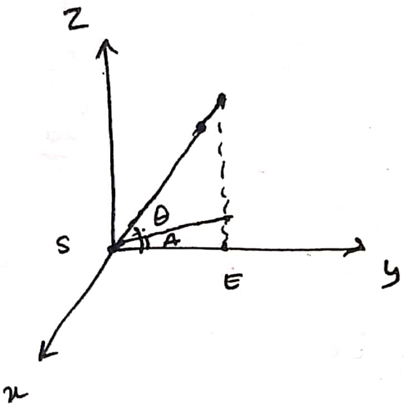
$$= \sum_{i=1}^{5} \left(\sqrt{x_i^2 + y_i^2} - r\right)^2$$

$$L = \sum_{i=1}^{5} r^2 + x_i^2 + y_i^2 - 2r\sqrt{x_i^2 + y_i^2}$$



$$\tan\alpha = \frac{\lambda}{R-1} \quad ; \quad \tan\theta = \frac{\lambda}{R}$$

$$\theta = \tan^{-1}\left(\frac{1}{R}(R-1)\tan\alpha\right)$$



θ = latitude

A = longitude

$$x = -R\cos\theta\,\sin A$$

$$z = R\sin\theta$$

$$y = R\cos\theta\,\cos A$$

Dis of point (x_i, y_i, z_i) from plane $d_i = \dfrac{Ax_i + By_i + Cz_i}{\sqrt{A^2+B^2+C^2}}$

$$L = \sum_{i=1}^{12} d_i^2$$

Let $S = A^2+B^2+C^2$

$$\frac{dS}{dA} = 2A, \quad \frac{dS}{dB} = 2B, \quad \frac{dS}{dC} = 2C$$

Let $V_i = (Ax_i + By_i + Cz_i)^2$

$$L = \frac{\sum V_i}{S}$$

$$\frac{dV_i}{dA} = \cancel{2V_i}\, x_i$$

$$\frac{dV_i}{dB} = \cancel{2V_i}\, y_i$$

$$\frac{dV_i}{dC} = \cancel{2V_i}\, z_i$$

$$\frac{\partial L}{\partial A} = \frac{1}{S}\left(\sum \frac{dV_i}{dA}\right) + \left(\frac{-1}{S^2}\frac{dS}{dA}\right) \sum V_i$$

$$\frac{\partial L}{\partial A} = \frac{1}{S}\Sigma(2V_i x_i) - \frac{1}{S^2}(2A)(\Sigma V_i)$$

By symmetry

$$\frac{\partial L}{\partial B} = \frac{1}{S}\Sigma(2V_i y_i) - \frac{1}{S^2}(2B)(\Sigma V_i)$$

$$\frac{\partial L}{\partial C} = \frac{1}{S}\Sigma(2V_i z_i) - \frac{1}{S^2}(2C)(\Sigma V_i)$$

3(ii)

Algorithm : Steepest Descent with exact line search using dichotomous search.

Let f to be the fn to be minimised,

$K=1$; $x_1 \leftarrow$ Random guess

while $|\nabla f_k| >$ tolerance :

$$\alpha_k^* = \underset{\alpha \geq 0}{\operatorname{argmin}} \; f(x_k - \alpha \nabla f_k)$$

where $\nabla f_k = \nabla f \big|_{x = x_k}$

$$x_{k+1} = x_k - \alpha_k \nabla f_k$$

$$K = K+1$$

Finding α_k : Line search using dichotomous search

Input : Interval of uncertainty $[a, b]$, tolerance

Init : $k = 0$

$a_k = a$

$b_k = b$

$\varepsilon \, (>0)$

while $b_k - a_k > l$

$$\lambda_k = \frac{a_k + b_k}{2} - \varepsilon, \quad \mu_k = \frac{a_k + b_k}{2} + \varepsilon$$

if $f(\lambda_k) > f(\mu_k)$

$a_{k+1} = \lambda_k$

$b_{k+1} = b_k$

else

$b_{k+1} = \mu_k$

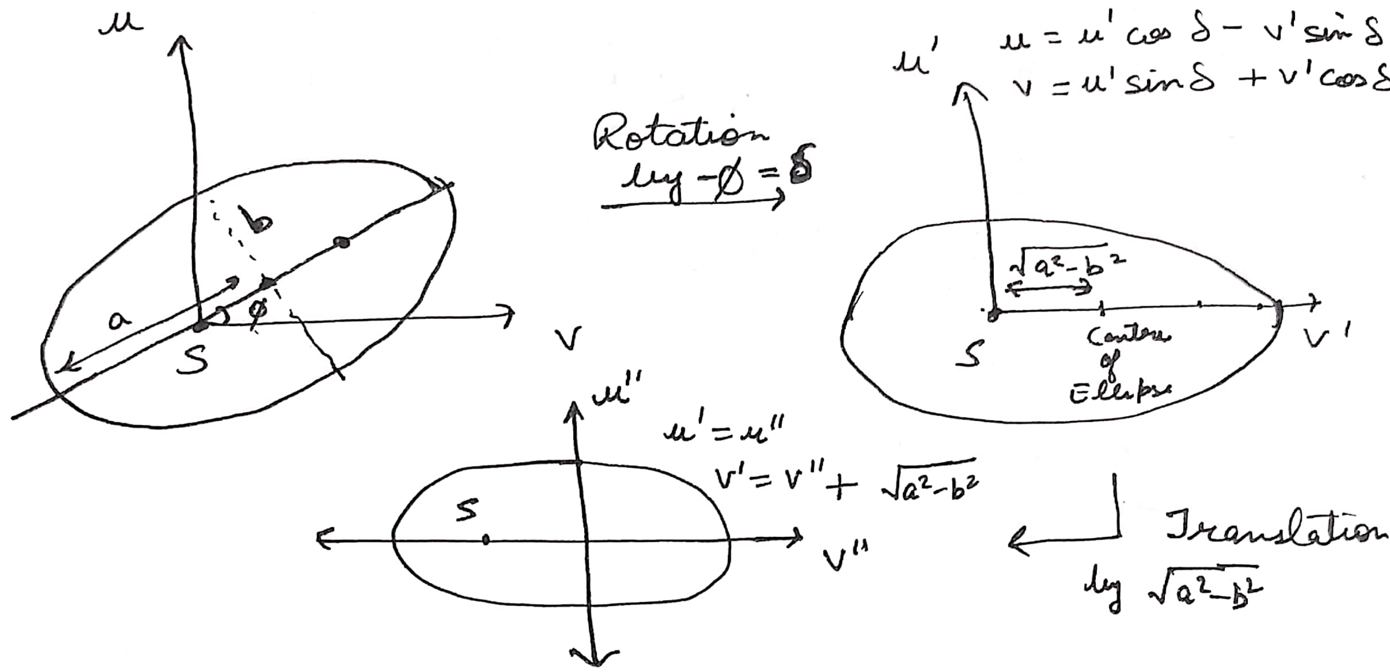
$a_{k+1} = a_k$

end if

$k \leftarrow k+1$

endwhile

return $x^* = \frac{a_k + b_k}{2}$



u
v
a
b
φ
S
Rotation
by −φ = δ
u'
u = u' cos δ − v' sin δ
v = u' sin δ + v' cos δ
√(a² − b²)
S
Centre of Ellipse
v'
u''
u' = u''
v' = v'' + √(a² − b²)
S
v''
Translation
by √(a² − b²)

Eq. of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \frac{u''^2}{a^2} + \frac{v''^2}{b^2} = 1$$

$$= \frac{u'^2}{a^2} + \frac{\left(v' + \sqrt{a^2 - b^2}\right)^2}{b^2} = 1$$

$$= \frac{(u \cos\delta + v \sin\delta)^2}{a^2} + \frac{\left((v\cos\delta - u\sin\delta) + \sqrt{a^2-b^2}\right)^2}{b^2} = 1$$

Let $f(u,v;\, a,b,\delta) = \dfrac{(u\cos\delta + v\sin\delta)^2}{a^2} + \dfrac{\left(v\cos\delta - u\sin\delta + \sqrt{a^2-b^2}\right)^2}{b^2} - 1$

Then Loss $\Rightarrow \sum_{\forall (u,v)} f(u,v;a,b,\delta)^2$

Reason: This loss fn is 0 for perfect fit and has less parameters than number of points.