Solution of 1D Solition Equation

Aman Abhishek Tiwari* and Shikher Verma†

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We present the numerical solution of the Solition equation using pseudo spectral method with periodic boundary conditions. We have tried to show the time evolution of a Gaussian wave packet satisfying the Solition equation and compare the solution for different diffusive parameter.

I. INTRODUCTION

Solition equation is given by

$$d_t u + d_x \frac{u^2}{2} = \nu d_x^3 u \tag{1}$$

where, ν is the diffusive parameter.

We tried to solve the Solition equation using the pseudo spectral method. Pseudo spectral methods are used for solving partial differential equations. The main advantage of pseudo spectral method over spectral method is evaluation of certain operators can speed up with the use of fast Fourier transform[1].

A Gaussian wave packet is given by the general equation

$$u(x) = e^{-\frac{(x-x_c)^2}{\sigma^2}} \tag{2}$$

Initial condition

For the initial conditions, we have taken a Gaussian wave packet in a domain $x\epsilon[-1,1]$ with periodic boundary conditions. The Gaussian wave packet that we have chosen is given by

$$u_0(x) = e^{-\frac{(x-0.5)^2}{0.1^2}} \tag{3}$$

II. CALCULATION

To find the numerical solution for the Solition Equation

$$d_t u + d_x \frac{u^2}{2} = \nu d_x^3 u \tag{4}$$

we are going to use the pseudo spectral method with periodic boundary conditions.

Let us call the term $\partial_x \frac{u^2}{2}$ as non-linear term and the $\frac{\partial^3 u}{\partial x^3}$ as the diffusion term.

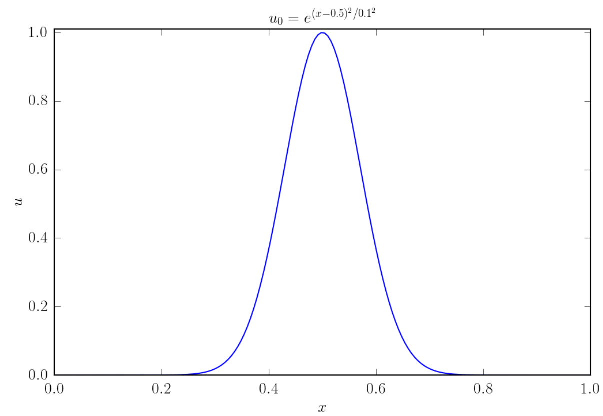

FIG. 1: Plot of the Gaussian distribution function which is taken as the initial condition for the solution of solition equation.

We choose initial wave to be a Gaussian function give by

$$u_0 = e^{-\frac{(x-x_c)^2}{\sigma^2}} \tag{5}$$

where σ^2 is the variance and x_c is the centre of the Gaussian wave. The figure of the Gaussian wave is shown in the figure 1

Now, to calculate the wave in the $(n+1)^{th}$ time step, represented as u_n, we followed the following procedures. Since we are going to be working in the Fourier domain, let's represent the Fourier transform of u_n, as $\hat{u_n}$.

The non-linear and the dissipation term in the Fourier domain is given by,

$$\partial_x \frac{u_n^2}{2} = ifft(\frac{2\pi ik\hat{u_n^2}}{2}) \tag{6}$$

and the diffusion term is given as,

$$\frac{\partial^3 u_n}{\partial x^3} = ifft((2\pi i)^3 k\hat{u_n}) \tag{7}$$

Working in the Fourier domain for calculating the next time step. The non-linear term has been stepped using the Predictor Corrector method[3] and the diffusion term is stepped using the Crank-Nicholson method[2] as shown in the following steps.

* amanabt@iitk.ac.in; At Department of Physics, IIT Kanpur.
† shikher@iitk.ac.in; Also at Department of Physics, IIT Kanpur.

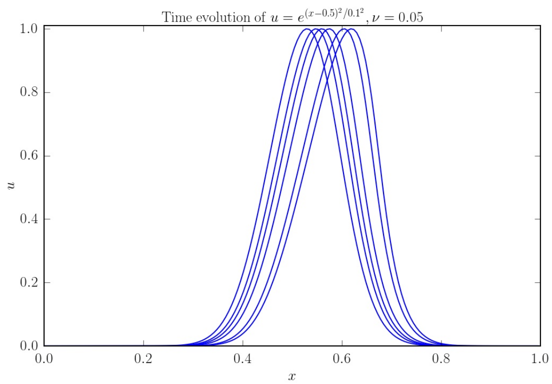

FIG. 2: Time evolution of the Gaussian wave $u_0 = e^{(x-0.5)^2/0.025^2}$ for $\nu = 0.05$ captured at $t = 0.010s, 0.015s, 0.020, 0.025s, 0.030s, 0.035s, 0.040s, 0.045s$. As we can clearly see from the figure that wave is propogating towards $x = 1.0$ and is tilting towards right.

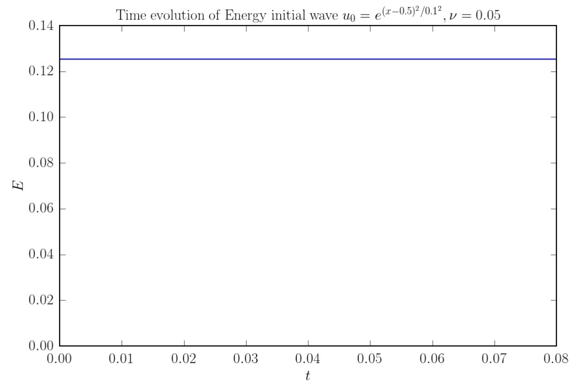

FIG. 3: Plot of time evolution of energy of the Gaussian wave $u_0 = e^{(x-0.5)^2/0.025^2}$ for $\nu = 0.05$. The Solition equation is an energy conserving equation and hence, the total energy of the system should remain constant, which is evident from the figure.

$$\frac{\hat{u_{n+\frac{1}{2}}} - \hat{u_n}}{0.5dt} = -\frac{2\pi ik\hat{u_n^2}}{2} + (2\pi i)^3 k\hat{u_{n+\frac{1}{2}}} \tag{8}$$

which, on solving for $\hat{u_{n+\frac{1}{2}}}$ gives

$$\hat{u_{n+\frac{1}{2}}} = \frac{\hat{u_n} - \pi ik\hat{u_n^2} 0.5dt}{1 - (2\pi i)^3 k 0.5dt} \tag{9}$$

and,

$$\frac{\hat{u}_{n+1} - \hat{u}_n}{dt} = -\frac{2\pi ik\hat{u^2_{n+\frac{1}{2}}}}{2} + 0.5((2\pi i)^3 k\hat{u}_{n+1} + (2\pi i)^3 k\hat{u}_n) \tag{10}$$

which on solving for $\hat{u}_{n+1}$ gives

$$\hat{u}_{n+1} = \frac{-2\pi ik\hat{u^2_{n+\frac{1}{2}}} dt + 0.5dt(2\pi i)^3 k\hat{u}_n + \hat{u}_n}{1 - 0.5dt(2\pi i)^3 k} \tag{11}$$

Now, to calculate u_{n+1}, we will take the inverse Fourier transform of $\hat{u}_{n+1}$. We will repeat these steps to calculate u to the desired time.

III. RESULTS

We implemented our numerical scheme to calculate the time evolution of Solition equation in $python3$ using standard libraries.

Here, we have calculated the time evolution of the Solition equations and plotted them at times $t = 0.010s, 0.015s, 0.020, 0.025s, 0.030s, 0.035s, 0.040s, 0.045s$ in the same figure.

The evaluated the time evolution of functions following Solition equations for $\nu = 0.05$, $\nu = 0.15$, $\nu = 0.25$ are shown in the figures 2, 4, 5 respectively.

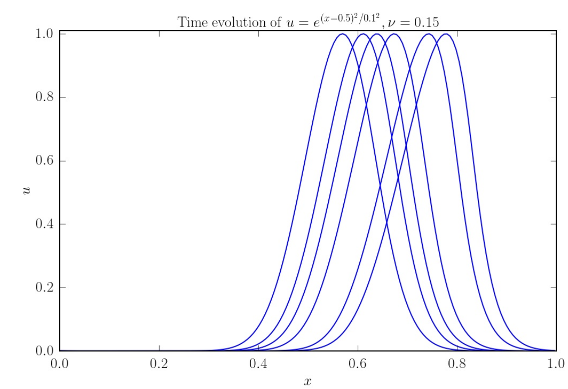

FIG. 4: Time evolution of the Gaussian wave $u_0 = e^{(x-0.5)^2/0.025^2}$ for $\nu = 0.05$ captured at $t = 0.010s, 0.015s, 0.020, 0.025s, 0.030s, 0.035s, 0.040s, 0.045s$. As we can clearly see from the figure that wave is propogating towards $x = 1.0$ and is tilting towards right. On comparing it with the time evolution obtained with for $\nu = 0.05$, shown in figure 2, it is propogating at a higher speed.

As we can see from the figure 2 that as the wave progresses, it starts tilting towards the right hand side of the plot.

We know that the Solition equation is energy conserving, so, the plot of energy or quantity on which energy depends should be constant. The energy of a wave

$$E \propto \int_0^1 u^2 dx \tag{12}$$

The figure , shows the plot of E vs time for $\nu = 0.05$ and we can see from the figure that it is constant.

To check the correctness of our numerical correctness we also calculated and compared the time evolution of wave for $\nu = 0.05$ for $dt = 10^{-5}$ as shown in figure 6. As

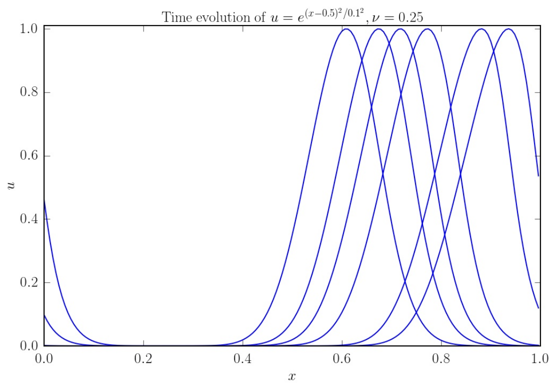

FIG. 5: Time evolution of the Gaussian wave $u_0 = e^{(x-0.5)^2/0.025^2}$ for $\nu = 0.05$ captured at $t = 0.010s, 0.015s, 0.020, 0.025s, 0.030s, 0.035s, 0.040s, 0.045s$. As we can clearly see from the figure that wave is propogating towards $x = 1.0$ and is tilting towards right. On comparing it with the time evolution obtained with for $\nu = 0.05$ and $\nu = 0.05$, shown in figure 2 and 4 it is propogating at a higher speed. We can also see that the program also respects the periodic boundary conditions, as we can see in this figure that for $t = 0.040$ and $t = 0.045$, the wave from the previous system is exiting and a new wave packet is emerging from $x = 0.0$

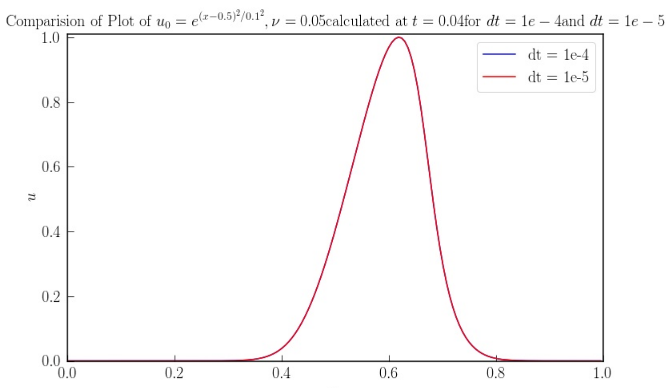

FIG. 6: Comparision of the capture of wave calculated at $t = 0.04$ calculated using $dt = 10^{-4}$ and $dt = 10^{-5}$. As we can see, both waves overlap.

we can see from the figure that they completely overlap each other.

IV. CONCLUSION

We created a computational simulation using Python3, Matplotlib, Numpy and Scipy. It showed how a gaussian wave packet would move in 1D solition equation under periodic boundary conditions. We found that packet moves towards right. And its velocity is increases as ν increases. We also showed that wave tilts right as it moves in time and it's energy remains constant.

[1] Wikipedia entry on Pseudo-spectral method https://en.wikipedia.org/wiki/Pseudo-spectral_method

[2] Wikipedia entry on CrankNicolson method https://en.wikipedia.org/wiki/CrankNicolson_method

[3] Wikipedia entry on Predictorcorrector method https://en.wikipedia.org/wiki/Predictorcorrector_method