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MESHLESS PSEUDO-POLYHARMONIC DIVERGENCE-FREE AND CURL-FREE VECTOR FIELDS APPROXIMATION*

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Abstract. In this paper, we propose a meshless approximation of a vector field in multidimensional space minimizing quadratic forms related to the divergence or to the curl of a vector field. Our approach guarantees the conservation of the divergence-free or the curl-free properties, which are of great importance in applications. For instance, divergence-free vector fields correspond to incompressible fluid flows, and curl-free vector fields correspond to magnetic fields in the equations of classical electrodynamics. Our construction is based on the meshless approximation by the pseudo-polyharmonic functions which are a class of radial basis functions. We provide an Helmholtz–Hodge decomposition of the used native functional space. Numerical examples are included to illustrate our approach.

Key words. meshless approximation, approximation theory, radial basis functions, curl-free, divergence-free, Helmholtz–Hodge decomposition

AMS subject classifications. 41A29, 41A15, 60E05, 65D15, 65D05

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1. Introduction.

Meshless methods have attracted much attention in recent years for a wide range of engineering sciences. Much effort has been particularly devoted to developing a variety of meshless approximation methods of the numerical solution of partial differential equations. The approximation methods by radial basis functions (RBFs) do not require the use of any mesh in the domain where the unknown function has to be approximated, so the RBF methods are inherently meshless. Furthermore, a variety of meshless methods are based on the RBFs. The RBF approximation is based on a set of scattered data points and is obtained by using a one-variable function to generate an approximation of the unknown function (see [10, 28], for instance). In this paper, we are interested in the approximation of vector fields by using the meshless pseudo-polyharmonic splines. The pseudo-polyharmonic splines are a class of RBFs obtained from variational problems. The theory of the pseudo-polyharmonic splines was introduced by Duchon [18, 19, 20, 21] (see also [3, 7, 8, 9]).

In [1], an approximation of a vector field based on the biharmonic thin plate splines in two variables and minimizing some energy is introduced. The energy used in [1] has divergence and rotational terms, each multiplied by a fixed real positive parameter that controls its relative weight. The meshless vector field approximation has been used successfully in many applications like reconstruction of wind velocity in meteorology, optical flow motion estimates, and human heart motion analysis, as well as image processing [2, 11, 12, 32, 33, 34].

Vector field approximation arises in a broad range of scientific applications. These include, for example, fluid mechanics, electromagnetics, meteorology, optic flow anal-

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ysis, and so on. Many physical phenomena can be characterized by vector fields with certain differential invariant properties. When considering the approximation of a vector field, a key task is how to correlate the components of the reconstructed vector field in order to conserve certain invariant differential characterizations (see [13]). For example, divergence-free vector fields occur naturally in incompressible fluid dynamics. The magnetic field from a system of currents and charges is also divergence free. The magnetic field in the equations of classical electrodynamics is curl free. The problem of reconstructing a vector field with the divergence free by using RBFs was already studied in [5, 16, 22, 25, 26]. In [25, 26], the reconstructed divergence free was obtained by using compactly supported RBFs, without minimizing any quadratic form.

In a recent paper [6], we have studied an extension of the concept given in [1]. The variational problem in [6] is given from a quadratic functional which linearly combines two energy terms related to the divergence and the curl of a vector field in multidimensional space. The approximation is constructed by using the pseudo-polyharmonic spline functions in multidimensional space. Unfortunately, this approximation does not guarantee certain invariant properties on the approximated vector fields. Some efforts are needed to adjust a parameter in order to reflect numerically some physical properties of the approximated vector field. The approach presented here is different from the one given in [6]. The objective of this paper is to provide a meshless approximation from a variational problem by using the pseudo-polyharmonic splines in any arbitrary dimensional space and by conserving the divergence-free or curl-free properties of the approximated vector field. The notations, the functional spaces and some arguments used in this paper are similar to those introduced in [6]. In order to facilitate the readability of this paper, we will recall some notations and properties. If necessary, we will establish some new properties.

In section 2, we give preliminary functional framework studies and some background. Some topological properties of the native space are given. We also give the Helmholtz–Hodge decomposition of the native space. In sections 3 and 4, the curl-free and divergence-free problems are studied, respectively. In section 5, we provide some numerical tests to illustrate the effectiveness of our approach. The conclusion and some extensions are given in section 6.

2. Background and preliminary study. Let s be a real number, let $n, m \in \mathbb{N}^* := \mathbb{Z}_+ \setminus \{0\}$ be two integers, and consider the (scalar) space (see [27])

$$\widetilde{H}^s(\mathbb{R}^n) = \left\{ v \in \mathscr{S}'(\mathbb{R}^n) : \widehat{v} \in L^1_{loc}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{v}(\xi)|^2 d\xi < +\infty \right\},$$

where $\mathscr{S}'(\mathbb{R}^n)$ is the Schwartz space of tempered distributions. The notation $\widehat{v}$ stands for the Fourier transform of the tempered distribution v, and $|\xi|$ stands for the classical Euclidian norm in $\mathbb{R}^n$. Consider the Beppo–Levi space [6, 15, 20]

$$X^{m,s}(\mathbb{R}^n) = \left\{ u \in \mathscr{D}'(\mathbb{R}^n) : \ \text{for all } \alpha \in \mathbb{Z}^n_+, \quad |\alpha| = m, \partial^\alpha u \in \widetilde{H}^s(\mathbb{R}^n) \right\}. \tag{2.1}$$

The notation ∂^α stands for the partial derivative of order $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the module of α. We also recall that $\alpha! = \alpha_1 \cdots \alpha_n$. The space $X^{m,s}(\mathbb{R}^n)$ is equipped with the following semiscalar product together with its associated seminorm given by

$$
\begin{aligned}
[u|v]_{m,s} &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \widehat{(\partial^\alpha u)}(\xi) \overline{\widehat{(\partial^\alpha v)}(\xi)} d\xi, \\
[u]_{m,s} &= \sqrt{[u|u]_{m,s}}
\end{aligned} \tag{2.2}
$$

for all $u, v \in X^{m,s}(\mathbb{R}^n)$. We assume that $m \geq 1$ and $s \in \mathbb{R}$ are such that

$$
-m + \frac{n}{2} < s < \frac{n}{2}. \tag{2.3}
$$

Then space $X^{m,s}(\mathbb{R}^n)$ endowed with the semiscalar product $[\,.|.\,]_{m,s}$ is a semi-Hilbert space with the null subspace the space $\Pi_{m-1}(\mathbb{R}^n)$ of polynomials of degree $\leq m-1$.

Consider the Beppo–Levi space of vector-valued distributions given by

$$
X^{m,s}(\mathbb{R}^n; \mathbb{R}^n) = \left[X^{m,s}(\mathbb{R}^n)\right]^n. \tag{2.4}
$$

We have $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n;\mathbb{R}^n)$, where $\mathscr{D}'(\mathbb{R}^n;\mathbb{R}^n)$ is the space of the vector-valued distributions on $\mathbb{R}^n$. The space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ equipped with the following semiscalar product and its associated seminorm

$$
(u|v)_{m,s} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \left\langle \widehat{\partial^\alpha u}(\xi) | \overline{\widehat{\partial^\alpha v}(\xi)} \right\rangle_n d\xi, \quad |u|_{m,s} = \sqrt{(u|u)_{m,s}} \tag{2.5}
$$

is a semi-Hilbert space. Here $\langle z|z'\rangle_n = z^T z'$, where z and z' are two vectors and z^T stands for the transpose of z. It is clear that the semiscalar (2.5) may be given in terms of the semiscalar product (2.2) as

$$
(u|v)_{m,s} = \sum_{i=1}^{n} [u_i|v_i]_{m,s},
$$

where $u = (u_1, \ldots, u_n)^T$ and $v = (v_1, \ldots, v_n)^T$. The null space associated to the semiscalar product is the space, denoted by $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$, of vector-valued polynomials of n-variables with degree $\leq m-1$. Let N be a nonnegative integer, and let $L : X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}^{N\times n}$ denote a linear operator such that $L(p) = 0$ for $p \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ implies that $p = 0$. We consider the following scalar product defined in the space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ and associated to the operator L:

$$
(u|v)_{L,m,s} = (u|v)_{m,s} + \left\langle Lu|Lv \right\rangle_{N\times n}. \tag{2.6}
$$

Its associated norm is denoted by $||\,.\,||_{L,m,s}$. Here the notation $\langle\,.|.\,\rangle_{N\times n}$ stands for the Frobenius scalar product $\langle Z|Z'\rangle_{N\times n} = trace(Z^T Z')$. Its associated norm is denoted by $||\,.\,||_{N\times n}$ in $\mathbb{R}^{N\times n}$. The following proposition gives some topological properties of the space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ equipped with the scalar product (2.6) and its associated norm.

Proposition 2.1. *The space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ endowed with the scalar product $(\,.|.\,)_{L,m,s}$ is a Hilbert space, and its topology is independent of the choice of L. The following continuous inclusions hold for all integers k such that $k < m + s - n/2$:*

$$
X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n;\mathbb{R}^n), \quad X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{C}^k(\mathbb{R}^n;\mathbb{R}^n).
$$

Furthermore, the space $\mathscr{D}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ is dense in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. See [6]. ∎

The operators div and curl are the divergence and the rotational operators, respectively. They will be of fundamental importance in this paper. Both operators div and curl owe much of their importance to the electromagnetic theory and Maxwell's equations. The divergence operator is a scalar distribution defined by

$$\operatorname{div} u = \nabla^T u = \sum_{i=1}^{n} \partial_i u_i, \tag{2.7}$$

where $\nabla = (\partial_1, \dots, \partial_n)^T$ stands for the nabla operator, $\partial_i = \frac{\partial}{\partial x_i}$ is the partial derivative with respect to the variable x_i, and $u = (u_1, \dots, u_n)^T$ is a vector-valued distribution. The classical definition of the curl is limited to only the spaces of dimensions 2 and 3 and is defined as a vector-valued distribution. In [6], we defined the curl as a matrix-valued distribution of size $n \times n$, given by

$$\operatorname{curl} u = \nabla u^T - (\nabla u^T)^T = (\partial_i u_j - \partial_j u_i)_{1 \le i,j \le n}. \tag{2.8}$$

The general definition (2.8) of the curl is standard in multidimensional harmonic analysis (see [23, 31], for instance, and the references therein).

We consider the bilinear forms $D_{m,s}$ and $R_{m,s}$ given by

$$\begin{aligned} D_{m,s}(u,v) &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \widehat{\partial^\alpha(\operatorname{div} u)}(\xi) \overline{\widehat{\partial^\alpha(\operatorname{div} v)}(\xi)} d\xi, \\ R_{m,s}(u,v) &= \frac{1}{2} \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \left\langle \widehat{\partial^\alpha(\operatorname{curl} u)}(\xi) \middle| \overline{\widehat{\partial^\alpha(\operatorname{curl} v)}(\xi)} \right\rangle_{n\times n} d\xi. \end{aligned} \tag{2.9}$$

It is obvious that

$$D_{m,s}(u,v) = [\operatorname{div} u | \operatorname{div} v]_{m-1,s},$$

$$R_{m,s}(u,v) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [(\operatorname{curl} u)_{ij} | (\operatorname{curl} v)_{ij}]_{m-1,s},$$

where $(\operatorname{curl} u)_{ij} = \partial_i u_j - \partial_j u_i$ are the components of the matrix $\operatorname{curl} u$. The quadratic forms associated to $D_{m,s}$ and $R_{m,s}$ are denoted by $D_{m,s}(u) = D_{m,s}(u,u)$ and $R_{m,s}(u) = R_{m,s}(u,u)$, respectively. They are called the divergence energy and the curl energy, respectively.

The classical $\operatorname{curl} u$ in the dimension $n = 2$ or $n = 3$ is a vector-valued distribution given by

$$\operatorname{curl} u = \begin{cases} (0, \partial_1 u_2 - \partial_2 u_1)^T & \text{for} \quad n = 2, \\ (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T & \text{for} \quad n = 3. \end{cases} \tag{2.10}$$

Thus, for the dimension $n = 2$ or $n = 3$, one can define the bilinear form $R_{m,s}$ by

$$R_{m,s}(u,v) = \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \left\langle \widehat{\partial^\alpha(\operatorname{curl} u)}(\xi) \middle| \overline{\widehat{\partial^\alpha(\operatorname{curl} v)}(\xi)} \right\rangle_{n} d\xi. \tag{2.11}$$

We notice that both definitions of $R_{m,s}$ in (2.9) and (2.11) give the same result for the dimension $n = 2$ or $n = 3$ and that, in this case, we can write $R_{m,s}$ in the following form:

$$R_{m,s}(u,v) = (\operatorname{curl} u | \operatorname{curl} v)_{m-1,s}.$$

Let us consider the following matrix-polynomials defined by

$$\begin{aligned} P_{\mathrm{div}}(\xi) &= \xi\,\xi^T = (\xi_k\xi_l)_{1\le k,l\le n}, \\ P_{\mathrm{curl}}(\xi) &= |\xi|^2 I_n - P_{\mathrm{div}}(\xi) = (\delta_{k,l}|\xi|^2 - \xi_k\xi_l)_{1\le k,l\le n} \end{aligned} \tag{2.12}$$

for all $\xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$, where I_n denotes the identity matrix of size $n \times n$ and the notation $\delta_{k,l}$ stands for the Kronecker symbol. An elementary computation shows that the matrix-polynomials P_{div} and P_{curl} satisfy the relations

$$\begin{aligned} P_{\mathrm{div}}(\xi)P_{\mathrm{div}}(\xi) &= |\xi|^2\, P_{\mathrm{div}}(\xi), \\ P_{\mathrm{div}}(\xi)P_{\mathrm{curl}}(\xi) &= P_{\mathrm{curl}}(\xi)P_{\mathrm{div}}(\xi) = 0_n, \\ P_{\mathrm{curl}}(\xi)P_{\mathrm{curl}}(\xi) &= |\xi|^2\, P_{\mathrm{curl}}(\xi), \end{aligned} \tag{2.13}$$

where 0_n denotes the zero matrix of size $n \times n$. The associated differential matrix-operators are defined as follows:

$$P_{\mathrm{div}}(\mathrm{i}\,\nabla) = (\mathrm{i}\,\nabla)\,(\mathrm{i}\,\nabla)^T = -\nabla\,\nabla^T = (-\partial^2_{k,l})_{1\le k,l\le n}, \tag{2.14}$$

$$\begin{aligned} P_{\mathrm{curl}}(\mathrm{i}\,\nabla) &= (\mathrm{i}\,\nabla)^T\,(\mathrm{i}\,\nabla) I_n - P_{\mathrm{div}}(\mathrm{i}\,\nabla) = -\Delta I_n - P_{\mathrm{div}}(\mathrm{i}\,\nabla) \\ &= -\Delta I_n + \nabla\,\nabla^T = (-\delta_{k,l}\Delta + \partial^2_{k,l})_{1\le k,l\le n}. \end{aligned} \tag{2.15}$$

The notation i refers to the complex number such that $\mathrm{i}^2 = -1$ and $\partial^2_{ij} = \partial_i\partial_j$ are the second partial derivatives. Let $u \in \mathscr{S}'(\mathbb{R}^n;\mathbb{R}^n)$ be a vector-tempered distribution. Then we have

$$\begin{aligned} P_{\mathrm{div}}(\mathrm{i}\,\nabla)u &= -\nabla(\operatorname{div} u), \\ P_{\mathrm{curl}}(\mathrm{i}\,\nabla)u &= -(\Delta I_n)u + \nabla(\operatorname{div} u). \end{aligned} \tag{2.16}$$

We notice that the differential matrix-operators $P_{\mathrm{div}}(\mathrm{i}\,\nabla)$ and $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)$ satisfy the following relations:

$$\begin{aligned} P_{\mathrm{div}}(\mathrm{i}\,\nabla)P_{\mathrm{div}}(\mathrm{i}\,\nabla) &= -\Delta\, P_{\mathrm{div}}(\mathrm{i}\,\nabla) = -P_{\mathrm{div}}(\mathrm{i}\,\nabla)\,\Delta, \\ P_{\mathrm{div}}(\mathrm{i}\,\nabla)P_{\mathrm{curl}}(\mathrm{i}\,\nabla) &= P_{\mathrm{curl}}(\mathrm{i}\,\nabla)P_{\mathrm{div}}(\mathrm{i}\,\nabla) = 0_n, \\ P_{\mathrm{curl}}(\mathrm{i}\,\nabla)P_{\mathrm{curl}}(\mathrm{i}\,\nabla) &= -\Delta\, P_{\mathrm{curl}}(\mathrm{i}\,\nabla) = -P_{\mathrm{curl}}(\mathrm{i}\,\nabla)\,\Delta. \end{aligned}$$

For a vector-tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n;\mathbb{R}^n)$, the Fourier transforms $\mathcal{F}[P_{\mathrm{div}}(\mathrm{i}\,\nabla)u]$ and $\mathcal{F}[P_{\mathrm{curl}}(\mathrm{i}\,\nabla)u]$ are given by

$$\begin{aligned} \mathcal{F}[P_{\mathrm{div}}(\mathrm{i}\,\nabla)u] &= P_{\mathrm{div}}(\xi)\widehat{u}, \\ \mathcal{F}[P_{\mathrm{curl}}(\mathrm{i}\,\nabla)u] &= P_{\mathrm{curl}}(\xi)\widehat{u}. \end{aligned} \tag{2.17}$$

The relationships between the bilinear forms defined in (2.9) and the differential matrix-polynomials defined in (2.12) are given in the following proposition.

PROPOSITION 2.2.

(1) *For all $u \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n;\mathbb{R}^n)$, we have*

$$\begin{aligned} D_{m,s}(u,\varphi) &= \langle |\xi|^{2(m-1)+2s} P_{\mathrm{div}}(\xi)\widehat{u}, \overline{\widehat{\varphi}}\rangle, \\ R_{m,s}(u,\varphi) &= \langle |\xi|^{2(m-1)+2s} P_{\mathrm{curl}}(\xi)\widehat{u}, \overline{\widehat{\varphi}}\rangle. \end{aligned} \tag{2.18}$$

(2) *For all $u, v \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$, we have*

$$D_{m,s}(u,v) + R_{m,s}(u,v) = (u|v)_{m,s}. \tag{2.19}$$

Proof. See [6]. ◻

PROPOSITION 2.3. *The operator $\nabla : X^{m+1,s}(\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ is continuous, and the operator $\Delta I_n : X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ is continuous and surjective. Furthermore, for all $u, v \in X^{m+1,s}(\mathbb{R}^n)$,*

$$[u|v]_{m+1,s} = (\nabla u|\nabla v)_{m,s}, \tag{2.20}$$

and for all $u, v \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$,

$$(u|v)_{m+2,s} = \Big((\Delta I_n)u|(\Delta I_n)v\Big)_{m,s}. \tag{2.21}$$

Proof. The continuity of the two operators is obviously obtained from the continuity of the operators $\partial_i : u \in X^{m+2,s}(\mathbb{R}^n) \longrightarrow \partial_i u \in X^{m,s}(\mathbb{R}^n)$ and $\partial^2_{ii} : u \in X^{m+2,s}(\mathbb{R}^n) \longrightarrow \partial^2_{ii} u \in X^{m,s}(\mathbb{R}^n)$ for $1 \le i \le n$. For all $u \in X^{m+1,s}(\mathbb{R}^n)$ and for all $v = \psi + p \in \mathscr{D}(\mathbb{R}^n) + \Pi_m(\mathbb{R}^n)$, we have

$$\begin{aligned}
[u|v]_{m+1,s} &= [u|\psi + p]_{m+1,s} = [u|\psi]_{m+1,s} = \langle |\xi|^{2(m+1)+2s}\widehat{u}, \overline{\widehat{\psi}}\rangle \\
&= \sum_{k=1}^{n} \langle |\xi|^{2m+2s}(\mathrm{i}\,\xi_k)\widehat{u}, \overline{(\mathrm{i}\,\xi_k)\widehat{\psi}}\rangle = \langle |\xi|^{2m+2s}\widehat{\nabla u}, \overline{\widehat{\nabla\psi}}\rangle \\
&= [\nabla u|\nabla\psi]_{m,s} = [\nabla u|\nabla(\psi+p)]_{m,s} = [\nabla u|\nabla v]_{m,s}.
\end{aligned}$$

The density of $\mathscr{D}(\mathbb{R}^n) + \Pi_m(\mathbb{R}^n)$ in $X^{m+1,s}(\mathbb{R}^n)$ together with the continuity of the operator $\nabla : X^{m+1,s}(\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n)$ imply that the relation (2.20) holds.

For the surjectivity of the operator ΔI_n, it is sufficient to show that the operator $\Delta : X^{m+2,s}(\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n)$ is surjective. According to [20], the quotient spaces

$$\dot{X}^{m+2,s}(\mathbb{R}^n) = X^{m+2,s}(\mathbb{R}^n)/\Pi_{m+1}(\mathbb{R}^n) \quad \text{and} \quad \dot{X}^{m,s}(\mathbb{R}^n) = X^{m,s}(\mathbb{R}^n)/\Pi_{m-1}(\mathbb{R}^n),$$

endowed with the scalar product $[\dot{u}|\dot{v}]_{m+2,s} = [u|v]_{m+2,s}$ for $\dot{u}, \dot{v} \in \dot{X}^{m+2,s}(\mathbb{R}^n)$ and $[\dot{u}|\dot{v}]_{m,s} = [u|v]_{m,s}$ for $\dot{u}, \dot{v} \in \dot{X}^{m,s}(\mathbb{R}^n)$, respectively, are Hilbert spaces. Consider the operator $\dot{\Delta} : \dot{X}^{m+2,s}(\mathbb{R}^n) \longrightarrow \dot{X}^{m,s}(\mathbb{R}^n)$ given by

$$\dot{\Delta}\dot{v} = \dot{\widehat{\Delta v}} \text{ for all } \dot{v} \in \dot{X}^{m+2,s}(\mathbb{R}^n).$$

Since $\Delta(\Pi_{m+1}(\mathbb{R}^n)) \subseteq \Pi_{m-1}(\mathbb{R}^n)$, it is obvious that $\dot{\Delta}\dot{v}$ does not depend on the choice of the representant of the class $\dot{v}$ and that the operator $\dot{\Delta}$ is well defined.

For all $u \in X^{m+2,s}(\mathbb{R}^n)$ and for all $v = \psi + p \in \mathscr{D}(\mathbb{R}^n) + \Pi_{m+1}(\mathbb{R}^n)$, we have

$$\begin{aligned}
[u|v]_{m+2,s} &= [u|\psi + p]_{m+2,s} = [u|\psi]_{m+2,s} \\
&= \langle |\xi|^{2(m+2)+2s}\widehat{u}, \overline{\widehat{\psi}}\rangle = \langle |\xi|^{2m+2s}|\xi|^2\widehat{u}, \overline{|\xi|^2\widehat{\psi}}\rangle = \langle |\xi|^{2m+2s}\widehat{\Delta u}, \overline{\widehat{\Delta\psi}}\rangle \\
&= [\Delta u|\Delta\psi]_{m,s} = [\Delta u|\Delta(\psi+p)]_{m,s} = [\Delta u|\Delta v]_{m,s}.
\end{aligned}$$

The density of $\mathscr{D}(\mathbb{R}^n) + \Pi_{m+1}(\mathbb{R}^n)$ in $X^{m+2,s}(\mathbb{R}^n)$ together with the continuity of the operator $\Delta : X^{m+2,s}(\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n)$ imply the relation (2.21). It follows that

$$[\dot{\Delta}\dot{u}|\dot{\Delta}\dot{v}]_{m,s} = [\dot{\widehat{\Delta u}}|\dot{\widehat{\Delta v}}]_{m,s} = [\Delta u|\Delta v]_{m,s} = [u|v]_{m+2,s} = [\dot{u}|\dot{v}]_{m+2,s}$$

for all $\dot{u}, \dot{v} \in \dot{X}^{m+2,s}(\mathbb{R}^n)$, and we deduce that the operator $\dot{\Delta}$ is an isometry. It follows that $\dot{\Delta}$ is continuous and its range $Im(\dot{\Delta})$ is closed in the Hilbert space $\dot{X}^{m,s}(\mathbb{R}^n)$.

For all $\dot{u} \in Im(\dot{\Delta})^{\perp}$, the orthogonal space to $Im(\dot{\Delta})$ for $[\,.|.\,]_{m,s}$, we have

$$0 = [\dot{u}|\dot{\Delta}\dot{v}]_{m,s} = [u|\Delta v]_{m,s} \qquad \text{for all } v \in X^{m+2,s}(\mathbb{R}^n).$$

Taking $v = \varphi \in \mathscr{D}(\mathbb{R}^n)$, we get

$$0 = [u|\Delta\varphi]_{m,s} = \langle |\xi|^{2m+2s}\widehat{u}, \overline{\widehat{\Delta\varphi}}\rangle = \langle |\xi|^{2m+2s+2}\widehat{u}, \overline{\widehat{\varphi}}\rangle.$$

The function $|\xi|^{2m+2s+2}$ vanishes only at zero. Thus, u is a polynomial belonging to $X^{m,s}(\mathbb{R}^n)$. Then $u \in \Pi_{m-1}(\mathbb{R}^n)$, and consequently $\dot{u} = \dot{0}$ in $\dot{X}^{m,s}(\mathbb{R}^n)$. Thus, the orthogonal $Im(\dot{\Delta})^{\perp}$ of the space $Im(\dot{\Delta})$ is $Im(\dot{\Delta})^{\perp} = \{\dot{0}\}$. Since $Im(\dot{\Delta})$ is closed, we obtain that $Im(\dot{\Delta}) = \dot{X}^{m,s}(\mathbb{R}^n)$. Hence, the operator $\dot{\Delta}$ is surjective. Let us notice that the operator $\dot{\Delta}$ is, in fact, bijective because it is an isometry.

Now we will prove that the operator $\Delta : X^{m+2,s}(\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n)$ is surjective. Let $u \in X^{m,s}(\mathbb{R}^n)$; the surjectivity of $\dot{\Delta}$ implies that there exists $v \in X^{m+2,s}(\mathbb{R}^n)$ such that $\dot{\Delta}\dot{v} = \dot{u}$. Then we have $\dot{\widehat{\Delta v}} = \dot{u}$, namely, there exists a polynomial $p \in \Pi_{m-1}(\mathbb{R}^n)$ such that $\Delta v = u + p$. According to [17, p. 168], for any homogenous polynomial p_k of degree k, there exists a homogenous polynomial p_{k+2} of degree $k+2$ such that $\Delta\, p_{k+1} = p_k$. This implies, obviously, that there exists a polynomial $q \in \Pi_{m+1}(\mathbb{R}^n)$ such that $\Delta\, q = p$. It follows that the element $w = v - q \in X^{m+2,s}(\mathbb{R}^n)$ satisfies $\Delta\, w = u$. $\square$

Now we introduce the two following spaces:

$$X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) = \{u \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) : \mathrm{div}\, u = 0\},$$
$$X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) = \{u \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) : \mathrm{curl}\, u = 0\}.$$

Since the operators div and curl are continuous on the space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$, then the spaces $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ and $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ are closed subspaces of $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. It follows that the spaces $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ and $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ endowed with the induced semiscalar product (2.5) are semi-Hilbert spaces.

The null spaces associated to the semiscalar product (2.5) in $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ and in $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ are the subspaces of vector-polynomials which are of divergence free and curl free and are denoted by

$$\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) = \{p \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n) : \mathrm{div}\, p = 0\} \tag{2.22}$$

and

$$\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) = \{p \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n) : \mathrm{curl}\, p = 0\}, \tag{2.23}$$

respectively. We have the following proposition.

PROPOSITION 2.4. *The dimension of the space* $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ *is*

$$d_1 = \dim\,[\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)] = \frac{(n+m)!}{n!m!} - 1,$$

and the dimension of the space $\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ *is*

$$d_2 = \dim\,[\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)] = \frac{(m+n-2)!}{n!(m-1)!}(n^2 + (n-1)(m-1)).$$

Proof. We consider the restriction $\mathscr{L}_1 : \Pi_m(\mathbb{R}^n) \longrightarrow \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ of the nabla operator ∇. Let $p \in Im(\mathscr{L}_1)$. Then there exists a polynomial $q \in \Pi_m(\mathbb{R}^n)$ such that $p = \nabla q$. It follows that $\operatorname{curl} p = \operatorname{curl}(\nabla q) = 0_n$. Conversely, if $p = (p_1, \dots, p_n)^T \in \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, we have $\operatorname{curl} p = 0$ and the compatibility conditions $\partial_i p_j = \partial_j p_i$ for $1 \le i, j \le n$ hold. Then there exists a polynomial $q \in \Pi_m(\mathbb{R}^n)$ such that $p = \nabla q$. In other words, $Im(\mathscr{L}_1) = \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. The dimension theorem of linear algebra gives $\dim[Im(\mathscr{L}_1)] + \dim[\ker(\mathscr{L}_1)] = \dim[\Pi_m(\mathbb{R}^n)]$. Since $\dim[\ker(\mathscr{L}_1)] = 1$ and $\dim[\Pi_m(\mathbb{R}^n)] = \frac{(m+n)!}{n!m!}$, then we obtain the first result.

Now we consider the restriction $\mathscr{L}_2 : \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow \Pi_{m-2}(\mathbb{R}^n)$ of the div operator to the subspace $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$. For a polynomial $p \in \Pi_{m-2}(\mathbb{R}^n)$, we consider the vector-polynomial $q = (q_1, \dots, q_n)^T \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ the vector-polynomial defined by $q_1(x) = \int_0^{t_1} p(t, t_2, \dots, t_n)dt$ for $x = (t_1, \dots, t_n)^T$ and $q_i = 0$ for $i = 2, \dots, n$. We have, $\partial_1 q_1 = p$ and $\mathscr{L}_2(q) = \operatorname{div} q = \partial_1 q_1 = p$. Then the linear operator $\mathscr{L}_2$ is surjective. The dimension theorem of linear algebra gives $\dim[Im(\mathscr{L}_2)] + \dim[\ker(\mathscr{L}_2)] = \dim[\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)]$. Since $\ker(\mathscr{L}_2) = \Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$, $Im(\mathscr{L}_2) = \Pi_{m-2}(\mathbb{R}^n)$, and the dimension of the polynomial space $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ is $\dim[\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)] = n\frac{(m+n-1)!}{(m-1)!n!}$, we obtain $d_2 = n\frac{(m+n-1)!}{n!(m-1)!} - \frac{(n+m-2)!}{n!(m-2)!}$. $\square$

Let us now give some examples of the basis of the spaces $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and $\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$.

Example 2.1.

(1) For $n = 2$ and $m = 2$, we have $d_1 = d_2 = 5$. The bases of the spaces $\Pi_{1,\mathrm{curl}}(\mathbb{R}^2;\mathbb{R}^2)$ and $\Pi_{1,\mathrm{div}}(\mathbb{R}^2;\mathbb{R}^2)$ are $(q_1, \dots, q_5)$ and $(p_1, \dots, p_5)$, respectively, and are given for $x = (t_1, t_2)$ by

$$\begin{aligned} &q_1(x) = (1,0)^T, \quad q_2(x) = (0,1)^T, \quad q_3(x) = (t_1,0)^T, \\ &q_4(x) = (0,t_2)^T, \quad q_5(x) = (t_2,t_1)^T, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &p_1(x) = (1,0)^T, \quad p_2(x) = (0,1)^T, \quad p_3(x) = (0,t_1)^T, \\ &p_4(x) = (t_2,0)^T, \quad p_5(x) = (t_1,-t_2)^T. \end{aligned} \tag{2.25}$$

(2) For $n = 2$ and $m = 3$, we have $d_1 = d_2 = 9$. The bases of the spaces $\Pi_{2,\mathrm{curl}}(\mathbb{R}^2;\mathbb{R}^2)$ and $\Pi_{2,\mathrm{div}}(\mathbb{R}^2;\mathbb{R}^2)$ are $(q_1, \dots, q_9)$ and $(p_1, \dots, p_9)$, respectively, and are given for $x = (t_1, t_2)$ by

$$\begin{aligned} &q_6(x) = (t_1^2,0)^T, \quad q_7(x) = (0,t_2^2)^T, \quad q_8(x) = (2t_1t_2,t_1^2)^T, \\ &q_9(x) = (t_2^2,2t_1t_2)^T, \end{aligned}$$

and

$$\begin{aligned} &p_6(x) = (0,t_1^2)^T, \quad p_7(x) = (t_2^2,0)^T, \quad p_8(x) = (t_1^2,-2t_1t_2)^T, \\ &p_9(x) = (2t_1t_2,-t_2^2)^T, \end{aligned}$$

$q_1, \dots, q_5$ are given in (2.24), and $p_1, \dots, p_5$ are given in (2.25).

(3) For $n = 3$ and $m = 2$, we have $d_1 = 9$ and $d_2 = 11$. The bases of the spaces $\Pi_{1,\mathrm{curl}}(\mathbb{R}^3;\mathbb{R}^3)$ and $\Pi_{1,\mathrm{div}}(\mathbb{R}^3;\mathbb{R}^3)$ are $(q_1, \dots, q_9)$ and $(p_1, \dots, p_{11})$, respectively, and are given for $x = (t_1, t_2, t_3)$ by

$$\begin{aligned} &q_1(x) = (1,0,0)^T, \quad q_2(x) = (0,1,0)^T, \quad q_3(x) = (0,0,1)^T, \\ &q_4(x) = (t_1,0,0)^T, \quad q_5(x) = (0,t_2,0)^T, \quad q_6(x) = (0,0,t_3)^T, \\ &q_7(x) = (0,t_3,t_2)^T, \quad q_8(x) = (t_3,0,t_1)^T, \quad q_9(x) = (t_2,t_1,0)^T, \end{aligned} \tag{2.26}$$

and

$$
(2.27)\qquad \begin{aligned}
&p_1(x) = (1,0,0)^T, \quad p_2(x) = (0,1,0)^T, \quad p_3(x) = (0,0,1)^T,\\
&p_4(x) = (0,t_1,0)^T, \quad p_5(x) = (0,0,t_1)^T, \quad p_6(x) = (t_2,0,0)^T,\\
&p_7(x) = (0,0,t_2)^T, \quad p_8(x) = (t_3,0,0)^T, \quad p_9(x) = (0,t_3,0)^T,\\
&p_{10}(x) = (-t_1,0,t_3)^T, \quad p_{11}(x) = (t_1,-t_2,0)^T.
\end{aligned}
$$

(4) For $n = 3$ and $m = 3$, we have $d_1 = 19$ and $d_2 = 26$. The bases of the spaces $\Pi_{2,\mathrm{curl}}(\mathbb{R}^3;\mathbb{R}^3)$ and $\Pi_{2,\mathrm{div}}(\mathbb{R}^3;\mathbb{R}^3)$ are $(q_1,\ldots,q_{19})$ and $(p_1,\ldots,p_{26})$, respectively, and are given for $x = (t_1,t_2,t_3)$ by

$$
\begin{aligned}
&q_{10}(x) = (t_1^2,0,0)^T, \quad q_{11}(x) = (0,t_2^2,0)^T, \quad q_{12}(x) = (0,0,t_3^2)^T,\\
&q_{13}(x) = (0,2t_2t_3,t_2^2)^T, \quad q_{14}(x) = (0,t_3^2,2t_2t_3)^T, \quad q_{15}(x) = (2t_1t_3,0,t_1^2)^T,\\
&q_{16}(x) = (t_3^2,0,2t_1t_3)^T, \quad q_{17}(x) = (t_2^2,2t_1t_2,0)^T, \quad q_{18}(x) = (2t_1t_2,t_1^2,0)^T,\\
&q_{19}(x) = (t_2t_3,t_1t_3,t_1t_2)^T,
\end{aligned}
$$

and

$$
\begin{aligned}
&p_{12}(x) = (0,t_1^2,0)^T, \quad p_{13}(x) = (0,0,t_1^2)^T, \quad p_{14}(x) = (t_2^2,0,0)^T,\\
&p_{15}(x) = (0,0,t_2^2)^T, \quad p_{16}(x) = (t_3^2,0,0)^T, \quad p_{17}(x) = (0,t_3^2,0)^T,\\
&p_{18}(x) = (t_1^2,-2t_1t_2,0)^T, \quad p_{19}(x) = (2t_1t_2,-t_2^2,0)^T,\\
&p_{20}(x) = (-2t_1t_3,0,t_3^2)^T, \quad p_{21}(x) = (-t_1t_2,0,t_2t_3)^T,\\
&p_{22}(x) = (0,t_1t_2,-t_1t_3)^T, \quad p_{23}(x) = (t_1t_3,-t_2t_3,0)^T,\\
&p_{24}(x) = (t_2t_3,0,0)^T, \quad p_{25}(x) = (0,t_1t_3,0)^T, \quad p_{26}(x) = (0,0,t_1t_2)^T,
\end{aligned}
$$

$q_1,\ldots,q_9$ are given by (2.26), and $p_1,\ldots,p_{11}$ are given by (2.27).

We consider the following subspaces of vector-valued compactly supported $\mathscr{C}^\infty$ functions with divergence free and curl free

$$
(2.28)\qquad \mathscr{D}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) = \{\varphi \in \mathscr{D}(\mathbb{R}^n;\mathbb{R}^n) : \operatorname{div}\varphi = 0\}
$$

and

$$
(2.29)\qquad \mathscr{D}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) = \{\varphi \in \mathscr{D}(\mathbb{R}^n;\mathbb{R}^n) : \operatorname{curl}\varphi = 0\},
$$

respectively. We have the following proposition.

PROPOSITION 2.5.

(1) *The space* $\mathscr{D}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ *is dense in* $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$.

(2) *The space* $\mathscr{D}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ *is dense in* $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. This is an immediate consequence of Proposition 2.1. ∎

In the following lemma we study some properties of the operators $P_{\mathrm{div}}(\mathrm{i}\,\nabla)$ and $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)$. These properties play an important role in the proof of some of the following propositions and theorems in the remainder of this paper.

LEMMA 2.6. *The following matrix-operators*

$$\begin{aligned}&(1)\quad P_{\mathrm{div}}(\mathrm{i}\,\nabla): X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n),\\ &(2)\quad P_{\mathrm{curl}}(\mathrm{i}\,\nabla): X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n),\end{aligned}$$

are continuous. Furthermore, for all $u, v \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\begin{aligned}&(P_{\mathrm{div}}(\mathrm{i}\,\nabla)u|P_{\mathrm{div}}(\mathrm{i}\,\nabla)v)_{m,s} = D_{m+2,s}(u,v),\\ &(P_{\mathrm{curl}}(\mathrm{i}\,\nabla)u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)v)_{m,s} = R_{m+2,s}(u,v),\\ &(P_{\mathrm{div}}(\mathrm{i}\,\nabla)u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)v)_{m,s} = 0.\end{aligned} \tag{2.30}$$

Proof. First, we will prove that $P_{\mathrm{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$.

Let $u \in P_{\mathrm{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. There exists $w = (w_1,\dots, w_n)^T$ in the space $X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $u = P_{\mathrm{div}}(\mathrm{i}\,\nabla)w = (-\sum_{k=1}^n \partial^2_{lk} w_k)_{1\le l\le n}$. It follows that

$$\begin{aligned}\mathrm{curl}\, u = (\partial_i u_j - \partial_j u_i)_{1\le i,j\le n} &= \left(-\partial_i\left(\sum_{k=1}^n \partial^2_{jk} w_k\right) + \partial_j\left(\sum_{k=1}^n \partial^2_{ik} w_k\right)\right)_{1\le i,j\le n}\\ &= \Big(-\partial^2_{ij}(\mathrm{div}\, w) + \partial^2_{ij}(\mathrm{div}\, w)\Big)_{1\le i,j\le n} = 0_n.\end{aligned}$$

Thus, $u \in X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, and $P_{\mathrm{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$.

Now let $u \in P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. Thus, there exists $w \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $u = P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w = -(\Delta I_n)w + \nabla(\mathrm{div}\, w)$. It follows that

$$\mathrm{div}\, u = \nabla^T\Big(-(\Delta I_n)w + \nabla(\mathrm{div}\, w)\Big) = -\Delta(\mathrm{div}\, w) + \Delta(\mathrm{div}\, w) = 0.$$

Then $u \in X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$, and $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$.

The continuity of the matrix-operators above is an immediate consequence of the continuity of the operators $\partial^2_{ij} : u \in X^{m+2,s}(\mathbb{R}^n) \longrightarrow \partial^2_{ij} u \in X^{m,s}(\mathbb{R}^n)$ for $1 \le i,j \le n$. Now we will give the proof of the relations in (2.30). According to (2.20), it follows that

$$\begin{aligned}D_{m+2,s}(u,v) = [\mathrm{div}\, u|\mathrm{div}\, v]_{m+1,s} &= \Big(\nabla(\mathrm{div}\, u)|\nabla(\mathrm{div}\, v)\Big)_{m,s}\\ &= (P_{\mathrm{div}}(\mathrm{i}\,\nabla)u|P_{\mathrm{div}}(\mathrm{i}\,\nabla)v)_{m,s}\end{aligned}$$

for all $u, v \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$.

The third relation in (2.30) is a consequence of the fact that $P_{\mathrm{curl}}(\xi)P_{\mathrm{div}}(\xi) = 0_n$. Now we show the second relation in (2.30). From the relation (2.21), we have

$$((\Delta I_n)u|(\Delta I_n)v)_{m,s} = (u|v)_{m+2,s}$$

for all $u, v \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$. The relations $P_{\mathrm{curl}}(\mathrm{i}\,\nabla) + P_{\mathrm{div}}(\mathrm{i}\,\nabla) = -(\Delta)I_n$ and the third relation in (2.30) provide

$$(P_{\mathrm{curl}}(\mathrm{i}\,\nabla)u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)v)_{m,s} + (P_{\mathrm{div}}(\mathrm{i}\,\nabla)u|P_{\mathrm{div}}(\mathrm{i}\,\nabla)v)_{m,s} = (u|v)_{m+2,s}.$$

It follows from (2.19) that

$$\begin{aligned}(P_{\mathrm{curl}}(\mathrm{i}\,\nabla)u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)v)_{m,s} &= (u|v)_{m+2,s} - (P_{\mathrm{div}}(\mathrm{i}\,\nabla)u|P_{\mathrm{div}}(\mathrm{i}\,\nabla)v)_{m,s}\\ &= (u|v)_{m+2,s} - D_{m+2,s}(u,v) = R_{m+2,s}(u,v). \qquad \square\end{aligned}$$

For $n = 2, 3$, the Hodge or Helmholtz-decomposition is given by (see, for instance, [14])

$$u = \nabla\psi \ + \ \operatorname{curl} v,$$

where $\operatorname{curl} v$ is defined by (2.10). We have $\operatorname{div}(\operatorname{curl} v) = 0$ and $\operatorname{curl}(\nabla\psi) = 0$. Namely, a distribution u is a sum of curl-free and divergence-free distributions.

The following proposition provides a divergence-curl decomposition in any dimensional space it is the Hodge or Helmholtz-like decomposition.

PROPOSITION 2.7. *We have the following decompositions*

$$\begin{aligned} X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) &= P_{\text{curl}}(\mathrm{i}\,\nabla)\left(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)\right) + P_{\text{div}}(\mathrm{i}\,\nabla)\left(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)\right) \\ &= X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n) + X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n), \end{aligned} \tag{2.31}$$

$$\begin{aligned} \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n) &= P_{\text{curl}}(\mathrm{i}\,\nabla)\left(\Pi_{m+1}(\mathbb{R}^n;\mathbb{R}^n)\right) + P_{\text{div}}(\mathrm{i}\,\nabla)\left(\Pi_{m+1}(\mathbb{R}^n;\mathbb{R}^n)\right) \\ &= \Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m-1,\text{curl}}(\mathbb{R}^n;\mathbb{R}^n). \end{aligned} \tag{2.32}$$

Proof. According to Proposition 2.3, for all $u \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$, there exists v in $X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $u = -(I_n\Delta)v$. Then

$$u = -(I_n\Delta)v = P_{\text{curl}}(\mathrm{i}\,\nabla)(v) + P_{\text{div}}(\mathrm{i}\,\nabla)(v),$$

and, consequently, we get the first equality of (2.31), namely,

$$X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) = P_{\text{curl}}(\mathrm{i}\,\nabla)\left(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)\right) + P_{\text{div}}(\mathrm{i}\,\nabla)\left(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)\right).$$

Since the inclusion $P_{\text{curl}}(\mathrm{i}\,\nabla)(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)) \subset X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$ and the second inclusion $P_{\text{div}}(\mathrm{i}\,\nabla)(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)) \subset X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ hold, then we get

$$\begin{aligned} X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) &= P_{\text{curl}}\left(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)\right) + P_{\text{div}}\left(X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)\right) \\ &\subset X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n) + X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n) \subset X^{m,s}(\mathbb{R}^n;\mathbb{R}^n), \end{aligned}$$

which proves the second equality of (2.31). To obtain the equalities (2.32), we again use similar arguments. ∎

The following proposition plays an important role in the proof of Proposition 2.9.

PROPOSITION 2.8. *We have the following characterizations:*

(1) $X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n) = P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] = P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)]$.
(2) $P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] = P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)]$.
(3) $X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n) = P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)] + \Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof.

(1) Using the relation (2.31) for $m+2$, we get

$$X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n) = X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n) + X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n).$$

Thus

$$\begin{aligned} &P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \\ &\qquad = P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)] + P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)]. \end{aligned}$$

Since $P_{\text{div}}(\mathrm{i}\,\nabla)w = -\nabla \text{div}\, w$, we obtain $P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)] = \{0\}$, and, consequently,

$$P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] = P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)].$$

From Lemma 2.6, we have $P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. Conversely, for $u = (u_1,\dots,u_n)^T \in X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, we have $\text{curl}\, u = 0_n$. The components of u satisfy the compatibility conditions $\partial_i u_j = \partial_j u_i$ for $1 \le i,j \le n$ (see [30]). It follows that there exists a scalar-valued function $f \in X^{m+1,s}(\mathbb{R}^n;\mathbb{R})$ such that $u = \nabla f$. Let $w = (w_1,\dots,w_n)^T$ be the vector-function defined by $w_1(t_1,\dots,t_n) = -\int_0^{t_1} f(s,t_2,\dots,t_n)ds$, and $w_i = 0$ for $i = 2,\dots,n$. Since f is continuous, the vector-function w is well defined. It is clear that $w \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$ and that it satisfies $\text{div}\, w = \partial_1 w_1 = -f$. It follows that $u = \nabla f = -\nabla(\text{div}\, w) = P_{\text{div}}(\mathrm{i}\,\nabla)w$, and, consequently,

$$X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n) = P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] = P_{\text{div}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)].$$

(2) Using again the decomposition

$$X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n) = X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n) + X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n),$$

we obtain

$$\begin{aligned} &P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \\ &\qquad = P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)] + P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)]. \end{aligned}$$

For all $w = (w_1,\dots,w_n)^T \in X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, we have $\partial_k w_l - \partial_l w_k = 0$ for $1 \le k,l \le n$, and applying ∂_k, we obtain $\partial^2_{kk} w_l - \partial_l \partial_k w_k = 0$ for $1 \le k,l \le n$. Summing over k, we get $\Delta w_l - \partial_l \text{div}\, w = 0$ for $1 \le l \le n$, namely, $P_{\text{curl}}(\mathrm{i}\,\nabla)w = -(\Delta I_n)w + \nabla(\text{div}\, w) = 0$. Thus, $P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)] = \{0\}$, and

$$P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] = P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)].$$

(3) Let $F = P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)]$. From Lemma 2.6, we obtain the following inclusion: $P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$. Using item 2 above, we obtain $F = P_{\text{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)] \subset X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$. It follows that

$$F + \Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n) \subset X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.33}$$

Now we will show that the space $X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$ is also included in the sum $F + \Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$. For all u and v in $X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$, we have the property $\text{div}\, u = \text{div}\, v = 0$, and then $D_{m+2,s}(u,v) = 0$. Using the relation (2.30), it follows that

$$\begin{aligned} (P_{\text{curl}}(\mathrm{i}\,\nabla)u | P_{\text{curl}}(\mathrm{i}\,\nabla)v)_{m,s} &= R_{m+2,s}(u,v) \\ &= R_{m+2,s}(u,v) + D_{m+2,s}(u,v) \\ &= (u|v)_{m+2,s}. \end{aligned} \tag{2.34}$$

The last relation (2.34) means that the operator

$$P_{\text{curl}}(\mathrm{i}\,\nabla) : X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$$

is an isometry (seminorm preserving linear map). As a consequence, this operator is continuous, and F is a closed subspace in the semi-Hilbert space $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ equipped with the semiscalar product $(\,.|.\,)_{m,s}$.

Let $F^\perp$ denote the orthogonal space of F in $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ equipped with the semiscalar product $(\,.|.\,)_{m,s}$. As F is a closed subspace in $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) = \overline{F} + F^\perp = F + F^\perp. \tag{2.35}$$

Now let u be any element in the orthogonal $F^\perp$ of F. Then

$$(u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)v)_{m,s} = 0 \tag{2.36}$$

for all $v \in X^{m+2,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$. According to the last item 2, we can see that the relation (2.36) also holds in the space $X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$; i.e.,

$$(u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)v)_{m,s} = 0 \quad \text{for all } v \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.37}$$

According to Proposition 2.3, there exists $w \in X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $u = -(\Delta I_n)\, w$. Thus

$$R_{m+2,s}(w,w) = (P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w)_{m,s}.$$

The third relation in (2.30) gives $(P_{\mathrm{div}}\,(\mathrm{i}\,\nabla)w|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w)_{m,s} = 0$. Then we get

$$R_{m+2,s}(w,w) = (P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w + P_{\mathrm{div}}\,(\mathrm{i}\,\nabla)w|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w)_{m,s}.$$

Using the fact that $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w + P_{\mathrm{div}}\,(\mathrm{i}\,\nabla)w = -(\Delta I_n)w = u$ and taking (2.37) into account, we obtain

$$R_{m+2,s}(w,w) = (u|P_{\mathrm{curl}}(\mathrm{i}\,\nabla)w)_{m,s} = 0.$$

Furthermore, $\mathrm{div}\, w \in X^{m+1,s}(\mathbb{R}^n)$, and $\Delta(\mathrm{div}\, w) = \mathrm{div}\,(\Delta I_n w) = -\mathrm{div}\, u = 0$. It follows that $|\xi|^2\widehat{\mathrm{div}\, w} = 0$. As $|\xi|^2$ vanishes only at the origin, then $\mathrm{div}\, w$ is a polynomial in $X^{m+1,s}(\mathbb{R}^n)$. Thus $\mathrm{div}\, w \in \Pi_m(\mathbb{R}^n)$, and

$$D_{m+2,s}(w,w) = (\mathrm{div}\, w|\mathrm{div}\, w)_{m+1,s} = 0.$$

We obtain

$$(w|w)_{m+2,s} = R_{m+2,s}(w,w) + D_{m+2,s}(w,w) = 0,$$

which implies that $w \in \Pi_{m+1}(\mathbb{R}^n;\mathbb{R}^n)$ and $u = -(\Delta I_n)w \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$. Since $\mathrm{div}\; u = 0$, then u is a polynomial in $\Pi_{m-1,\mathrm{div}}\,(\mathbb{R}^n;\mathbb{R}^n)$ and the inclusion $F^\perp \subset \Pi_{m-1,\mathrm{div}}\,(\mathbb{R}^n;\mathbb{R}^n)$ holds. From (2.35), we obtain

$$X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) = F + F^\perp \subset F + \Pi_{m-1,\mathrm{div}}\,(\mathbb{R}^n;\mathbb{R}^n). \tag{2.38}$$

We finally obtain from (2.33) and (2.38) that

$$X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) = P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)] + \Pi_{m-1,\mathrm{div}}\,(\mathbb{R}^n;\mathbb{R}^n),$$

which concludes the proof. ∎

Remark 2.1. One can notice that the sum in Item 3 of Proposition 2.8 is not direct. For instance, for $m \geq 1$ and $n = 2$, we consider the polynomial p in $\Pi_{m+1}(\mathbb{R}^n; \mathbb{R}^n)$ given by $p(x,y) = (x^2 - 2xy + 2x, y^2 - 2xy - 2y)^T$. We have $\operatorname{div} p = 0$, and then $p \in \Pi_{m+1,\mathrm{div}}(\mathbb{R}^n; \mathbb{R}^n)$. Let q be the polynomial given by $q = P_{\mathrm{curl}}(\mathrm{i}\,\nabla)p = -(\Delta I_n)p + \nabla(\operatorname{div} p) = -(\Delta I_n)p$. Then $q(x,y) = (-2,-2)^T$, and $q \in P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)] \cap \Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$. This example shows that $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)] \cap \Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) \neq \{0\}$, which means that the sum $P_{\mathrm{curl}}(\mathrm{i}\,\nabla)[X^{m+2,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)] + \Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$ is not direct.

Consider the function $K_{m,s}$ defined by

$$K_{m,s}(x) = \begin{cases} c_{1,m,s}|x|^{2m+2s-n}\log(|x|) & \text{if} \quad m+s-n/2 \in \mathbb{N}^\star, \\ c_{2,m,s}|x|^{2m+2s-n} & \text{if} \quad m+s-n/2 \notin \mathbb{N}^\star, \end{cases} \tag{2.39}$$

where $c_{1,m,s}$ and $c_{2,m,s} > 0$ are real constants. It is clear that the function $K_{m,s}$ is in $\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$. It was shown in [3, 29] that $\partial^\alpha K_{m,s}(0) = 0$ for $|\alpha| \leq \eta$, where

$$\eta = \begin{cases} 2m+2s-n-1 & \text{for} \quad 2m+2s-n \in \mathbb{N}^\star, \\ \lfloor 2m+2s-n \rfloor & \text{otherwise.} \end{cases} \tag{2.40}$$

Here, the notation $\lfloor r \rfloor$ stands for the integer part of the real r. Thus, the function $K_{m,s}$ belongs to $\mathscr{C}^\eta(\mathbb{R}^n)$. It is well known that the function $K_{m,s}$ defines a tempered distribution on $\mathbb{R}^n$ also denoted by $K_{m,s}$. The exact values of the constants $c_{1,m,s}$ and $c_{2,m,s}$ appearing in the expression (2.39) are of purely theoretical importance and are not needed in practice. In fact, the values of $c_{1,m,c}$ and $c_{2,m,s}$ are such that the Fourier transform $\widehat{K}_{m,s}$ satisfies the following relation:

$$|\xi|^{2m+2s}\widehat{K}_{m,s}(\xi) = 1. \tag{2.41}$$

Let us now introduce the matrix-functions $G_{m,s}$ and $H_{m,s}$ defined by

$$G_{m,s} = P_{\mathrm{div}}(\mathrm{i}\,\nabla)K_{m+1,s} = \left(-\nabla\,\nabla^T\right)K_{m+1,s} = \left(-\partial^2_{l,k}K_{m+1,s}\right)_{1\leq l,k\leq n} \tag{2.42}$$

and

$$\begin{aligned} H_{m,s} = P_{\mathrm{curl}}(\mathrm{i}\,\nabla)K_{m+1,s} &= \left(-\Delta I_n + \nabla\,\nabla^T\right)K_{m+1,s} \\ &= \left(-\delta_{l,k}\Delta K_{m+1,s} + \partial^2_{l,k}K_{m+1,s}\right)_{1\leq l,k\leq n}, \end{aligned} \tag{2.43}$$

where $K_{m+1,s}$ is given by (2.39).

PROPOSITION 2.9.

(1) *For any vector-measure $\omega = (\omega_1, \ldots, \omega_n)^T$ with compact support and orthogonal to the space $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, the convolution matrix-vector product $G_{m,s} * \omega$ belongs to $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, and for all $u \in X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$,*

$$D_{m,s}(G_{m,s} * \omega,\, u) = \langle \omega, u \rangle. \tag{2.44}$$

(2) *For any vector-measure $\omega = (\omega_1, \ldots, \omega_n)^T$ with compact support and orthogonal to the space $\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$, the convolution matrix-vector product $H_{m,s} * \omega$ belongs to $X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$, and for all $u \in X^{m,s}_{\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$,*

$$R_{m,s}(H_{m,s} * \omega,\, u) = \langle \omega, u \rangle. \tag{2.45}$$

Proof.

(1) For all $p \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$, the polynomial $\mathrm{q} = P_{\mathrm{div}}(\mathrm{i}\,\nabla)p \in \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and ω is orthogonal to the space $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. Thus, the vector-measure $\omega' = P_{\mathrm{div}}(\mathrm{i}\,\nabla)\omega$ satisfies

$$\langle \omega', p\rangle = \langle P_{\mathrm{div}}(\mathrm{i}\,\nabla)\omega, p\rangle = \langle \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)p\rangle = 0 \quad \text{for all } p \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n).$$

Namely, ω' is orthogonal to the space $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$. We have

$$\begin{aligned} G_{m,s} * \omega &= P_{\mathrm{div}}(\mathrm{i}\,\nabla)K_{m+1,s} * \omega = K_{m+1,s} * P_{\mathrm{div}}(\mathrm{i}\,\nabla)\omega \\ &= K_{m+1,s} * \omega' = (K_{m+1,s} * \omega_1', \ldots, K_{m+1,s} * \omega_1')^T, \end{aligned}$$

where ω_i' are the components of ω'. According to Proposition 2 in [6], the convolution matrix-vector product $G_{m,s} * \omega$ belongs to $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. Since

$$G_{m,s} * \omega = P_{\mathrm{div}}(\mathrm{i}\,\nabla)(K_{m+1,s} * \omega),$$

it follows that $\mathrm{curl}(G_{m,s} * \omega) = 0_n$ and $G_{m,s} * \omega$ is in $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$.
Now for all $v = \varphi + p \in \mathscr{D}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m+1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\begin{aligned} D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) &= D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)(\varphi + p)) \\ &= D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)\varphi). \end{aligned}$$

According to Proposition 2.2, we get

$$D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) = \langle |\xi|^{2(m-1)+2s} P_{\mathrm{div}}(\xi)\mathcal{F}(G_{m,s} * \omega), P_{\mathrm{div}}(\xi)\overline{\widehat{\varphi}}\rangle.$$

By using the relation (2.17), the Fourier transform of the convolution product $G_{m,s} * \omega$ is $\mathcal{F}(G_{m,s} * \omega) = P_{\mathrm{div}}(\xi)\widehat{K}_{m+1,s}\widehat{\omega}$. Furthermore, using the first relation in (2.13), we get

$$\begin{aligned} D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) &= \langle P_{\mathrm{div}}(\xi)|\xi|^{2m+2s}\widehat{K}_{m+1,s}\widehat{\omega}, P_{\mathrm{div}}(\xi)\overline{\widehat{\varphi}}\rangle \\ &= \langle P_{\mathrm{div}}(\xi)|\xi|^{2(m+1)+2s}\widehat{K}_{m+1,s}\widehat{\omega}, \overline{\widehat{\varphi}}\rangle. \end{aligned}$$

By taking the relations (2.41) and (2.17) into account, we obtain

$$D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) = \langle \widehat{\omega}, P_{\mathrm{div}}(\xi)\overline{\widehat{\varphi}}\rangle = \langle \widehat{\omega}, \overline{\widehat{P_{\mathrm{div}}(\mathrm{i}\,\nabla)\varphi}}\rangle.$$

According to the fact that $\langle \widehat{T}, \overline{\widehat{\varphi}}\rangle = \langle T, \varphi\rangle$ for a tempered distribution T, we get $D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) = \langle \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)\varphi\rangle$. Since the vector-measure ω is orthogonal to $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and $P_{\mathrm{div}}(\mathrm{i}\,\nabla)p \in \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, we have $\langle \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)p\rangle = 0$. Thus

$$D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) = \langle \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)(\varphi + p)\rangle = \langle \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v\rangle$$

for all $v \in \mathscr{D}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m+1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. The density of the subspace $\mathscr{D}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) + \Pi_{m+1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ in $X^{m+2,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ together with the continuity of the operator $P_{\mathrm{div}}(\mathrm{i}\,\nabla) : X^{m+2,s}(\mathbb{R}^n;\mathbb{R}^n) \to X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ provide that the relation

$$D_{m,s}(G_{m,s} * \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v) = \langle \omega, P_{\mathrm{div}}(\mathrm{i}\,\nabla)v\rangle$$

holds for all $v \in X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. According to Proposition 2.8, for all $u \in X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, there exists $v \in X^{m+2,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ such that $u = P_{\text{div}}(\mathrm{i}\,\nabla)v$. Thus

$$D_{m,s}(G_{m,s} * \omega, u) = \langle \omega, u\rangle \quad \text{for all } u \in X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n).$$

(2) For the second item, by similar arguments as in item 1, we show that $H_{m,s} * \omega$ belongs to $X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$ for any vector-measure ω which is orthogonal to $\Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$ and that the relation

$$R_{m,s}(H_{m,s} * \omega, P_{\text{curl}}(\mathrm{i}\,\nabla)v) = \langle \omega, P_{\text{curl}}(\mathrm{i}\,\nabla)v\rangle$$

holds for all $v \in X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$. According to Proposition 2.8, for all $u \in X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$, there exist $v \in X^{m+2,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$ and $p \in \Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$ such that $u = P_{\text{curl}}(\mathrm{i}\,\nabla)v + p$. Thus

$$\begin{aligned} R_{m,s}(H_{m,s} * \omega, u) &= R_{m,s}(H_{m,s} * \omega, P_{\text{curl}}(\mathrm{i}\,\nabla)v + p) \\ &= R_{m,s}(H_{m,s} * \omega, P_{\text{curl}}(\mathrm{i}\,\nabla)v) \\ &= \langle \omega, P_{\text{curl}}(\mathrm{i}\,\nabla)v\rangle. \end{aligned}$$

Since ω is orthogonal to the space $\Pi_{m-1,\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$, then $\langle \omega, p\rangle = 0$. It follows that

$$R_{m,s}(H_{m,s} * \omega, u) = \langle \omega, P_{\text{curl}}(\mathrm{i}\,\nabla)v + p\rangle = \langle \omega, u\rangle$$

for all $u \in X^{m,s}_{\text{div}}(\mathbb{R}^n;\mathbb{R}^n)$. □

3. Pseudo-polyharmonic curl-free approximation.

In this section, we study the curl-free approximation problem. Suppose we are given a collection of N distinct points $x_1, \dots, x_N$ in $\mathbb{R}^n$, such that the set $\Omega_N = \{x_1, \dots, x_N\}$ contains a $\Pi_{m-1}(\mathbb{R}^n)$-unisolvent subset. We recall that a set Ω is $\Pi_{m-1}(\mathbb{R}^n)$-unisolvent if any polynomial in $\Pi_{m-1}(\mathbb{R}^n)$ which vanishes on Ω is identically zero. We consider the linear Lagrange operator $L : X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}^{N\times n}$ given for $u \in X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ by

$$Lu = \begin{pmatrix} u_1(x_1) & \dots & u_n(x_1) \\ \vdots & & \vdots \\ u_1(x_N) & \dots & u_n(x_N) \end{pmatrix}.$$

Let $Z \in \mathbb{R}^{N\times n}$ be an $N \times n$ real matrix, and let $\varepsilon \geq 0$. We consider the following approximation problem

$$\min_{v \in \mathcal{I}^{m,s}_{\varepsilon}(Z)} \Big(D_{m,s}(v,v) + \varepsilon ||Lv - Z||^2_{N\times n} \Big), \tag{3.1}$$

where

$$\mathcal{I}^{m,s}_{\varepsilon}(Z) = \begin{cases} L^{-1}(Z) & \text{for} \quad \varepsilon = 0 \quad \text{(interpolating problem)}, \\ X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n) & \text{for} \quad \varepsilon > 0 \quad \text{(smoothing problem)}, \end{cases} \tag{3.2}$$

and

$$L^{-1}(Z) = \{ v \in X^{m,s}_{\text{curl}}(\mathbb{R}^n;\mathbb{R}^n) : L\,v = Z \}. \tag{3.3}$$

We have the following theorem.

THEOREM 3.1. *For all $Z \in \mathbb{R}^{N\times n}$ and $\varepsilon \geq 0$, there is a unique solution $\sigma^{c,\varepsilon}$ of the problem* (3.1). *The solution $\sigma^{c,\varepsilon}$ is the unique element of $X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ characterized by*

$$D_{m,s}(\sigma^{c,0}, v) = 0 \text{ for all } v \in \ker(L) \tag{3.4}$$

for the interpolating problem ($\varepsilon = 0$) and

$$D_{m,s}(\sigma^{c,\varepsilon}, v) + \varepsilon\langle L\sigma^{c,\varepsilon} - Z|Lv\rangle_{N\times n} = 0 \text{ for all } v \in X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) \tag{3.5}$$

for the smoothing problem ($\varepsilon > 0$). Furthermore, there are unique coefficients $\alpha^{\varepsilon}_{ij}$ for $i = 1,\dots,N$ and $j = 1,\dots,n$ such that the unique vector-measure $\omega^{\varepsilon,c}$ of the form

$$\omega^{\varepsilon,c} = \left(\sum_{i=1}^{N} \alpha^{(\varepsilon)}_{i1}\delta_{x_i},\dots,\sum_{i=1}^{N} \alpha^{(\varepsilon)}_{in}\delta_{x_i}\right)^T$$

is orthogonal to the space $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and satisfies

$$D_{m,s}(\sigma^{c,\varepsilon}, v) = \langle\omega^{\varepsilon,c}, v\rangle = \sum_{j=1}^{n}\sum_{i=1}^{N} \alpha^{(\varepsilon)}_{ij} v_j(x_i) \text{ for all } v \in X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n). \tag{3.6}$$

Proof. According to inequality (29) of Proposition 7 in [6], the symmetric positive bilinear form $D_{m,s}$ is continuous. Then there exists a positive and symmetric continuous linear operator $S_d : X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) \to X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$D_{m,s}(u, v) = (S_d u|v)_{m,s} \text{ for all } u, v \in X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n).$$

The operator S_d admits a symmetric positive square root. Namely, there exists a symmetric and positive continuous linear operator $T_d : X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) \to X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ such that $S_d = T_d^2$. In consequence, we have

$$D_{m,s}(u, u) = (T_d^2 u|u)_{m,s} = (T_d u|T_d u)_{m,s} = |T_d u|^2_{m,s} \text{ for all } u \in X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n).$$

The operators L and T_d satisfy the following properties:

(1) The operator $L : X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}^{N\times n}$ is continuous.
(2) The spaces $\ker(T_d) = \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and $T_d(X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n))$ are closed.
(3) The space $\ker(T_d) + \ker(L)$ is closed.
(4) $\ker(T_d) \cap \ker(L) = \{0\}$.

The proof of the previous properties is similar to the one of Theorem 1 in [6]. Now we just have to prove that the operator $L : X^{m,s}_{\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}^{N\times n}$ is surjective. As the points $x_1,\dots,x_N$ are distinct, then it is possible to find functions $\phi_1,\dots,\phi_N$ in $\mathscr{D}(\mathbb{R}^n)$ such that the following conditions hold:

$$\phi_i(x_j) = \delta_{ij}, \quad (\partial_k\phi_i)(x_j) = 0, \quad i, j = 1,\dots,N, \quad k = 1,\dots,n. \tag{3.7}$$

Let $Z = (Z_{i,j})_{\substack{1\leq i\leq N\\ 1\leq j\leq n}}$ be any $N \times n$ matrix in $\mathbb{R}^{N\times n}$, and let $Z_i = (Z_{i1},\dots,Z_{in})$ be the ith row of Z. We introduce the functions θ_i in $\mathscr{D}(\mathbb{R}^n)$ and the functions Φ_i in $\mathscr{D}(\mathbb{R}^n;\mathbb{R}^n)$ given for $1 \leq i \leq N$ by

$$\begin{aligned}\theta_i(x) &= \langle Z_i|x\rangle_n\phi_i(x),\\ \Phi_i(x) &= (\nabla\theta_i)(x) = Z_i\phi_i(x) + \langle Z_i|x\rangle_n(\nabla\phi_i)(x).\end{aligned}$$

Then $\Phi_i(x_j) = \delta_{ij} Z_i$ and $\operatorname{curl}\Phi = \operatorname{curl}(\nabla\theta_i) = 0_n$. It follows that the function $\Phi(x) = \sum_{i=1}^N \Phi_i(x)$ is an element of $\mathscr{D}(\mathbb{R}^n;\mathbb{R}^n)$ satisfying $L\Phi = Z$ and $\operatorname{curl}\Phi = 0$.

According to the general spline theory (see [4, 7, 24]), the variational problem (3.1) has a unique solution $\sigma^{c,\varepsilon}$. Since $(T_d\sigma^{c,\varepsilon}|T_d v)_{m,s} = D_{m,s}(\sigma^{c,\varepsilon}, v)$, the solution $\sigma^{c,\varepsilon}$ satisfies the characterization given by (3.4) and (3.5). The characterization (3.6) is obtained by similar arguments as in the proof of Proposition 3.2 in [5]. ∎

Remark 3.1. For $m \geq 2$, using the first relation in (2.30), we get that $\sigma^{c,\varepsilon}$ is also the solution of the curl-free approximation problem (3.1) by taking the quadratic form $(P_{\mathrm{div}}\,(\mathrm{i}\,\nabla)u|P_{\mathrm{div}}\,(\mathrm{i}\,\nabla)u)_{m-2,s}$ instead of $D_{m,s}(u,u)$.

Let $d_1 = \frac{(m+n)!}{n!(m-1)!} - 1$ be the dimension and $(P_1, \ldots, P_{d_1})$ be a basis of the space $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, respectively. The vector-polynomial P_l is given by its components $P_l = (p_{l,1}, \ldots, p_{l,n})^T$ for $l = 1, \ldots, d_1$.

THEOREM 3.2. *There are unique vectors $V_i^{(\varepsilon)} \in \mathbb{R}^n$ for $i = 1, \ldots, N$ and a unique vector $\beta^{(\varepsilon)} = (\beta_1^{(\varepsilon)}, \ldots, \beta_{d_1}^{(\varepsilon)})^T \in \mathbb{R}^{d_1}$ such that the unique solution $\sigma^{c,\varepsilon}$ of the problem* (3.1) *is explicitly given by*

$$\sigma^{c,\varepsilon}(x) = \sum_{i=1}^N G_{m,s}(x - x_i) V_i^{(\varepsilon)} + \sum_{l=1}^{d_1} \beta_l^{(\varepsilon)} P_l(x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{3.8}$$

Proof. Using Theorem 3.1, there exists a unique vector-measure $\omega^{\varepsilon,c}$ of the form

$$\omega^{\varepsilon,c} = \left(\sum_{j=1}^N \alpha_{1j}^{(\varepsilon)} \delta_{x_j}, \ldots, \sum_{j=1}^N \alpha_{nj}^{(\varepsilon)} \delta_{x_j}\right)^T,$$

which is orthogonal to the space $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ and satisfies the condition (3.6). Let $u^{\varepsilon,c} = G_{m,s} * \omega^{\varepsilon,c}$. Proposition 2.9 implies that $u^{\varepsilon,c} \in X_{\mathrm{curl}}^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ and satisfies

$$D_{m,s}(u^{\varepsilon,c}, v) = D_{m,s}(G_{m,s} * \omega^{\varepsilon,c}, v) = \langle \omega^{\varepsilon,c}, v\rangle \text{ for all } v \in X_{\mathrm{curl}}^{m,s}(\mathbb{R}^n;\mathbb{R}^n).$$

Hence, taking (3.6) into account, we obtain

$$D_{m,s}(u^{\varepsilon,c}, v) = \langle \omega^{\varepsilon,c}, v\rangle = D_{m,s}(\sigma^{c,\varepsilon}, v) \text{ for all } v \in X_{\mathrm{curl}}^{m,s}(\mathbb{R}^n;\mathbb{R}^n).$$

Thus $D_{m,s}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}, v) = 0$ for all $v \in X_{\mathrm{curl}}^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. In particular, for $v = \sigma^{c,\varepsilon} - u^{\varepsilon,c}$, we get $D_{m,s}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}, \sigma^{c,\varepsilon} - u^{\varepsilon,c}) = 0$. As $\operatorname{curl}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}) = 0_n$, it follows that $R_{m,s}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}, \sigma^{c,\varepsilon} - u^{\varepsilon,c}) = 0$. Thus

$$|\sigma^{c,\varepsilon} - u^{\varepsilon,c}|_{m,s}^2 = D_{m,s}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}, \sigma^{c,\varepsilon} - u^{\varepsilon,c}) + R_{m,s}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}, \sigma^{c,\varepsilon} - u^{\varepsilon,c}) = 0,$$

and $\sigma^{c,\varepsilon} - u^{\varepsilon,c}$ is a polynomial in $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ satisfying $\operatorname{curl}(\sigma^{c,\varepsilon} - u^{\varepsilon,c}) = 0_n$. Namely, $\sigma^{c,\varepsilon} - u^{\varepsilon,c} \in \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. Then there exists a polynomial $P_\varepsilon \in \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$ such that $\sigma^{c,\varepsilon} = u^{\varepsilon,c} + P_\varepsilon = G_{m,s} * \omega^{\varepsilon,c} + P_\varepsilon$. It is easy to see that $u^{\varepsilon,c}(x) = G_{m,s} * \omega^{\varepsilon,c}(x) = \sum_{i=1}^N G_{m,s}(x - x_i) V_i^{(\varepsilon)}$, where the vectors $V_i^{(\varepsilon)} = (\alpha_{i1}^{(\varepsilon)}, \ldots, \alpha_{in}^{(\varepsilon)})^T$ are in $\mathbb{R}^n$ for $i = 1, \ldots, N$ and the coefficients $\alpha_{ij}^{(\varepsilon)}$ are given in Theorem 3.1. Since $P_\varepsilon \in \Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)$, there exists a unique vector $\beta^{(\varepsilon)} = (\beta_1^{(\varepsilon)}, \ldots, \beta_{d_1}^{(\varepsilon)})^T \in \mathbb{R}^{d_1}$ such that $P_\varepsilon = \sum_{l=1}^{d_1} \beta_l^{(\varepsilon)} P_l$. ∎

The computational method for solving the problem (3.1) may be summarized in the following theorem.

THEOREM 3.3. *Let $Z = (z_{ij})_{\substack{1\le i\le N\\ 1\le j\le n}}$ be a given real matrix of size $N \times n$. The solution $\sigma^{c,\varepsilon} = (\sigma_1^{c,\varepsilon}, \dots, \sigma_n^{c,\varepsilon})^T$ of the problem* (3.1) *relative to Z is explicitly given by*

$$\sigma_k^{c,\varepsilon}(x) = \sum_{j=1}^{n}\sum_{i=1}^{N} \alpha_{i,j}^{(\varepsilon)} \left[\partial_{j,k}^2 K_{m+1,s}(x - x_i)\right] + \sum_{l=1}^{d_1} \beta_l^{(\varepsilon)} p_{l,k}(x) \tag{3.9}$$

for $k = 1, \dots, n$. The coefficients $\alpha_{i,j}^{(\varepsilon)}$ and $\beta_l^{(\varepsilon)}$ are computed by solving the following nonsingular linear system of size $(nN + d_1) \times (nN + d_1)$

$$\begin{pmatrix} \mathbb{K} + c_\varepsilon \mathbb{I}_{nN} & \mathbb{M} \\ \mathbb{M}^T & \mathbb{O} \end{pmatrix} \begin{pmatrix} \alpha^{(\varepsilon)} \\ \beta^{(\varepsilon)} \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{0} \end{pmatrix} \quad \text{with} \quad c_\varepsilon = \begin{cases} 0 \text{ if } \varepsilon = 0, \\ \dfrac{1}{\varepsilon} \text{ if } \varepsilon > 0, \end{cases} \tag{3.10}$$

where $\alpha^{(\varepsilon)}$ and $\beta^{(\varepsilon)}$ are the vectors given by

$$\alpha^{(\varepsilon)} = (\alpha_{1,1}^{(\varepsilon)}, \dots, \alpha_{N,1}^{(\varepsilon)}, \dots, \alpha_{1,n}^{(\varepsilon)}, \dots, \alpha_{N,n}^{(\varepsilon)})^T \in \mathbb{R}^{nN},$$
$$\beta^{(\varepsilon)} = (\beta_1^{(\varepsilon)}, \dots, \beta_{d_1}^{(\varepsilon)})^T \in \mathbb{R}^{d_1}.$$

The vector $\mathbf{z} = (z_{1,1}, \dots, z_{N,1}, \dots, z_{1,n}, \dots, z_{N,n})^T \in \mathbb{R}^{nN}$ is obtained by stacking the columns of the matrix Z, $\mathbf{0}$ is the zero vector in $\mathbb{R}^{d_2}$, $\mathbb{I}_{nN}$ is the identity matrix of size $nN \times nN$, and $\mathbb{O}$ is the zero matrix of size $d_1 \times d_1$. The $nN \times nN$ matrix $\mathbb{K} = (\mathbb{K}_{l,k})_{1\le l,k\le n}$ is given by the blocks $\mathbb{K}_{l,k}$ of size $N \times N$, where

$$\mathbb{K}_{l,k} = \left[\partial_{l,k}^2 K_{m+1,s}(x_i - x_j)\right]_{\substack{1\le i\le N\\ 1\le j\le N}},$$

and the $nN \times d_1$ matrix $\mathbb{M} = (\mathbb{M}_1^T, \dots, \mathbb{M}_n^T)^T$ is given by the $N \times d_1$ blocks $\mathbb{M}_k = [p_{j,k}(x_i)]_{\substack{1\le i\le N\\ 1\le j\le d_1}}$ for $k = 1, \dots, n$.

Proof. From Theorem 3.2 we have the explicit expression (3.8). With respect to the interpolating (respectively, smoothing) conditions together with the orthogonality conditions, we obtain the linear system (3.10). ◻

The following theorem gives some results about the regularity of $\sigma^{c,\varepsilon}$ and the convergence as $\varepsilon \to \infty$ or $\varepsilon \to 0$.

THEOREM 3.4. *The solution $\sigma^{c,\varepsilon}$ of the problem* (3.1) *has the following properties:*

(1) *The solution $\sigma^{c,\varepsilon}$ of the problem* (3.1) *belongs to $\mathscr{C}^\eta(\mathbb{R}^n; \mathbb{R}^n)$, where η is the integer given by* (2.40).

(2) $\lim_{\varepsilon\to+\infty} \sigma^{c,\varepsilon} = \sigma^{c,0}$ *and* $\lim_{\varepsilon\to 0} \sigma^{c,\varepsilon} = p^{c,0}$ *in $X_{\mathrm{curl}}^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$, where the vector-polynomial $p^{c,0}$ belonging to $\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n; \mathbb{R}^n)$ is the unique solution of the least-squares problem*

$$\min_{p\in\Pi_{m-1,\mathrm{curl}}(\mathbb{R}^n;\mathbb{R}^n)} ||Lp - Z||_{N\times n}.$$

Proof.

(1) The function $K_{m+1,s}$ belongs to $\mathscr{C}^{\eta+2}(\mathbb{R}^n)$, and taking (2.42) and (3.9) into account, we obtain that $\sigma^{c,\varepsilon} \in \mathscr{C}^\eta(\mathbb{R}^n; \mathbb{R}^n)$.

(2) It is a consequence of general spline theory (see [4, 24]). ◻

4. Pseudo-polyharmonic divergence-free approximation. In this section, we study the divergence-free approximation problem. As the proofs in this section are similar to those given in section 3, we just give here the results concerning the divergence-free problem. Suppose we are given a collection of N distinct points $x_1, \ldots, x_N$ in $\mathbb{R}^n$ such that the set $\Omega_N = \{x_1, \ldots, x_N\}$ contains a $\Pi_{m-1}(\mathbb{R}^n)$-unisolvent subset.

We consider the linear Lagrange operator $L : X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}^{N\times n}$ given for $u \in X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n)$ by

$$Lu = \begin{pmatrix} u_1(x_1) & \ldots & u_n(x_1) \\ \vdots & & \vdots \\ u_1(x_N) & \ldots & u_n(x_N) \end{pmatrix}.$$

Let $Z \in \mathbb{R}^{N\times n}$ be an $N \times n$ real matrix, and let $\varepsilon \geq 0$. We consider the following approximation problem

$$\min_{v \in \mathcal{I}^{m,s}_{\varepsilon}(Z)} \Big(R_{m,s}(v, v) + \varepsilon ||Lv - Z||^2_{N\times n} \Big), \tag{4.1}$$

where

$$\mathcal{I}^{m,s}_{\varepsilon}(Z) = \begin{cases} L^{-1}(Z) & \text{for} \quad \varepsilon = 0 \quad \text{(interpolating problem)}, \\ X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n) & \text{for} \quad \varepsilon > 0 \quad \text{(smoothing problem)} \end{cases} \tag{4.2}$$

and

$$L^{-1}(Z) = \{v \in X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n) : \, L\, v = Z\}. \tag{4.3}$$

We have the following theorem.

THEOREM 4.1. *For all $Z \in \mathbb{R}^{N\times n}$ and $\varepsilon \geq 0$, there is a unique solution $\sigma^{d,\varepsilon}$ of the problem* (4.1). *The solution $\sigma^{d,\varepsilon}$ is the unique element of $X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n)$ characterized by*

$$R_{m,s}(\sigma^{d,0}, v) = 0 \textit{ for all } v \in \ker(L) \tag{4.4}$$

for the interpolating problem ($\varepsilon = 0$) *and*

$$R_{m,s}(\sigma^{d,\varepsilon}, v) + \varepsilon \langle L\sigma^{d,\varepsilon} - Z | Lv \rangle_{N\times n} = 0 \textit{ for all } v \in X^{m,s}_{\text{curl}}(\mathbb{R}^n; \mathbb{R}^n) \tag{4.5}$$

for the smoothing problem ($\varepsilon > 0$). *Furthermore, there are unique coefficients $\alpha^{\varepsilon}_{ij}$ such that the unique vector-measure $\omega^{\varepsilon,d}$ of the form*

$$\omega^{\varepsilon,d} = \left(\sum_{i=1}^{N} \alpha^{(\varepsilon)}_{i1} \delta_{x_i}, \ldots, \sum_{i=1}^{N} \alpha^{(\varepsilon)}_{in} \delta_{x_i} \right)^T$$

is orthogonal to the space $\Pi_{m-1,\text{div}}(\mathbb{R}^n; \mathbb{R}^n)$ and satisfies

$$R_{m,s}(\sigma^{d,\varepsilon}, v) = \langle \omega^{\varepsilon,d}, v \rangle = \sum_{j=1}^{n} \sum_{i=1}^{N} \alpha^{\varepsilon}_{ij} v_j(x_i) \textit{ for all } v \in X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n). \tag{4.6}$$

Proof. The proof of this theorem is similar to one of Theorem 3.1. We just have to prove that the operator $L : X^{m,s}_{\text{div}}(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}^{N\times n}$ is surjective. As the points $x_1, \ldots, x_N$ are distinct, then it is possible to find functions $\phi_1, \ldots, \phi_N$ in $\mathscr{D}(\mathbb{R}^n)$ satisfying the conditions given by (3.7). Let $Z = (Z_{i,j})_{\substack{1\leq i\leq N \\ 1\leq j\leq n}}$ be any $N \times n$ matrix in

$\mathbb{R}^{N\times n}$, and let $Z_i = (Z_{i1}, \ldots, Z_{in})$ be the ith row of Z. We introduce the functions V_i, W_i, and Φ_i in $\mathscr{D}(\mathbb{R}^n; \mathbb{R}^n)$ given for $1 \le i \le N$ and $x = (t_1, \ldots, t_n)^T \in \mathbb{R}^n$ by

$$\begin{aligned} V_i(x) &= (Z_{i,1}t_2\partial_2\phi_i(x), Z_{i,2}t_3\partial_3\phi_i(x), \ldots, Z_{i,n-1}t_n\partial_n\phi_i(x), Z_{i,n}t_1\partial_1\phi_i(x))^T,\\ W_i(x) &= (t_1Z_{i,n}\partial_n\phi_i(x), t_2Z_{i,1}\partial_1\phi_i(x), t_3Z_{i,2}\partial_2\phi_i(x), \ldots, t_nZ_{i,n-1}\partial_{n-1}\phi_i(x))^T,\\ \Phi_i(x) &= Z_i\phi_i(x) + V_i(x) - W_i(x). \end{aligned}$$

Then $\Phi_i(x_j) = \delta_{ij}Z_i$ and

$$\begin{aligned} \operatorname{div}(Z_i\phi_i) &= \sum_{j=1}^{n} Z_{ij}\partial_j\phi_i,\\ \operatorname{div}(V_i) &= \sum_{j=1}^{n-1} Z_{ij}t_{j+1}\partial^2_{j,j+1}\phi_i + Z_{in}t_1\partial^2_{1n}\phi_i,\\ \operatorname{div}(W_i) &= \sum_{j=1}^{n} Z_{ij}\partial_j\phi_i + \sum_{j=1}^{n-1} Z_{ij}t_{j+1}\partial^2_{j,j+1}\phi_i + Z_{in}t_1\partial^2_{1n}\phi_i. \end{aligned}$$

It follows that $\operatorname{div}(\Phi_i) = 0$ and that the function $\Phi(x) = \sum_{i=1}^N \Phi_i(x)$ is an element of $\mathscr{D}(\mathbb{R}^n; \mathbb{R}^n)$ satisfying $L\Phi = Z$ and $\operatorname{div}\Phi = 0$. ∎

Remark 4.1. For $m \ge 2$, using the second relation in (2.30), we get that $\sigma^{d,\varepsilon}$ is also the solution of the divergence-free approximation problem (4.1) by taking the quadratic form $(P_{\mathrm{curl}}u|P_{\mathrm{curl}}u)_{m-2,s}$ instead of $R_{m,s}(u,u)$.

Let $d_2 = \frac{(m+n-2)!}{n!(m-1)!}(n^2 + (n-1)(m-1))$ be the dimension and $(P_1, \ldots, P_{d_2})$ be a basis of the space $\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n; \mathbb{R}^n)$, respectively. For $l = 1, \ldots, d_2$, the vector-polynomial P_l is given by its components $P_l = (p_{l,1}, \ldots, p_{l,n})^T$.

Theorem 4.2. *There are unique vectors $V_i^{(\varepsilon)} \in \mathbb{R}^n$ for $i = 1, \ldots, N$ and a unique vector $\beta^{(\varepsilon)} = (\beta_1^{(\varepsilon)}, \ldots, \beta_{d_2}^{(\varepsilon)})^T \in \mathbb{R}^{d_2}$ such that the unique solution $\sigma^{d,\varepsilon}$ of the problem* (4.1) *is explicitly given by*

$$\sigma^{d,\varepsilon}(x) = \sum_{i=1}^{N} H_{m,s}(x-x_i)V_i^{(\varepsilon)} + \sum_{l=1}^{d_2} \beta_l^{(\varepsilon)}P_l(x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{4.7}$$

Proof. The proof is similar to the one of Theorem 3.2. ∎

The computational method for solving the problem (4.1) may be summarized in the following theorem.

Theorem 4.3. *Let $Z = (z_{ij})_{\substack{1\le i\le N\\ 1\le j\le n}}$ be a given real matrix of size $N \times n$. The solution $\sigma^{d,\varepsilon} = (\sigma_1^{d,\varepsilon}, \ldots, \sigma_n^{d,\varepsilon})^T$ of the problem* (4.1) *relative to Z is explicitly given by*

(4.8)

$$\sigma_k^{d,\varepsilon}(x) = \sum_{j=1}^{n}\sum_{i=1}^{N} \alpha_{i,j}^{(\varepsilon)}\Big[-\delta_{j,k}\Delta K_{m+1,s}(x-x_i) + \partial^2_{j,k}K_{m+1,s}(x-x_i)\Big] + \sum_{l=1}^{d_2}\beta_l^{(\varepsilon)}p_{l,k}(x)$$

for $k = 1, \ldots, n$. The coefficients $\alpha_{i,j}^{(\varepsilon)}$ and $\beta_l^{(\varepsilon)}$ are computed by solving the following nonsingular linear system of size $(nN + d_2) \times (nN + d_2)$

$$\begin{pmatrix} \mathbb{K} + c_\varepsilon \mathbb{I}_{nN} & \mathbb{M} \\ \mathbb{M}^T & \mathbb{O} \end{pmatrix} \begin{pmatrix} \alpha^{(\varepsilon)} \\ \beta^{(\varepsilon)} \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{0} \end{pmatrix} \quad \textit{with} \quad c_\varepsilon = \begin{cases} 0 \ \textit{if}\ \varepsilon = 0, \\ \dfrac{1}{\varepsilon} \ \textit{if}\ \varepsilon > 0, \end{cases} \tag{4.9}$$

where $\alpha^{(\varepsilon)}$ and $\beta^{(\varepsilon)}$ are the vectors given by

$$\alpha^{(\varepsilon)} = (\alpha_{1,1}^{(\varepsilon)}, \dots, \alpha_{N,1}^{(\varepsilon)}, \dots, \alpha_{1,n}^{(\varepsilon)}, \dots, \alpha_{N,n}^{(\varepsilon)})^T \in \mathbb{R}^{nN},$$
$$\beta^{(\varepsilon)} = (\beta_1^{(\varepsilon)}, \dots, \beta_{d_2}^{(\varepsilon)})^T \in \mathbb{R}^{d_2}.$$

The vector $\mathbf{z} = (z_{1,1}, \dots, z_{N,1}, \dots, z_{1,n}, \dots, z_{N,n})^T \in \mathbb{R}^{nN}$ is obtained by stacking the columns of the matrix Z, $\mathbf{0}$ is the zero vector in $\mathbb{R}^{d_2}$, $\mathbb{I}_{nN}$ is the identity matrix of size $nN \times nN$, and $\mathbb{O}$ is the zero matrix of size $d_2 \times d_2$. The $nN \times nN$ matrix $\mathbb{K} = (\mathbb{K}_{l,k})_{1 \le l,k \le n}$ is given by the blocks $\mathbb{K}_{l,k}$ of size $N \times N$, where

$$\mathbb{K}_{l,k} = \left[-\delta_{l,k} \Delta K_{m+1,s}(x_i - x_j) + \partial_{l,k}^2 K_{m+1,s}(x_i - x_j) \right]_{\substack{1 \le i \le N \\ 1 \le j \le N}},$$

and the $nN \times d_2$ matrix $\mathbb{M} = (\mathbb{M}_1, \dots, \mathbb{M}_n)$ is given by the $N \times d_2$ blocks $\mathbb{M}_k = [p_{j,k}(x_i)]_{\substack{1 \le i \le N \\ 1 \le j \le d_2}}$ for $k = 1, \dots, n$.

Proof. From Theorem 4.2 we have the explicit expression (4.7). With respect to the interpolating (respectively, smoothing) conditions together with the orthogonality conditions, we obtain the linear system (4.9). □

The following theorem gives some results about the regularity of $\sigma^{d,\varepsilon}$ and the convergence as $\varepsilon \to \infty$ or $\varepsilon \to 0$.

THEOREM 4.4. *The solution $\sigma^{d,\varepsilon}$ of the problem* (4.1) *has the following properties:*

(1) *The solution $\sigma^{d,\varepsilon}$ belongs to $\mathscr{C}^\eta(\mathbb{R}^n; \mathbb{R}^n)$, where η is the integer given by* (2.40).

(2) $\lim_{\varepsilon \to +\infty} \sigma^{d,\varepsilon} = \sigma^{d,0}$ *and* $\lim_{\varepsilon \to 0} \sigma^{d,\varepsilon} = p^{d,0}$ *in $X_{\mathrm{div}}^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$, where the vector polynomial $p^{d,0}$ belonging to $\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n; \mathbb{R}^n)$ is the unique solution of the least-squares problem*

$$\min_{p \in \Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)} ||Lp - Z||_{N \times n}.$$

Proof.

(1) The function $K_{m+1,s}$ belongs to $\mathscr{C}^{\eta+2}(\mathbb{R}^n)$, and taking (2.42) and (4.7) into account, we obtain that $\sigma^{d,\varepsilon} \in \mathscr{C}^\eta(\mathbb{R}^n; \mathbb{R}^n)$.

(2) It is a consequence of general spline theory (see [4, 24]). □

5. Numerical results. In this section, we give some numerical examples to illustrate our approach. The examples proposed here deal with the reconstruction of two- and three-dimensional vector fields from scattered data points by smoothing or interpolating polyharmonic divergence-free. We have also tested the curl-free case, but the results are not reported here. All computations are carried out using MATLAB 7 and are executed on a machine with 2.9 GB RAM and 1.86GHz processor.

5.1. Example 5.1. In this example, we consider the original divergence-free vector field $F = (-\partial_2 f, \partial_1 f)$ as the curl of the scalar function

$$f(x, y) = \sum_{i=1}^{4} e^{-((x-a_i)^2 + (y-b_i)^2)/4},$$

where $(a_1, a_2, a_3, a_4) = (3, 3, -3, -3)$ and $(b_1, b_2, b_3, b_4) = (3, -3, -3, 3)$. Indeed, we have $\operatorname{div} F = -\partial_{12} f + \partial_{12} f = 0$. To show the effectiveness of our proposed method, we interpolated the original divergence-free vector field F on scattered random data points in $\mathbb{R}^2$. We assume that the points are located inside the square $\Omega = [-7, 7] \times [-7, 7]$, for instance. The proposed method holds in the whole space, and there are no boundary conditions to consider. The original vector field is plotted in the left side of Figure 5.1.

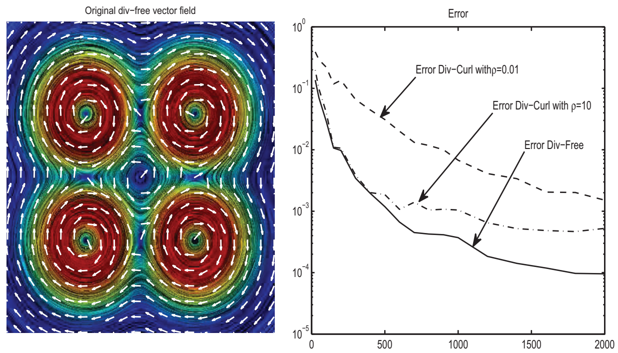

FIG. 5.1. *Original divergence-free field (left) and relative error curves (right).*

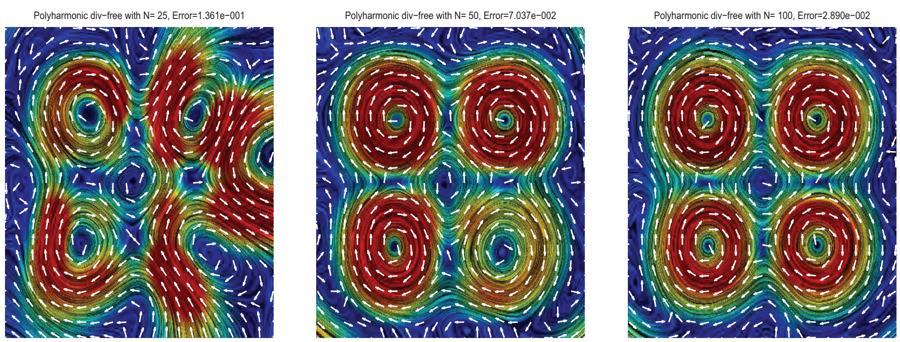

FIG. 5.2. *Reconstructed fields by polyharmonic divergence-free for $N = 25$ (left), $N = 50$ (center), and $N = 100$ (right).*

It was interpolated at different values of N randomly scattered data points in the square Ω by the polyharmonic divergence-free vector spline (with $m = 2$ and $s = 0$). For comparison, we have also interpolated the original divergence-free vector field, at the same scattered data points, by the polyharmonic divergence-curl vector spline corresponding to a small value of the parameter $\rho = 0.01$ and to a relatively big value of the parameter $\rho = 10$; see [6] for more details on the parameter ρ. Some tests are reported here for $N = 25, 50, 100$ and $N = 1000$. The polyharmonic divergence-free vector spline is represented in Figure 5.2, the polyharmonic divergence-curl for $\rho = 0.01$ is represented in Figure 5.3, and the polyharmonic divergence-curl vector spline for $\rho = 10$ is represented in Figure 5.4 for $N = 25, 50, 50$ with the same points. In Figure 5.5, we give the reconstruction of the original vector field interpolated at $N = 1000$ at the same points by divergence-free, divergence-curl ($\rho = 0.01$), and divergence-curl ($\rho = 10$).

The computed relative error R_e is defined by

$$N \longmapsto R_e(N) = \sqrt{\frac{\sum_{ij}^{25} ||F(t_{ij}) - S_N(F)(t_{ij})||^2}{\sum_{ij}^{25} ||F(t_{ij})||^2}},$$

where the points (t_{ij}) are the nodes of a 25×25 uniform grid on the domain Ω and $S_N(F)$ is a vector spline interpolating F at N scattered data points and corresponding to the polyharmonic divergence-free or divergence-curl splines. The different values

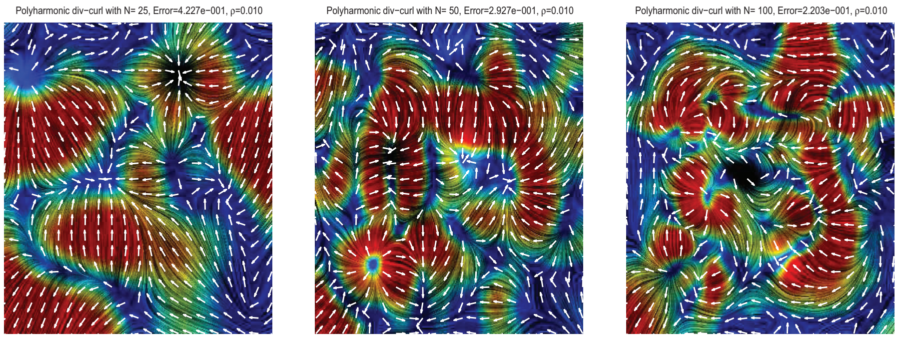

FIG. 5.3. *Reconstructed fields by polyharmonic divergence-curl for* $N = 25$ *(left),* $N = 50$ *(center), and* $N = 100$ *(right) all with the parameters* $\rho = 0.01$.

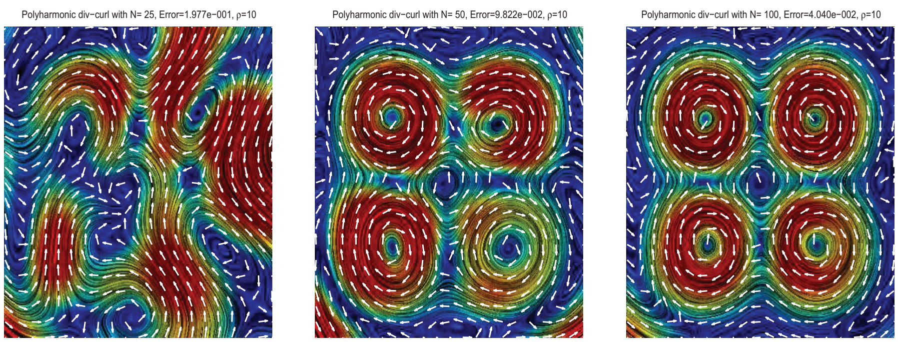

FIG. 5.4. *Reconstructed fields by polyharmonic divergence-curl for* $N = 25$ *(left),* $N = 50$ *(center), and* $N = 100$ *(right) all with the parameters* $\rho = 10$.

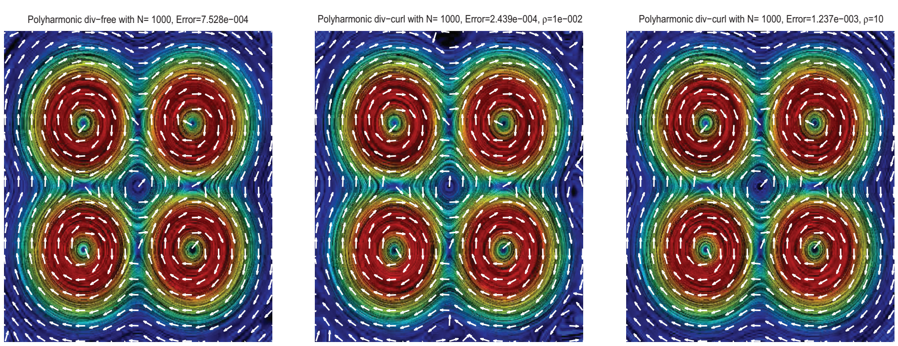

FIG. 5.5. *Reconstructed fields by polyharmonic divergence-free (left), divergence-curl with* $\rho = 0.01$ *(center) and divergence-curl with* $\rho = 10$ *for* $N = 1000$.

of the relative error are summarized in Table 5.1, and the corresponding curves are plotted in the right side of Figure 5.1.

In Figure 5.3, we observe that the divergence-curl method does not reflect any information about the divergence-free property for the small value $\rho = 0.01$ and the small values of $N = 25, 50, 100$. This result is expected because the original vector field is of a divergence-free and a small value of the parameter ρ reduces the influence of the divergence property. A best approximation by the divergence-curl method of a vector field with divergence-free, is obtained by a big value of the parameter ρ which

TABLE 5.1
Relative error.

N	R_e for Div-Free	R_e for Div-Curl $\rho = 0.01$	R_e for Div-Curl $\rho = 10$
25	1.36121e−01	3.92153e−01	1.97666e−01
50	7.03701e−02	2.92656e−01	9.82164e−02
100	2.88957e−02	2.20305e−01	4.03951e−02
150	1.06530e−02	1.17703e−01	1.09822e−02
300	3.44187e−03	6.57653e−02	3.86662e−03
500	1.18627e−03	3.10455e−02	1.86293e−03
1000	2.43918e−04	7.01390e−03	1.2370e−03
2000	9.57691e−05	1.51252e−03	5.25717e−04

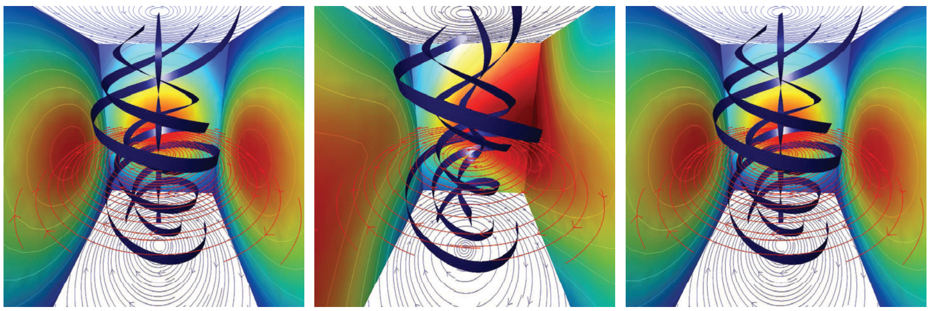

FIG. 5.6. *Original field (left) and reconstructed fields with $N = 20$ (center) and $N = 200$ (right).*

makes the divergence more important (see Figure 5.4). The choice of the optimal value of the parameter ρ is crucial. In a situation where one has the information of the divergence-free, it is better to approximate the original vector field directly by the divergence-free method. We observe in Figure 5.2 that the polyharmonic divergence-free is more closed to the original divergence-free vector field shown in Figure 5.1. We can also compare Figure 5.2 with Figures 5.3 and 5.4. The curves of the relative errors (see Figure 5.1) and the results summarized in Table 5.1 show also that, in the case of a vector field with a divergence-free, it is better to use the divergence-free method.

It is also interesting to notice that the computation of the coefficients, as was mentioned, yields to a linear system of size $(2N+6)\times(2N+6)$. In this paper we do not give more attention to the numerical aspect of solving a such linear system. The computation was done by using the backslash division of MATLAB. The drawback of such a method is in the fact that the method arises to a matrix A which is dense and ill-conditioned, with the condition number $\kappa_1(A) = ||A||\,||A^{-1}|| \simeq 10^9$ for the divergence-free method and $\kappa_2(A) \simeq 10^{12}$ for the divergence-curl method. We notice that the condition number was computed by MATLAB.

5.2. Example 5.2. We also examined and tested our approach in the three-dimensional space, $n = 3$. As in two-dimensional space, we interpolated the original divergence-free vector field,

$$F : (x,y,z) \in \Omega \subset \mathbb{R}^3 \longrightarrow (u,v,w) = F(x,y,z) = (2ye^{(-x^2-y^2-z^2)/49}, -2xe^{(-x^2-y^2-z^2)/49}, 0) \in \mathbb{R}^3. \tag{5.1}$$

The original vector field is represented in the left side of Figure 5.6 in the cube $\Omega = [-7,7]\times[-7,7]\times[-7,7]$. Now we give some details about the representation of the vector fields given in Figure 5.6. A similar representation is given for the other vector fields. In the left side of Figure 5.6, three slices representing F are given in the bottom, in the middle, and in the top of the cube. We can see the function F as a field of a wind, where the vectors are the velocity of the wind and $t = \sqrt{u^2 + v^2 + w^2}$ represents the module of the speed. The contour of the projections of the module t are presented on the wholes of equations $x = -7$, $x = 7$, and $y = 7$, respectively. Four streams of ribbons are immersed in the field of wind, and we can observe the effect of the wind on the four streams of ribbons. The polyharmonic divergence-free interpolating the original vector field from $N = 20$ scattered data points is represented in the center of Figure 5.3, and the root-mean-square error was 2.42908×10^{-1}. While for $N = 1000$ the root-mean-square error was 2.24757×10^{-3}, and the polyharmonic divergence-free interpolating the original vector field is given in the right side of Figure 5.3. The divergence-free nature is always reproduced, and, for a large value of N, the interpolating polyharmonic divergence-free vector field is closed to the original one. The goal of this example is only to show that our method may be numerically used for the reconstruction of a vector field in $\mathbb{R}^3$.

6. Conclusion. We have introduced a new method for interpolating and smoothing divergence-free and curl-free in multidimensional space by minimizing a seminorm related to the pseudo-polyharmonic splines. The method is meshless and preserves the divergence-free or irrotational-free nature. The explicit solution is given and is computed by solving a linear system.

We conclude our article by giving hereafter some extensions to the case of Hermite interpolation instead of Lagrange interpolation, and also we give an extension to the case of another kernel from the RBFs.

6.1. Hermite approximation. Under the assumption $-m + 1 + \frac{n}{2} < s < \frac{n}{2}$, the continuous imbedding inclusion $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$ holds. Thus, one can use the Hermite approximation as follows. Let us consider the operator $L : X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}^{N\times n(n+1)}$ given by

$$Lu = \begin{pmatrix} u_1(x_1) & \cdots & u_n(x_1) & \nabla u_1(x_1) & \cdots & \nabla u_n(x_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ u_1(x_N) & \cdots & u_n(x_N) & \nabla u_1(x_N) & \cdots & \nabla u_n(x_N) \end{pmatrix}, \tag{6.1}$$

where $\nabla u_k(x_i) = (\partial_1 u_k(x_i), \ldots, \partial_n u_k(x_i)) \in \mathbb{R}^n$. We can establish the following results:

1. The curl-free approximation problem (3.1) with L now given by (6.1) has also a unique solution $\sigma^{c,\varepsilon}$ with the expression

$$\sigma^{c,\varepsilon}(x) = \sum_{i=1}^{N} G_{m,s}(x - x_i)V_i^{(c,\varepsilon)} + \sum_{j=1}^{n}\sum_{i=1}^{N} \partial_j G_{m,s}(x - x_i)W_{i,j}^{(c,\varepsilon)} + P^{c,\varepsilon}(x),$$

where $V_i^{(c,\varepsilon)}$, $W_{i,j}^{(c,\varepsilon)}$ for $i = 1,\ldots,N$ and $j = 1,\ldots,n$ are vectors in $\mathbb{R}^n$ and $P^{c,\varepsilon}$ is a polynomial in $\Pi_{m-1,\text{curl}}(\mathbb{R}^n;\mathbb{R}^n)$. The notation $\partial_j G_{m,s}$ stands for the matrix-function whose components are the partial derivative ∂_j of each of the components of the matrix-function $G_{m,s}$.

2. The divergence-free approximation problem (4.1) with L now given by (6.1) has also a unique solution $\sigma^{d,\varepsilon}$ with the expression

$$\sigma^{d,\varepsilon}(x) = \sum_{i=1}^{N} H_{m,s}(x-x_i)V_i^{(d,\varepsilon)} + \sum_{j=1}^{n}\sum_{i=1}^{N} \partial_j H_{m,s}(x-x_i)W_{i,j}^{(d,\epsilon)} + P^{d,\varepsilon}(x),$$

where $V_i^{(d,\epsilon)}$, $W_{i,j}^{(d,\epsilon)}$ for $i = 1,\dots,N$ and $j = 1,\dots,n$ are vectors in $\mathbb{R}^n$ and $P^{d,\varepsilon}$ is a polynomial in $\Pi_{m-1,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n)$. The notation $\partial_j H_{m,s}$ stands for the matrix-function whose components are the partial derivative ∂_j of each of the components of the matrix-function $H_{m,s}$.

6.2. Curl-free and divergence-free approximation by RBF. In the relation (2.42) (respectively, (2.43)), instead of the function $K_{m,s}$, we can use any other function Φ which is a radial and conditionally positive definite function on $\mathbb{R}^n$ of order $m+1$ and of class at least $\mathcal{C}^2$ on $\mathbb{R}^n$. In this case, we also obtain a system of the form (3.10) (respectively, (4.9)) which is also nonsingular for both curl-free (respectively, divergence-free) interpolating and smoothing cases. It is also possible to replace the function $K_{m,s}$ by the shift-function $\phi_{m,s}(x) = K_{m,s}(\sqrt{|\ x\ |^2 + c^2})$. But it is important to notice that, in this case, the interpolating (or smoothing) problem does not derive from any minimization problem of certain energy in some Hilbert space.

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