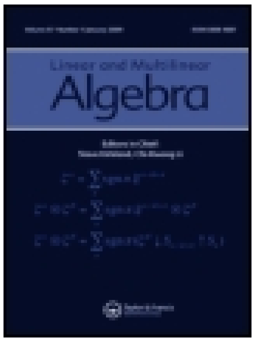

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The Sherman–Morrison–Woodbury formula for the Moore–Penrose metric generalized inverse

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ABSTRACT
In this paper, we focus on the Moore–Penrose metric generalized inverse of the modified operator $B = A + UGV$, where A, U, G, V are bounded linear operators between some Banach spaces. We establish conditions that guarantee the existence of the Moore–Penrose metric generalized inverse of B. As a consequence, we present an extension of the so-called classical Sherman–Morrison–Woodbury formula for the Moore–Penrose metric generalized inverse. Some particular cases and applications will be also considered.

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1. Introduction

Let A and G be $n \times n$ and $r \times r$ nonsingular matrices with $r \leq n$. Also, let Y and Z be $n \times r$ matrices such that $G^{-1} + Z^*A^{-1}Y$ is invertible. Then, there is a well known and perhaps the most widely used formula called the Sherman–Morrison–Woodbury (for short SMW) formula. It gives an explicit formula for the inverse of matrices of the form $A + YGZ^*$:

$$(A + YGZ^*)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1}. \tag{1}$$

The formula (1) for matrices was first presented by Sherman and Morrison [1] and extended by Woodbury [2]. An excellent review by Hager [3] described some of the applications to statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations. The SMW formula (1) is valid only if the matrices A and $G^{-1} + Z^*A^{-1}Y$ are invertible. Over the years, generalizations have been considered in the case of singular or rectangular matrices using the concept of Moore–Penrose generalized inverses (see [4–6]). Certain results on extending the SMW formula to operators on Hilbert spaces are also considered by many authors (see [7,8]).

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Let X and Y be Banach spaces, and let $B(X, Y)$ be the Banach space consisting of all bounded linear operators from X to Y. For $A \in B(X, Y)$, let $\mathcal{N}(A)$ (resp., $\mathcal{R}(A)$) denote the kernel (resp., range) of A. It is well known that for $A \in B(X, Y)$, if $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are topologically complemented in the spaces X and Y, respectively, then there exists a linear projector generalized inverse $A^+ \in B(Y, X)$ defined by

$$A^+Ax = x,\ x \in \mathcal{N}(A)^c \quad \text{and } A^+y = 0,\ y \in \mathcal{R}(A)^c,$$

where $\mathcal{N}(A)^c$ and $\mathcal{R}(A)^c$ are topologically complemented subspaces of $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. In this case, A^+A is the projection from X onto $\mathcal{N}(A)^c$ along $\mathcal{N}(A)$ and AA^+ is the projection from Y onto $\mathcal{R}(A)^c$ along $\mathcal{R}(A)$. However, it is generally well known that not every closed subspace in a Banach space is complemented, thus, the linear generalized inverse A^+ of A may not exist. In this case, in order to solve some approximation problems in Banach spaces, we may seek other types of generalized inverses for A. For example, generally speaking, the linear projector generalized inverse cannot deal with the extremal solution, or the best approximation solution of an ill-posed operator equation in Banach spaces. In order to solve the best approximation problems for an ill-posed linear operator equation in Banach spaces, Nashed and Votruba [9] introduced the concept of the (set-valued) metric generalized inverse of a linear operator in Banach spaces. Later, in 2003, Wang and Wang [10] defined the Moore–Penrose metric generalized inverse for a linear operator with closed range in Banach spaces and gave some useful characterizations. Then, Ni [11] defined and characterized the Moore–Penrose metric generalized inverse for an arbitrary linear operator in a Banach space. From then on, many research papers about the Moore–Penrose metric generalized inverses have appeared in literature such as [12–18].

The objectives of this paper are concerned with certain extensions of the so-called Sherman–Morrison–Woodbury formula to operators between some Banach spaces. We consider the SMW formula in which the inverse is replaced by the Moore–Penrose metric generalized inverse. More precisely, let $X,\ Y$ be reflexive strictly convex Banach spaces. Denote the set of all bounded linear operators from X into Y by $B(X, Y)$ and by $B(X)$ when $X = Y$. Let $A \in B(X)$, $G \in B(Y)$, and $U \in B(Y, X)$, $V \in B(X, Y)$ such that $\mathcal{R}(A)$ and $\mathcal{R}(G)$ are closed. In the main part of this paper, we will develop some conditions under which the Sherman–Morrison–Woodbury formula can be represented as

$$(A + UGV)^M = A^M - A^M U(G^M + VA^M U)^M VA^M.$$

where A^M is the Moore–Penrose metric generalized inverse of A. As a consequence, some particular cases and applications will be also considered. Our results generalize the results of many authors for finite dimensional matrices and Hilbert space operators in the literature. In the next section, we give some necessary concepts and preliminary results. We prove our main results in Section 3.

2. Preliminaries

We first recall the definition of set-valued metric projection.

Definition 2.1 ([19, Definition 4.1]): Let $G \subset X$ be a subset. The set-valued mapping $P_G : X \to G$ defined by

$$P_G(x) = \{s \in G \mid \|x - s\| = \operatorname{dist}(x, G)\}, \quad \forall\, x \in X$$

is called the set-valued metric projection, where $\operatorname{dist}(x, G) = \inf_{z \in X} \|x - z\|$.

For a subset $G \subset X$, if $P_G(x) \neq \emptyset$ for each $x \in X$, then G is said to be approximal; if $P_G(x)$ is at most a singleton for each $x \in X$, then G is said to be semi-Chebyshev; if G is simultaneously approximal and a semi-Chebyshev set, then G is called a Chebyshev set. We denote by π_G any selection for the set-valued mapping P_G, i.e., any single-valued mapping $\pi_G : \mathcal{D}(\pi_G) \to G$ with the property that $\pi_G(x) \in P_G(x)$ for any $x \in \mathcal{D}(\pi_G)$, where $\mathcal{D}(\pi_G) = \{x \in X : P_G(x) \neq \emptyset\}$. For the particular case, when G is a Chebyshev set, then $\mathcal{D}(\pi_G) = X$ and $P_G(x) = \{\pi_G(x)\}$. In this case, the mapping π_G is called the metric projector from X onto G.

Remark 2.2 ([19, Section 3.3])**:** Let $G \subset X$ be a closed convex subset. It is well known that if X is reflexive, then G is an approximal set; If X is a strictly convex, then G is a semi-Chebyshev set. Thus, every closed convex subset in a reflexive and strictly convex Banach space is a Chebyshev set, and the metric projector is just the linear orthogonal projector in Hilbert space.

The following lemma gives some important properties of the metric projectors.

Lemma 2.3 ([19, Theorem 4.1])**:** *Let X be a Banach space, and let L be a subspace of X. Then*

(1) $\pi_L^2(x) = \pi_L(x)$ *for any* $x \in X$, *i.e.* π_L *is idempotent*;
(2) $\|x - \pi_L(x)\| \leq \|x\|$ *for any* $x \in X$, *i.e.* $\|\pi_L\| \leq 2$.

In addition, if L is a semi-Chebyshev subspace, then

(3) $\pi_L(\lambda x) = \lambda \pi_L(x)$ *for any* $x \in X$ *and* $\lambda \in \mathbb{R}$, *i.e.* π_L *is homogeneous*;
(4) $\pi_L(x + z) = \pi_L(x) + z$ *for any* $x \in X$ *and* $z \in L$.

We also need some concepts about homogeneous operators and the geometry of Banach spaces. For more information about the geometric properties of Banach space, such as strict convexity, reflexivity, we refer to [20]. Let $X,\ Y$ be Banach spaces, let $A : X \to Y$ be a mapping, and let D be a subset of X. Recall from [21] that D is said to be homogeneous if $\lambda\, x \in D$ whenever $x \in D$ and $\lambda \in \mathbb{R}$, and a mapping $A\colon X \to Y$ is said to be a bounded homogeneous operator if A maps every bounded set in X into a bounded set in Y and $A(\lambda\, x) = \lambda\, A(x)$ for every $x \in X$ and every $\lambda \in \mathbb{R}$. We denote the set of all bounded homogeneous operators from X to Y by $H(X, Y)$ and by $H(X)$ when $X = Y$. Equipped with the usual linear operations on $H(X, Y)$ and the norm on $A \in H(X, Y)$ defined by $\|A\| = \sup\{\|Ax\| \mid \|x\| = 1, x \in X\}$, we can easily prove that $(H(X, Y), \|\cdot\|)$ is a Banach space (see [21]). Obviously, $B(X, Y) \subset H(X, Y)$.

For a bounded homogeneous operator $A \in H(X, Y)$, we always denote by $\mathcal{D}(A), \mathcal{N}(A)$, and $\mathcal{R}(A)$ the domain, the null space, and the range of A, respectively. One important concept in this paper is the following so-called quasi-additivity.

Definition 2.4: Let $M \subset X$ be a subset and let $A : X \to Y$ be a mapping. Then we call A quasi-additive on M if A satisfies

$$A(x+z) = A(x) + A(z), \quad \forall\, x \in X, \quad \forall\, z \in M.$$

For a homogeneous operator $A \in H(X)$, if A is quasi-additive on $\mathcal{R}(A)$, then we simply say that A is a quasi-linear operator.

Definition 2.5: Let $P \in H(X)$. If $P^2 = P$, we call P a homogeneous projector. In addition, if P is also quasi-additive on $\mathcal{R}(P)$, that is, for any $x \in X$ and any $z \in \mathcal{R}(P)$,

$$P(x+z) = P(x) + P(z) = P(x) + z,$$

then we call P a quasi-linear projector.

The following concept of bounded homogeneous generalized inverse is also a generalization of bounded linear generalized inverse.

Definition 2.6 ([22, Definition 3.1]): Let $A \in B(X, Y)$. If there is $A^h \in H(Y, X)$ such that

$$AA^hA = A, \quad A^hAA^h = A^h,$$

then we call A^h a bounded homogeneous generalized inverse of A.

Now, we present the definition of the Moore–Penrose metric generalized inverse, which is a special kind of bounded homogeneous generalized inverse.

Definition 2.7 ([21, Definition 4.3.1] or [10, Definition 2.1]): Let $A \in B(X, Y)$. Suppose that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are Chebyshev subspaces of X and Y, respectively. If there exists a bounded homogeneous operator $A^M : Y \to X$ such that

$$\begin{array}{ll} (1)\ AA^MA = A; & (2)\ A^MAA^M = A^M; \\ (3)\ A^MA = I_X - \pi_{\mathcal{N}(A)}; & (4)\ AA^M = \pi_{\mathcal{R}(A)}. \end{array}$$

Then A^M is called the Moore–Penrose metric generalized inverse of A, where $\pi_{\mathcal{N}(A)}$ and $\pi_{\mathcal{R}(A)}$ are the metric projectors onto $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.

When X and Y are Hilbert spaces, then from Definition 2.7, we see obviously that the Moore–Penrose metric generalized inverse A^M of A is indeed the Moore–Penrose orthogonal projection generalized inverse $A^\dagger$ of A under usual sense. Please see [21] for more information about the Moore–Penrose metric generalized inverses and related knowledge. Here we only need the following results in reflexive strictly convex Banach spaces.

Proposition 2.8 ([10, Corollary 2.1] or [23])**:** *Let X and Y be reflexive and strictly convex Banach spaces, let $A \in B(X, Y)$ with $\mathcal{R}(A)$ is closed. Then A^M uniquely exists. In addition, if one homogeneous generalized inverse $A^h \in H(Y, X)$ exists, then $A^M = (I - \pi_{\mathcal{N}(A)})A^h\pi_{\mathcal{R}(A)}$.*

3. Main results

In this section, we mainly study the SMW formula for the Moore–Penrose metric generalized inverse of a bounded linear operator in reflexive and strictly convex Banach spaces. In order to prove our main theorems, we first need to generalize some well known results for bounded linear operators to bounded homogeneous operators. Although the following lemma has appeared in [24], we give its proof here for completeness.

Lemma 3.1: *Let $A, B \in H(X)$ such that A is quasi-additive on $\mathcal{R}(B)$ and B is quasi-additive on $\mathcal{R}(A)$, then $I+AB$ is invertible if and only if $I+BA$ is invertible. Specially, when $A \in B(X)$ and $B \in H(X)$, if B is quasi-additive on $\mathcal{R}(A)$, then $I+AB$ is invertible if and only if $I+BA$ is invertible.*

Proof: If $I+AB$ is invertible, then there is a $\Phi \in H(X)$ such that $(I + AB)\Phi = \Phi(I + AB) = I$. Consequently, we have

$$\begin{aligned} I &= I + BA - BA = I + BA - B((I + AB)\Phi)A \\ &= I + BA - (B + BAB)\Phi A \quad (using\ B\ quasi-additive\ on\ \mathcal{R}(A)) \\ &= I + BA - (I + BA)B\Phi A \\ &= (I + BA)(I - B\Phi A) \quad (using\ A\ quasi-additive\ on\ \mathcal{R}(B)). \end{aligned}$$

Similarly, using $\Phi(I + AB) = I$, we can also check that $(I - B\Phi A)(I + BA) = I$. Thus, $I+BA$ is invertible on X with $(I + BA)^{-1} = (I - B\Phi A)$.

The converse can also be proved by using the same way as above. ■

The following Lemma 3.2 is very useful and will be frequently used in this section.

Lemma 3.2: *Let $A \in H(X)$. Let $L \subset X$ be a closed subspace such that π_L is the metric projector from X onto L.*

(1) *$\pi_L A = A$ if and only if $\mathcal{R}(A) \subset L$;*
(2) *$A\pi_L = 0$ if and only if $L \subset \mathcal{N}(A)$.*

Proof: Here, we only prove (1), then (2) can be proved in the same way. On the one hand, if $\pi_L A = A$, then $\mathcal{R}(A) = \mathcal{R}(\pi_L A) \subset \mathcal{R}(\pi_L) = L$. On the other hand, for any $x \in X$, since $\mathcal{R}(A) \subset L$, we can get that $(A - \pi_L A)x = (I - \pi_L)Ax = 0$, thus, $\pi_L A = A$. This completes the proof. ■

Lemma 3.3: *Let $A \in B(X, Y)$ such that A^M exists. Then $\mathcal{R}(A^M A) = \mathcal{R}(A^M)$ and $\mathcal{N}(AA^M) = \mathcal{N}(A^M)$.*

Proof: Obviously, we have $\mathcal{R}(A^M A) \subset \mathcal{R}(A^M)$ and $\mathcal{N}(A^M) \subset \mathcal{N}(AA^M)$. Since we also have $AA^M A = A$ and $A^M AA^M = A^M$, then we can get our desired results. ■

Unless stated otherwise, in the remainder of this paper, for convenience, we always assume that X and Y are reflexive and strictly convex Banach spaces. The following Theorem 3.4 is one of our main results, which establishes the Sherman–Morrison–Woodbury formula for the Moore–Penrose metric generalized inverse.

Theorem 3.4: *Let $A \in B(X)$, $G \in B(Y)$, and $U \in B(Y, X)$, $V \in B(X, Y)$ such that $\mathcal{R}(A)$ and $\mathcal{R}(G)$ are closed, also, let $B = A + UGV \in B(X)$ and $C = G^M + VA^M U \in H(Y)$ such that $\mathcal{R}(B)$ and $\mathcal{R}(C)$ are closed. If B^M is quasi-additive on both $\mathcal{R}(A)$ and $\mathcal{R}(U)$, and*

$$\begin{aligned} &\mathcal{R}(A^M) \subset \mathcal{R}(B^M), \quad \mathcal{N}(A^M) \subset \mathcal{N}(B^M), \\ &\mathcal{N}(G^M) \subset \mathcal{N}(U), \quad \mathcal{N}(C^M) \subset \mathcal{N}(G). \end{aligned} \tag{2}$$

Then $(A + UGV)^M = A^M - A^M U(G^M + VA^M U)^M VA^M$.

Proof: From Lemma 3.3 and (2), we have that

$$\begin{aligned} &\mathcal{R}(A^M) \subset \mathcal{R}(B^M) = \mathcal{R}(B^M B), \\ &\mathcal{R}(I_X - AA^M) = \mathcal{N}(AA^M) = \mathcal{N}(A^M) \subset \mathcal{N}(B^M). \end{aligned} \tag{3}$$

Then, by Lemma 3.2, we can obtain that

$$A^M = B^M BA^M, \quad B^M(I_X - AA^M) = 0.$$

Note that B^M is quasi-additive on $\mathcal{R}(A)$, thus, we have $B^M = B^M AA^M$, and then

$$\begin{aligned} A^M U &= B^M BA^M U = B^M BA^M U - B^M AA^M U + B^M AA^M U \\ &= B^M(B - A)A^M U + B^M U. \end{aligned}$$

Similarly, by Lemma 3.2, and also note that $U \in B(Y, X)$, then, from $\mathcal{N}(G^M) \subset \mathcal{N}(U)$, we get that $U = UGG^M$. Now, since B^M is also quasi-additive on $\mathcal{R}(U)$ and $B - A = UGV$, thus

$$\begin{aligned} A^M U &= B^M(B - A)A^M U + B^M U = B^M UGVA^M U + B^M UGG^M \\ &= B^M UGC. \end{aligned} \tag{4}$$

Using Lemma 3.2 again, then, from $\mathcal{N}(C^M) \subset \mathcal{N}(G)$ and (4), we get that

$$B^M UG = B^M UGCC^M = A^M UC^M. \tag{5}$$

Now, using (5), by simple computation, we can obtain that

$$\begin{aligned} (A + UGV)^M &= B^M AA^M = B^M(B - UGV)A^M \\ &= B^M BA^M - B^M UGVA^M \\ &= A^M - A^M UC^M VA^M \\ &= A^M - A^M U(G^M + VA^M U)^M VA^M. \end{aligned}$$

This completes the proof. ■

Specially, if the operators C and G in Theorem 3.4 above are all invertible, then we have $\mathcal{N}(C) = \mathcal{N}(G) = \{0\}$. In this case, obviously, we can obtain the following result.

Corollary 3.5: *Let $A \in B(X)$, $G \in B(Y)$, and $U \in B(Y,X)$, $V \in B(X,Y)$ such that $\mathcal{R}(A)$ is closed and G is invertible, also, let $B = A + UGV \in B(X)$ and $C = G^{-1} + VA^M U \in H(Y)$ such that $\mathcal{R}(B)$ is closed and C is invertible. If B^M is quasi-additive on both $\mathcal{R}(A)$ and $\mathcal{R}(U)$, and*

$$\mathcal{R}(A^M) \subset \mathcal{R}(B^M), \quad \mathcal{N}(A^M) \subset \mathcal{N}(B^M).$$

Then $(A + UGV)^M = A^M - A^M U(G^{-1} + VA^M U)^{-1} VA^M$.

In addition, if we assume that A is invertible and $G = I_Y$ in Corollary 3.5, then we get the following result, which is one of the main results in [25].

Corollary 3.6 ([25, Theorem 2.1])**:** *Let $A \in B(X)$, $U \in B(Y,X)$, and $V \in B(X,Y)$ such that A is invertible. Then $A+UV$ is invertible if and only if $I_Y + VA^{-1}U$ is invertible. Furthermore, if $A+UV$ is invertible, then*

$$(A + UV)^{-1} = A^{-1} - A^{-1}U(I_Y + VA^{-1}U)^{-1}VA^{-1}.$$

Proof: Since A^{-1} exists, then, from Lemma 3.1, we can get that $I_X + A^{-1}UV$ is invertible if and only if $I_Y + VA^{-1}U$ is invertible. Now, using the equality $A + UV = A(I_X + A^{-1}UV)$, we see that $A+UV$ is invertible if and only if $I_X + A^{-1}UV$ is invertible. The inverse formula can be obtained by some simple computations. ■

The following Theorem 3.7 gives some conditions that can guarantee $(A + UGV)^M$ exists, in addition, we can also get its representation.

Theorem 3.7: *Let $A \in B(X)$, $G \in B(Y)$, and $U \in B(Y,X)$, $V \in B(X,Y)$ such that $\mathcal{R}(A)$ and $\mathcal{R}(G)$ are closed, also, let $B = A + UGV \in B(X)$ and $C = G^M + VA^M U \in H(Y)$ such that $\mathcal{R}(C)$ is closed. If $I_Y + VA^M UG \in H(Y)$ is invertible and*

$$\mathcal{N}(A) \subset \mathcal{N}(V), \quad \mathcal{R}(U) \subset \mathcal{R}(A),$$

then

(1) *The Moore–Penrose metric generalized inverse B^M exists and*

$$\mathcal{N}(B) = \mathcal{N}(A), \quad \mathcal{R}(B) = \mathcal{R}(A). \tag{6}$$

In addition, if A^M is quasi-additive on $\mathcal{R}(A)$, C^M is quasi-additive on $\mathcal{R}(C)$ and

$$\mathcal{N}(C) \subset \mathcal{N}(U), \quad \mathcal{R}(V) \subset \mathcal{R}(G^M), \quad \mathcal{N}(G^M) \subset \mathcal{N}(U), \quad \mathcal{R}(V) \subset \mathcal{R}(C),$$

then we have the following representation

(2) $(A + UGV)^M = A^M - A^M U(G^M + VA^M U)^M VA^M$.

Proof: (1) Since X, Y are reflexive and strictly convex Banach spaces and A^M exists, thanks to Proposition 2.8, we only need to prove $\mathcal{N}(B) = \mathcal{N}(A)$ and $\mathcal{R}(B) = \mathcal{R}(A)$. First, from

$\mathcal{N}(A) \subset \mathcal{N}(V)$, $\mathcal{R}(U) \subset \mathcal{R}(A)$, it is easy to see that $\mathcal{N}(A) \subset \mathcal{N}(B)$ and $\mathcal{R}(B) \subset \mathcal{R}(A)$. On the other hand, note that

$$\mathcal{N}(A^M A) = \mathcal{N}(A) \subset \mathcal{N}(V) \subset \mathcal{N}(UGV), \quad \mathcal{R}(UGV) \subset \mathcal{R}(U) \subset \mathcal{R}(A) = \mathcal{R}(AA^M).$$

Thus, by using Lemma 3.2, we can get that

$$\begin{aligned} B = A + UGV = A + UGVA^M A = (I_X + UGVA^M)A, \\ B = A + UGV = A + AA^M UGV = A(I_X + A^M UGV). \end{aligned} \tag{7}$$

Since $I_Y + VA^M UG \in H(Y)$ is invertible, then it follows from Lemma 3.1 that both $I_X + UGVA^M$ and $I_X + A^M UGV$ are invertible. Thus, from (7), we get that $\mathcal{N}(B) \subset \mathcal{N}(A)$, $\mathcal{R}(A) \subset \mathcal{R}(B)$.

(2) We need to check that the formula $A^M - A^M U(G^M + VA^M U)^M VA^M$ satisfies Definition 2.7. For convenience, put $X = A^M - A^M U(G^M + VA^M U)^M VA^M$. Since

$$\begin{aligned} &\mathcal{R}(I_X - A^M A) = \mathcal{N}(A^M A) = \mathcal{N}(A) \subset \mathcal{N}(V), \quad \mathcal{R}(V) \subset \mathcal{R}(G^M) = \mathcal{R}(G^M G), \\ &\mathcal{R}(I_Y - C^M C) = \mathcal{N}(C^M C) = \mathcal{N}(C) \subset \mathcal{N}(U), \end{aligned}$$

then, from Lemma 3.2, we can get that

$$V = VA^M A = G^M GV, \quad U = UC^M C.$$

Note that A^M is quasi-additive on $\mathcal{R}(A)$ and C^M is quasi-additive on $\mathcal{R}(C)$, now, we can compute by using the equality $VA^M U = C - G^M$.

$$\begin{aligned} XB &= (A^M - A^M UC^M VA^M)(A + UGV) \\ &= A^M A + A^M UGV - A^M UC^M VA^M A - A^M UC^M VA^M UGV \\ &= A^M A + A^M UGV - A^M UC^M V - A^M UC^M CGV + A^M UC^M G^M GV \\ &= A^M A = I_X - \pi_{\mathcal{N}(A)}. \end{aligned}$$

Since

$$\begin{aligned} &\mathcal{R}(I_Y - GG^M) = \mathcal{N}(GG^M) = \mathcal{N}(G^M) \subset \mathcal{N}(U), \quad \mathcal{R}(U) \subset \mathcal{R}(A) = \mathcal{R}(AA^M), \\ &\mathcal{R}(V) \subset \mathcal{R}(C) = \mathcal{R}(CC^M), \end{aligned}$$

then, by Lemma 3.2 again, we can get that

$$U = UGG^M = AA^M U, \quad V = CC^M V.$$

Now, similarly, using the equality $VA^M U = C - G^M$, we can obtain that

$$\begin{aligned} BX &= (A + UGV)(A^M - A^M UC^M VA^M) \\ &= AA^M - AA^M UC^M VA^M + UGVA^M - UGVA^M UC^M VA^M \\ &= AA^M - UC^M VA^M + UGVA^M - UGCC^M VA^M + UGG^M C^M VA^M \\ &= AA^M = \pi_{\mathcal{R}(A)}. \end{aligned}$$

From (6) in (1), we know that $\mathcal{N}(B) = \mathcal{N}(A)$, $\mathcal{R}(B) = \mathcal{R}(A)$, thus,

$$XB = I_X - \pi_{\mathcal{N}(A)} = I_X - \pi_{\mathcal{N}(B)}, \quad BX = \pi_{\mathcal{R}(A)} = \pi_{\mathcal{R}(B)}.$$

Moreover, using Lemma 3.2, we can get that

$$BXB = BA^M A = B, \quad XBX = XAA^M = X.$$

This completes the proof. ■

Now, we formulate some important consequences of Theorem 3.7. If we let $G = I_Y$ and assume that $I_Y + VA^M U$ is invertible in Theorem 3.7, then we can get the following result.

Corollary 3.8: *Let $A \in B(X)$, $U \in B(Y,X)$ and $V \in B(X,Y)$ such that $\mathcal{R}(A)$ is closed. Suppose that A^M is quasi-additive on $\mathcal{R}(A)$. If $C = I_Y + VA^M U \in H(Y)$ is invertible and $N(A) \subset \mathcal{N}(V)$, $\mathcal{R}(U) \subset \mathcal{R}(A)$, then $(A + UV)^M = A^M - A^M U(I_Y + VA^M U)^{-1} VA^M$ and*

$$\|(A + UV)^M - A^M\| \leq \frac{\|A^M U\| \|VA^M\|}{1 - \|VA^M U\|}.$$

In addition, if $U = I$ and $\|V\| \|A^M\| < 1$, then from Corollary 3.8, we can get the following result, which is some generalization of the main results in [23,26].

Corollary 3.9: *Let $A \in B(X)$, $V \in B(X, Y)$ with $\mathcal{R}(A)$ be closed. Suppose that A^M is quasi-additive on $\mathcal{R}(A)$. If $\|V\| \|A^M\| < 1$ and $N(A) \subset \mathcal{N}(V)$, $\mathcal{R}(V) \subset \mathcal{R}(A)$, then*

$$(A + V)^M = A^M (I_X + VA^M)^{-1} \quad \text{and} \quad \|(A + V)^M - A^M\| \leq \frac{\|A^M\|}{1 - \|V\| \|A^M\|}. \tag{8}$$

It is well known that Hilbert spaces are reflexive and strictly convex Banach spaces. Then, from the proof of Theorem 3.7, we can also get the following classic result in Hilbert space.

Corollary 3.10 ([27, Theorem 3.1])**:** *Let H,K be Hilbert spaces. Let $A \in B(H, K)$ have the Moore–Penrose generalized inverse $A^\dagger \in B(K, H)$. Let $\delta A \in B(H, K)$ with $\|\delta AA^\dagger\| < 1$. Then $G = A^\dagger (I_X + \delta AA^\dagger)^{-1}$ is the Moore–Penrose generalized inverse of $\bar{A} = A + \delta A$ if and only if $\mathcal{R}(\bar{A}) = \mathcal{R}(A)$ and $\mathcal{N}(\bar{A}) = \mathcal{N}(A)$.*

For the modified operator $B = A + UGV$, if we let $A = I_Y$, and $G = -I_Y$, then, under some conditions, we can obtain the following result, which has been obtained by some authors for finite matrices. Here, we can also get the error estimate for the corresponding perturbed bound.

Theorem 3.11: *Let $U \in B(Y, X)$ and $V \in B(X, Y)$ such that $UV = I_X$. Put $W = I_Y - VU$, then U^M and V^M exist. Moreover, if V^M is quasi-additive on $\mathcal{R}(V)$, then W^M exists and*

$$W^M = (I_Y - VV^M)(I_Y - U^M U). \tag{9}$$

Moreover, $\|W^M\| \leq 2$.

Proof: We first prove that U^M and V^M exist. From $UV = I_X$, we get $UV\,U = U$ and $V\,UV = V$, then we can check that the range $\mathcal{R}(U)$ and $\mathcal{R}(V)$ are closed. In fact, let $\{x_n\} \subset X$ such that $Vx_n \to y_0$ as $n \to \infty$ for some $y_0 \in Y$. Since $V\,UV = V$, we have $Vx_n = (VU)Vx_n \to VUy_0$ and hence $y_0 = VUy_0 \in \mathcal{R}(V)$. Thus, $\mathcal{R}(V)$ is closed. From $UV = I_X$, we can obtain that $\mathcal{R}(U) = X$, i.e., $\mathcal{R}(U)$ is closed. Since $U \in B(Y, X)$ and $V \in B(X, Y)$, we get that both $\mathcal{N}(U)$ and $\mathcal{N}(V)$ are closed. Now, from Proposition 2.8, we obtain that U^M and V^M exist.

In order to prove the existence of W^M, by using Proposition 2.8 again, we only need to show that $\mathcal{R}(I_Y - VU) = \mathcal{N}(U)$, but, this is obvious from $UV = I_X$. In fact, from $UV = I_X$, we can also obtain that $\mathcal{N}(I_Y - VU) = \mathcal{R}(V)$.

Put $X = (I_Y - VV^M)(I_Y - U^M U)$. Now, we show that X satisfies Definition 2.7, i.e., $W^M = X$. First, note that, we have already proved that $\pi_{\mathcal{R}(W)} = \pi_{\mathcal{N}(U)}$ and $\pi_{\mathcal{N}(W)} = \pi_{\mathcal{R}(V)}$. Using $UV = I_X$ again, we can obtain $WV = 0$ and $UW = 0$. Then,

$$\begin{aligned} &WX = (I_Y - VU)(I_Y - VV^M)(I_Y - U^M U) = (I_Y - VU)(I_Y - U^M U) \\ &= I_Y - U^M U = \pi_{\mathcal{N}(U)} = \pi_{\mathcal{R}(W)}, \\ &XW = (I_Y - VV^M)(I_Y - U^M U)(I_Y - VU) = (I_Y - VV^M)(I_Y - VU) \\ &= I_Y - VV^M = I_Y - \pi_{\mathcal{R}(V)} = I_Y - \pi_{\mathcal{N}(W)}. \end{aligned} \tag{10}$$

Now, using (10), we can also obtain

$$WXW = W(I_Y - \pi_{\mathcal{N}(W)}) = W,$$
$$XWX = (I_Y - VV^M)(I_Y - VV^M)(I_Y - U^M U) = X.$$

Finally, from Lemma 2.3, we can get that

$$\|I_Y - VV^M\| = \|I_Y - \pi_{\mathcal{R}(V)}\| \le 1, \quad \|I_Y - U^M U\| = \|\pi_{\mathcal{N}(U)}\| \le 2.$$

Consequently, we obtain that $\|W^M\| \le 2$. This completes the proof. ∎

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