BASIC LINEAR ALGEBRA AND MATRIX COMPUTATION

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1. Sum, product and particular matrices. We begin this section by reminding some classical definitions about matrices. Let $A = [a_{ij}]$ be a matrix in $\mathbb{C}^{n \times m}$ (whose ij-th element is a_{ij}) and let $A = [a_{ij}]$ be a matrix in $\mathbb{C}^{p \times q}$, then

- If $n = p$ and $m = q$, the ij-th element of the matrix sum $C = \alpha A + \beta B = [c_{ij}]$ is defined by $c_{ij} = \alpha a_{ij} + \beta b_{ij}$.
- If $m = p$, then the product $C = AB$ is defined by $C = [c_{ij}]$ such that

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj},\ i = 1, \ldots, n;\ j = 1, \ldots, q.$$

- The transpose of the matrix A is the $m \times n$ matrix $A^T = [a_{ji}]$ and the transpose conjugate is defined by $A^* = \bar{A}^T$ where the bar denotes the complex conjugaison.
- The ith row of A is defined by $a_{i*} = [a_{i1}, a_{i2}, \ldots, a_{im}]$
- The jth column is defined by

$$a_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{nj} \end{pmatrix}.$$

Matrices are related to linear mapping between vector spaces. Let f be a linear mapping from $\mathbb{R}^m$ onto $\mathbb{R}^n$ and consider two bases $B_1 = \{v_1, \ldots, v_m\}$ and $B_2 = \{w_1, \ldots, w_n\}$ then

$$A = Mat_{B_1, B_2} f$$

where the j-th column of A is expressed as $a_{*j} = f(v_j) = a_{1j} w_1 + \ldots + a_{nj} w_n$, $j = 1, \ldots, m$.
In matrix computation, we usually use the following particular matrices. The square matrix $(m = n)$ A is said to be

- Diagonal iff $a_{ij} = 0$ if $i \neq j$ and we denote $A = diag(a_{11}, \ldots, a_{nn})$.
- Identity is defined by $I = diag(1, 1, \ldots, 1)$.
- Hermitian iff $A^* = A$ and skew-Hermitian
- Symmetric iff $A^T = A$ and skew-symmetric iff $A^T = -A$.
- Normal iff $A^* A = A A^*$.
- Nonnegative if $a_{ij} \geq 0$ for all i, j.
- Unitary iff $A^* A = I$ (orthonormal in the real case) and A is said to be orthogonal if A is not necessarily square and $A^T A = I$
- Upper triangular iff $a_{ij} = 0$ for $i > j$.

- Lower triangular if $a_{ij} = 0$ for $i < j$.
- Tridiagonal if $a_{ij} = 0$ for i, j such that $|i - j| > 1$ and we denote $A = tridiag(a_{i,i-1}, a_{ii}, a_{i,i+1})$.
- Permutation if the column (or rows) of A are a permutation of the identity matrix.
- A Toeplitz matrix is a matrix whose each diagonal is constant. If $T = [t_{ij}]$, we have $t_{ij} = t_{i+1,j+1} = a_{i-j}$.
- A Hankel matrix $H = [h_{ij}]$ is a square matrix such that $h_{i,j} = h_{i-1,j+1}$.

REMARK 1.

1. *The inverse of an upper (resp. lower) triangular matrix is also an upper (resp. lower) triangular matrix.*
2. *If P is a permutation matrix, then the product PA is obtained by permuting the rows of A while the product AP is obtained by permuting the columns of A.*
3. $(AB)^T = B^T A^T$.

2. Range space, null space and matrix inversion.

2.1. Range and null spaces. Let $\{u_1, u_2, \ldots, u_p\}$ be p vectors in $\mathbb{R}^n$, the subspace generated by these vectors is defined and denoted as follows

$$span\{u_1, \ldots, u_p\} = \{\alpha_1 u_1 + \ldots + \alpha_p u_p,\ \alpha_i \in \mathbb{R}\} \subset \mathbb{R}^n.$$

There are two important subspaces that are associated with the matrix A.

DEFINITION 2.1.

1. *The* **range** *of $A \in \mathbb{R}^{n\times m}$ is defined by*

$$\mathbf{range}(A) = \{y \in \mathbb{R}^n :\ \exists x \in \mathbb{R}^m \text{ such that } y = Ax\}.$$

2. *The* **null** *(or* **Kernel***) space of A is defined by*

$$\text{null}(A) = \text{Ker}(A) = \{x \in \mathbb{R}^m :\ Ax = 0\}.$$

3. *The* **rank** *of the matrix A is defined by*

$$\text{rank}(A) = dim(\text{range}(A)).$$

It is not difficult to show that

- $rank(A) = rank(A^T)$.
- The rank of the matrix A is equal to the maximum number of independent vector columns (or rows) of the matrix A.
- We have the classical relation: $dim(null(A)) + rank(A) = m$.
- If $n = m$ and A nonsingular, then $rank(A) = n$ and $range(A) = \mathbb{R}^n$.
- A square matrix A of size $n \times n$ is singular iff $rank(A) < n$.

2.2. Matrix inverse. Let A the square matrix $A \in \mathbb{R}^{n\times n}$. Then A is said to be invertible (regular, nonsingular) iff there exists a matrix $X \in \mathbb{R}^{n\times m}$ such that

$$AX = XA = I_n.$$

In this case the matrix X is the inverse of A and is denoted by $X = A^{-1}$ and we have the following properties

PROPOSITION 2.2. *If A and B are two nonsingular $n \times n$ matrices, then*

1. $(AB)^{-1} = B^{-1}A^{-1}$.
2. $(A^{-1})^T = (A^T)^{-1}$.
3. The Sherman-Morrison-Woodbury formula:

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1}U)^{-1}V^T A^{-1},$$

where U and V are two $n \times k$ matrices.

If the matrix $A \in \mathbb{R}^{n\times m}$ is not square, then we can define a generalized inverse also called the **Pseudo-inverse** or the **Moore-Penrose** inverse which is the unique matrix $X \in \mathbb{R}^{m\times n}$ satisfying the following conditions

$$AXA = A, \;\; XAX = X, \;\; (AX)^T = AX, \;\; \text{and} \;\; (XA)^T = XA.$$

In this case the Pseudo-inverse is denoted by $X = A^+$.

Remark 2.

- *If $m = n$ then $A^+ = A^{-1}$.*
- *If $rank(A) = m$, then $A^+ = (A^T A)^{-1}A^T$.*
- $(AB)^+ = B^+A^+$
- $(A^+)^+ = A$
- *AA^+ and A^+A are orthogonal projection.*

3. Eigenvalues of a square matrix. Let A be a square matrix in $\mathbb{R}^{n\times n}$. Then a scalar $\lambda \in \mathbb{C}$ is called an eigenvalue of the matrix A if and only if there exists a nonzero vector $u \in \mathbb{C}^n$ such that

$$Au = \lambda u.$$

The vector u is called and eigenvector associated to the eigenvalue λ and the set of all eigenvalues of A will be denoted by $\Lambda(A)$. We notice that a scalar λ is an eigenvalue of A iff it is a zero of the characteristic polynomial P_A defined by

$$P_A(\lambda) = det(A - \lambda I_n),$$

where $det(Z)$ denotes for the determinant of the square matrix $Z = [z_{ij}]$ defined by

$$det(Z) = \sum_{j=1}^{n} (-1)^{j+1} \, z_{1j} \, det(Z_{1j}),$$

where $det(Z_{1j})$, is the $(n-1)\times(n-1)$ determinant obtained by deleting the first row and the j-th column.

Remark 3.

1. The square matrix A is singular iff $det(A) = 0$ which is equivalent to $\lambda = 0$ is an eigenvalue of A.
2. The $n \times n$ matrix A has exactly n complex eigenvalues (the n roots of the characteristic polynomial P_A).
3. The maximum modulus of the eigenvalues is called the spectral radius of A and denoted by $\rho(A)$.
4. The subspace $E_\lambda = Ker(A - \lambda I)$ (of all eigenvectors associated to λ + the null vector) is invariant under A which means that $AE_\lambda \subset E_\lambda$.

5. *The characteristic polynomial P_A can be expressed as*

$$P_A(\lambda) = det(A - \lambda I) = \prod_{i=1}^{p} (\lambda_i - \lambda)^{m_i},$$

where m_i is the multiplicity of the eigenvalue λ_i with $\sum_{i=1}^{p} \lambda_i = n$.

6. $dim(E_{\lambda_i}) \leq m_i$.
7. *Cayley-Hamilton* $P_A(A) = 0$.
8. $trace(A) = \sum_{i=1}^{n} \lambda_i$ *and* $det(A) = \prod_{i=1}^{n} \lambda_i$.
9. *A and A^T have the same eigenvalues.*
10. *A unitary* $\Longrightarrow |\lambda| = 1, \forall \lambda \in \Lambda(A)$.
11. *Two matrices A and B are similar iff there exists a nonsingular matrix P such that $A = PBP^{-1}$ and then A and B have the same eigenvalues.*

DEFINITION 3.1. *The $n \times n$ matrix A is diagonalizable iff there exists a nonsingular matrix P such that*

$$A = P \; diag(\lambda_1, \ldots, \lambda_n) \; P^{-1}$$

which is also equivalent to the fact the eigenvectors of A form a basis of $\mathbb{C}^n$. It is shown that the matrix A is diagonalizable iff $dim(E_{\lambda_i}) = m_i$, for $i = 1, \ldots, p$. The dimension of E_{λ_i} is called the geometric multiplicity.

THEOREM 3.2. *The eigenvalues of a symmetric matrix are all real.* Symmetric matrices is subclass of a class of matrices called normal matrices and defined in the following definiton.

DEFINITION 3.3. *A square matrix A is normal if and only if $AA^T = A^T A$.*

THEOREM 3.4. *The following properties are equivalent*

1. *A is normal.*
2. *A is diagonalizable by a unitary matrix: $A = UDU^{-1}$ with $D = diag(\lambda_1, \ldots, \lambda_n)$ and $U^T U = UU^T = I$.*
3. *There exists a polynomilal p such that $A^T = p(A)$.*

DEFINITION 3.5. *Let A in $\mathbb{R}^{n \times n}$. Then*

1. *The field of values (also called numerical range) of the matrix A is defined by*

$$\mathcal{W}(A) = \{x^* Ax, \; x \in \mathbb{C}^n, \; x^T x = 1\}.$$

2. *The numerical radius of A is defined by*

$$r(A) = sup\{|x^* Ax|, \; x^T x = 1\}.$$

We have the followng properties:

1. $\Lambda(A) \subset \mathcal{W}(A)$.
2. $\rho(A) \leq r(A)$ and if A is normal $r(A) = \rho(A) = \|A\|_2$.
3. $r(A) \leq \|A\|_2 \leq 2r(A)$.
4. If A is normal, $\mathcal{W}(A) = \mathcal{W}(diag(\lambda_1, \ldots, \lambda_n))$.

Next, we give two eigenvalue-localisation results.

THEOREM 3.6. *(Gershgorin 1931). Let $A = [a_{ij}]$ be an $n \times n$ matrix and let D_i denotes the following disc*

$$D_i = \{z \in \mathbb{C} : \ |z - a_{ii}| \leq \sum_{i \neq j=1}^{n} |a_{ij}|\}.$$

Then the spectrum $\Lambda(A)$ satisfies

$$\Lambda(A) \subset \bigcup_{i=1}^{n} D_i.$$

We also have

$$\rho(A) \leq \min\{\max_i(\sum_{j=1}^{n} |a_{ij}|), \ \max_j(\sum_{i=1}^{n} |a_{ij}|)\}.$$

4. Positive matrices. The matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite ($A \geq 0$) iff

$$x^T Ax \geq 0, \ \forall x \in \mathbb{R}^n$$

and it said positive definite iff $x^T AX > 0$, $\forall x \neq 0$ and in this case we denote $A > 0$. We have the following properties

PROPOSITION 4.1. *Let $A \in \mathbb{R}^{n \times n}$. Then*

1. *If A is positive definite and if $X \in \mathbb{R}^{n \times k}$ has rank k, then $X^T AX$is also positive definite.*
2. *All the principal submatrices of A are positive definite and in particular, the diagonal entries are positive.*
3. *A is poitive definite iff the symmetric part of A ($A_s = (1/2)(A + A^T)$) is positive definite.*
4. *If A is definite then A is nonsingular.*
5. *If A is symmetric and positive definite then all its eigenvalues are real postive .*

5. Vector and Matrix norms.

5.1. Vector norms. A norm in vector $\mathbb{R}$-space E is a function from E onto $\mathbb{R}^+$ satisfying the following properties

1. $\|x\| = 0$ iff $x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$, $\forall \lambda \in \mathbb{R}$ and $\forall x \in E$.
3. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in E$

In $\mathbb{R}^n$, a useful class of vector norms are the p-norms defined as follows. For a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, we define $\|x\|_p$ as

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{\frac{1}{p}}.$$

In particular, we have the following well known 1,2 and ∞ norms

$$\begin{aligned}
\|x\|_1 &= |x_1| + \ldots + |x_n| \\
\|x\|_2 &= (x^T x)^{1/2} = \sqrt{\sum_{i=1}^{n} |x_i|^2} \\
\|x\|_\infty &= \max_{1 \le i \le n} |x_i|.
\end{aligned}$$

REMARK 4. *The 2-norm is associated to the scalar inner product in $\mathbb{R}^n$ defined as follows. For two vectors $x, y \in \mathbb{R}^n$, the Euclidian inner product is defined by*

$$\langle x, y \rangle_2 = \sum_{i=0}^{n} x_i y_i = x^T y$$

where $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$. We have the following properties

- *For all x, $y \in \mathbb{R}^n$, we have $\langle x, Ay \rangle_2 = \langle A^T x, y \rangle_2$.*
- *The Cauchy-Schwartz inequality*

$$\langle x, y \rangle_2 \le \|x\|_2 \|y\|_2.$$

- *If Q is an orthogonal matrix, then*

$$\|Qx\|_2 = \|x\|_2.$$

- *All the norms in $\mathbb{R}^n$ are equivalent and we have*

$$\begin{array}{ccccc}
\|x\|_2 & \le & \|x\|_1 & \le & \sqrt{n}\|x\|_2 \\
\|x\|_\infty & \le & \|x\|_2 & \le & \sqrt{n}\|x\|_\infty \\
\|x\|_\infty & \le & \|x\|_1 & \le & n\|x\|_\infty
\end{array}$$

5.2. Matrix norms. Usually in matrix computation, one needs the knowledge of the norm of a matrix. For a general matrix $A \in \mathbb{R}^{n \times m}$, we consider the following induced norm

$$\|A\|_{pq} = \max_{x \in \mathbb{R}^n / \{0\}} \frac{\|Ax\|_p}{\|x\|_q}.$$

These norms satisfy the usual properties of the norm and If $p = q$, then we have the following property

$$\|AB\|_p \le \|A\|_p \, \|B\|_p,$$

and in this case, the norm is called consistent. If the matrix A is square, then we have

$$\|A^k\|_p \le \|A\|_p^k, \; k = 1, 2, \ldots.$$

Notice also that $\|I_n\|_p = 1$. Another non-induced but consistent matrix norm is the well known Frobenius norm defined as follows

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$

This is not an induced norm since $\|I_n\|_F = \sqrt{n}$. The Frobenius norm is associated to the scalar product in the space of matrices in $\mathbb{R}^{n\times m}$ defined as follows. Let X and Y be two matrices in $\mathbb{R}^{n\times m}$, then

$$\langle X, Y\rangle_F = trace(X^T Y),$$

where $trace(Z)$ denotes the sum of the elements on the diagonal of the square matrix Z. In this case, we have

$$\|A\|_F = \sqrt{\langle X, Y\rangle_F} = \sqrt{trace(X^T Y)}.$$

If $vec(A)$ denotes the nm vector obtained from the matrix A by stacking all the columns of A, then

$$\|A\|_F = \|vec(A)\|_2.$$

For the classical 1, 2 and ∞ matrix norms, we have the following expressions

$$\begin{aligned}
\|A\|_1 &= \max_{1\le j\le m} \sum_{i=1}^{n} |a_{ij}|,\\
\|A\|_\infty &= \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|,\\
\|A\|_2 &= \sqrt{\rho(AA^T)},
\end{aligned}$$

where $\rho(AA^T)$ denotes the spectral radius of the matrix AA^T.

6. Matrix products.

6.1. The Kronecker product. Let $A = [a_{ij}] \in \mathbb{R}^{n\times m}$and $B = [b_{ij}] \in \mathbb{R}^{p\times q}$, then the Kronecker product of these two matrices in the $np \times mq$ matrix defined as follows

$$A \otimes B = [a_{ij}B] = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}.$$

For this product, we have the following properties [9]

- $(A \otimes B)^T = A^T \otimes B^T$,
- $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.
- If A and B are nonsingular matrices of dimension $n\times n$ and $p\times p$ respectively, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
- If A and B are $n \times n$ and $p \times p$ matrices, then $det(A \otimes B) = det(A)^p det(B)^n$ and $tr(A \otimes B) = tr(A)tr(B)$.
- $\mathrm{vec}(AXC) = (C^T \otimes A)\,\mathrm{vec}(X)$,
- $\mathrm{vec}(A)^T\,\mathrm{vec}(B) = \mathrm{trace}(A^T B) = \langle A, B\rangle_F$,
- $vec(AX + XB) = (I \otimes A) + (B^T \otimes I))vec(X)$,

where $vec(X) \in \mathbb{R}^{np}$ is the long vector obtained by stacking the columns of the matrix $X \in \mathbb{R}^{n\times p}$.

PROPOSITION 6.1. *If A and B are two $n \times n$ and $p \times p$ matrices, respectively, and if λ_i, $i = 1, \ldots, n$ are the eigenvalues of A and $\mu_1, \ldots, \mu_p$ are the eigenvalues of the matrix B, then*

1. *The eigenvalues of the $np \times nm$ matrix $A \otimes B$ are the np scalars $\lambda_i \mu_j$, $i = 1, \ldots, n$ and $j = 1, \ldots, p$.*
2. *The eigenvalues of $(I \otimes A) + (B^T \otimes I))$ are the scalars $\lambda_i + \mu_j$.*

6.2. The Hadamard product. The Hadamard product of the matrices $A = [a_{ij}] \in \mathbb{R}^{n\times m}$and $B = [b_{ij}] \in \mathbb{R}^{n\times m}$ is the $n \times m$ matrix defined by

$$A \circ B = [a_{ij}b_{ij}] = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \ldots & a_{1m}b_{1m} \\ a_{21}b_{21} & a_{22}b_{22} & \ldots & a_{2m}b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \ldots & a_{nm}b_{nm} \end{pmatrix}.$$

The Hadamard product is carried out by multiplying the matrices entry by entry. We have the following properties

PROPOSITION 6.2. *[4] Let A and B be two matrices of sizes $(n \times m$. Then*

$$rank(A \circ B) \leq rank(A)rank(B).$$

6.3. The $\diamond$ product. In the following we consider the product denoted by $\diamond$ defined as follows [7]

DEFINITION 6.3. *Let $A = [A_1, A_2, \ldots, A_p]$ and $B = [B_1, B_2, \ldots, B_l]$ be matrices of dimension $n \times ps$ and $n \times ls$ respectively where A_i and B_j $(i = 1, \ldots, p; j = 1, \ldots, l)$ are $n \times s$ matrices. Then the $p \times l$ matrix $A^T \diamond B$ is defined by:*

$$A^T \diamond B = \begin{pmatrix} \langle A_1, B_1\rangle_F & \langle A_1, B_2\rangle_F & \ldots & \langle A_1, B_l\rangle_F \\ \langle A_2, B_1\rangle_F & \langle A_2, B_2\rangle_F & \ldots & \langle A_2, B_l\rangle_F \\ \vdots & \vdots & \vdots & \vdots \\ \langle A_p, B_1\rangle_F & \langle A_p, B_2\rangle_F & \ldots & \langle A_p, B_l\rangle_F \end{pmatrix}.$$

It is not difficult to show the following properties satisfied by the product $\diamond$.

PROPOSITION 6.4. *Let $A, B, C \in \mathbb{R}^{n\times ps}$, $D \in \mathbb{R}^{n\times n}$, and $L \in \mathbb{R}^{p\times p}$. Then we have*

1. $(A + B)^T \diamond C = A^T \diamond C + B^T \diamond C$.
2. $A^T \diamond (B + C) = A^T \diamond B + A^T \diamond C$.
3. $(A^T \diamond B)^T = B^T \diamond A$.
4. $(DA)^T \diamond B = A^T \diamond (D^T B)$.
5. $A^T \diamond (B(L \otimes I_s)) = (A^T \diamond B)L$.
6. $||A^T \diamond B||_F \leq ||A||_F\, ||B||_F$.

The $\diamond$-product is related to the inner product $\langle ., .\rangle_F$ on matrix subspaces. In fact if $\mathcal{V} = [V_1, V_2, \ldots, V_m]$ where each $V_i \in \mathbb{R}^{n\times s}$, then the matrix $\mathcal{V}$ is orthonormal with respect to the inner product $\langle ., .\rangle_F$, that is

$$\langle V_i\,,\, V_j\rangle_F = \delta_{ij}$$

if and only if

$$\mathcal{V}^T \diamond \mathcal{V} = I.$$

7. The Schur complement. We first recall the definition of the Schur complements and give some of their properties [1]..

DEFINITION 7.1. *Let M be a matrix partitioned in four blocks*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the submatrix D is assumed to be square and nonsingular. The Schur complement of D in M, denoted by (M/D), is defined by

$$(M/D) = A - BD^{-1}C.$$

Moreover, since

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (M/D) & B \\ O & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix},$$

we get

$$det(M) = det(M/D) \times det(D).$$

If D is not a square matrix then a Pseudo-Schur complement of D in M can still be defined. Let us remark that having the nonsingular submatrix D in the lower right-hand corner of M is a matter of convention. We can similarly define the following Schur complements

$$(M/A) = D - CA^{-1}B,$$

$$(M/B) = C - DB^{-1}A,$$

$$(M/C) = B - AC^{-1}D.$$

If the two matrices A and D are square and nonsingular, we have the following relation

$$(M/D)^{-1} = A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1}.$$

We also can show the Guttman rank additivity formula

$$rank(M) = rank(D) + rank(M/D).$$

Now we give some other classical algebraic properties of the Schur complements.

PROPOSITION 7.2. *Let us assume that the submatrix D is nonsingular, then*

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} / D\right) = \left(\begin{bmatrix} D & C \\ B & A \end{bmatrix} / D\right) = \left(\begin{bmatrix} B & A \\ D & C \end{bmatrix} / D\right) = \left(\begin{bmatrix} C & D \\ A & B \end{bmatrix} / D\right). \tag{2.1}$$

PROPOSITION 7.3. *Assuming that the matrix D is nonsingular and E is a matrix such that the product EA is well defined, then*

$$\left(\begin{bmatrix} EA & EB \\ C & D \end{bmatrix} / D\right) = E\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} / D\right). \tag{2.2}$$

PROPOSITION 7.4. *Assuming that the matrix D is nonsingular and the matrices A and A' have the same dimension, then*

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} / D\right) + \left(\begin{bmatrix} A' & B' \\ C & D \end{bmatrix} / D\right) = \left(\begin{bmatrix} A+A' & B+B' \\ C & D \end{bmatrix} / D\right), \tag{2.3}$$

and

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} / D\right) + \left(\begin{bmatrix} A' & B \\ C' & D \end{bmatrix} / D\right) = \left(\begin{bmatrix} A+A' & B \\ C+C' & D \end{bmatrix} / D\right). \tag{2.4}$$

The proofs of these propositions are easily derived from the definition of the Schur complement.

Consider the matrices K, M_1, M_2, M_3 and M_4 partitioned as follows

$$K = \begin{bmatrix} A & B & E \\ C & D & F \\ G & H & L \end{bmatrix}, \quad M_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

$$M_2 = \begin{bmatrix} B & E \\ D & F \end{bmatrix}, \quad M_3 = \begin{bmatrix} D & F \\ H & L \end{bmatrix}, \quad M_4 = \begin{bmatrix} C & D \\ G & H \end{bmatrix}.$$

Let n_1, n_2 and n_3 denote the number of rows of the matrices A, C and G respectively. We also denote by p_1, p_2 and p_3 the number of columns of the matrices A, B and E respectively.

Assume that the matrices A and M_1 are square ($n_1 = p_1$ and $n_2 = p_2$) and nonsingular.The we have the following theorem.

THEOREM 7.5. *(The quotient property)*

$$\begin{aligned} (K/M_1) &= ((K/A)/(M_1/A)) \\ &= \left(\begin{bmatrix} A & E \\ G & L \end{bmatrix} / A\right) - \left(\begin{bmatrix} A & B \\ G & H \end{bmatrix} / A\right) (M_1/A)^{-1} \left(\begin{bmatrix} A & E \\ C & F \end{bmatrix} / A\right). \end{aligned} \tag{2.5}$$

8. Orthogonal vectors and matrices. Two vectors $u = (u^1, \ldots, u^n)^T$ and $v = (v^1, \ldots, v^n)^T$ of $\mathbb{R}^n$ are orthogonal iff

$$\langle u, v\rangle_2 = \sum_{i=1}^{n} u^i v^j = 0.$$

A set of vectors $F = span\{u_1, u_2, \ldots, u_p\}$ is orthogonal iff

$$\langle u_i, u_j \rangle_2 = 0,\ i, j = 1, \ldots, p,\ i \neq j.$$

Two subspaces $F = span\{u_1, u_2, \ldots, u_p\}$ and $G = span\{v_1, v_2, \ldots, v_q\}$ are orthogonal iff

$$\langle u_i, v_j \rangle_2 = 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q.$$

The vectors $\{u_1, u_2, \ldots, u_p\}$ are orthonormal iff

$$\langle u_i, u_j \rangle_2 = \delta_{ij} (= 1 \text{ if } i = j \text{ and } 0 \text{ elsewhere}),\ i, j = 1, \ldots, p.$$

In this case, tha matrix $U = [u_1, u_2, \ldots, u_p]$ is said to be orthogonal and we have

$$U^T U = I.$$

The orthogonal of the subspace $F = span\{u_1, u_2, \ldots, u_p\}$ is the orthogonal-subspace of F defined as

$$F^\perp = \{y \in \mathbb{R}^n \ / \langle y, u_i \rangle_2 = 0,\ i = 1, \ldots, p\}.$$

Remark 5. *If Q, Z are orthogonal matrix ($Q^T Q = I; Z^T Z = I$), then for any vector x, and for any matrix A, with appropriate sizes, we have*

$$\|Qx\|_2 = \|x\|_2, \text{ and } \|QAZ\|_F = \|A\|_F.$$

In the complex case, a matrix $Q \in \mathbb{C}^{n \times n}$ is **unitary** if and only if

$$Q^T Q = Q Q^T = I_n.$$

Definition 8.1. *(Invariant subspaces) The subspace F is an invariant subspace of A iff $AF \subset F$.* We have the following result

Theorem 8.2. *If V is a matrix whose columns form a basis of the invariant subspace F, then there exists a unique matrix L such that*

$$AV = VL,$$

and we also have (u, λ) is an eigenpair of L if and only if (Vu, λ) is an eigenpair of A.

9. The Schur decomposition. An important theme of matrix theory is the reduction of matrices to a simple form such as diagonal or triangular by similarity transformations. In particular, unitary transformations are particularly desirable. The following theorem shows that any matrix A can be reduced to an upper triangular matrix by a unitary similarity.

Theorem 9.1. *Let $A \in \mathbb{R}^{n \times n}$, then there exists an $n \times n$ unitary matrix such that*

$$T = U^T A U = \begin{pmatrix} \lambda_1 & t_{12} & t_{13} & \ldots & t_{1n} \\ 0 & \lambda_2 & t_{23} & \ldots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \lambda_n \end{pmatrix}.$$

is upper triangular. The matrix U may be chosen such that the eigenvalues of A appear on the diagonal of T in any order. For normal matrices ($AA^T = A^T A$), we have the following result

THEOREM 9.2. *If A is a normal matrix, then there exists a unitary matrix U and a diagonal matrix $\Lambda = diag(\lambda_1, \dots, \lambda_n)$ such that*

$$\Lambda = U^T AU,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix A. The preceding theorem shows that the columns u_i of the unitary matrix U are eigenvectors. For the particular case of symmetric matrices, we have the following result

THEOREM 9.3. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix.*

- *The eigenvalues of A are all real.*
- *There exists a unitary matrix (whose columns are orthonormal eigenvectors of A) such that*

$$A = U\Lambda U^T, \text{ with } \Lambda = diag(\lambda_1, \dots, \lambda_n).$$

This decomposition is usually called the spectral decomposition of A.

The last result shows that any symmetric matrix A can be written in the form

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T.$$

THEOREM 9.4. *(The real Schur decomposition)*
Let $A \in \mathbb{R}^{n\times n}$, then there exists an orthonormal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^T AQ = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ 0 & R_{22} & \dots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{nn} \end{pmatrix}$$

where R_{ii} is a square block of size one or two having complex conjugate eigenvalues.

10. The Singular Value Decomposition. The following theorem shows that every matrix could be decomposed as a product of orthogonal matrices and a diagonal matrix.

THEOREM 10.1. *Let A be a real $n \times m$ matrix. Then there exist two orthogonal matrices*

$$U = [u_1, u_2, \dots, u_n] \in \mathbb{R}^{n\times n} \text{ and } V = [v_1, v_2, \dots, v_m] \in \mathbb{R}^{m\times m}$$

and a diagonal matrix

$$\Sigma = diag(\sigma_1, \dots, \sigma_p), \text{ with } \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0, \quad p = min\{n, m\},$$

such that

$$A = U\Sigma V^T.$$

Proof. As the matrix A^TA is symmetric and postive, it can be diagonalisable in an orthonormal basis of eigenvectors and the eigenvalues are potitive.
Let the eigenvalues of A^TA be $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_r^2 > 0 = \sigma_{r+1}^2 = \ldots = \sigma_n^2$. If $V = [V_1, V_2]$ is an ortogonal matrix formed from the corresponding eigenvectors. Then

$$V^TA^TAV = \begin{pmatrix} \Sigma_+^2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\Sigma_+^2 = diag(\sigma_1^2, \ldots, \sigma_r^2)$. Then we have

$$V_1^TA^TAV_1 = \Sigma_+^2, \text{ and } V_2^TA^TAV_2 = 0.$$

From the second relation, we conclude that

$$AV_2 = 0.$$

We set

$$U_1 = AV_1\Sigma_+^{-1}.$$

Then, we get $U_1U_1 = I$. We choose U_2 such that the square matrix $U = [U_1, U_2]$ is orthogonal. Therefore

$$U^TAV = \Sigma = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{pmatrix},$$

wich ends the proof □

This decomposition is called the Singular Value Decomposition (SVD) of the matrix A, see [3]. The σ_i's are called the singular values of the matrix A. Notice that

$$Av_i = \sigma_i u_i, \ ; \ A^Tu_j = \sigma_j v_j,$$

and

$$A^TAv_i = \sigma_i^2 v_i, \ \ AA^Tu_i = \sigma_i^2 u_i.$$

The u_i's and v_i's are called left singular vectors and right singular vectors, respectively. The largest singular value is denoted by $\sigma_{max}(A)$ while the smallest one is denoted by $\sigma_{min}(A)$.
The singular value decomposition gives many important informations about the matrix A. Some of these properties are listed in the following theorem.

Theorem 10.2. *Consider the SVD given by Theorem 10.1 and define r by*

$$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_p = 0.$$

Then

1. $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$, *with* $r = rank(A)$.

2. $Ker(A) = span\{v_{r+1}, \ldots, v_n\}$.
3. $Range(A) = span\{u_1, \ldots, u_r\}$.
4. $\|A\|_2 = \sigma_1 = \sigma_{max}(A)$.
5. $\|A\|_F^2 = \sigma_1^2 + \ldots + \sigma_r^2$.
6. *The condition number* $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \dfrac{\sigma_{max}}{\sigma_{min}}$.

Another important result on SVD is stated in the following theorem

THEOREM 10.3. *(Theorem of Eckart-Young) Let the SVD of $A \in \mathbb{R}^{n\times m}$ be given as in Theorem 10.2, then if $k < r = rank(A)$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ then*

$$\min_{rank(X)=k} \|A - X\|_2 = \|A - A_k\|_2 = \sigma_{k+1}.$$

This gives the best approximation of rank k. Theorem 10.3 can be used for applications in image processing (compression, transmission). This is shown in the following example:

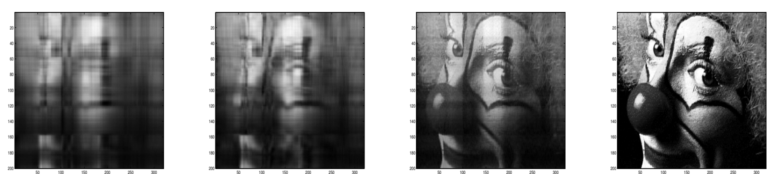

FIG. 10.1. *Low-rank TSVD approximations: $k = 5$, $k = 10$, $k = 50$ and the exact image (200×320)*

The Pseudo-inverse (called the Moore-Penrose inverse) can be expressed in terms of SVD. This is stated in the following theorem

THEOREM 10.4. *Let A be a real $n \times m$ matrix and consider the SVd of A as*

$$A = U\Sigma V^T,$$

then the $m \times n$ pseudo-inverse of the matrix A can be written as

$$A^+ = V\Sigma^+ U^T,$$

where

$$\Sigma^+ = diag\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_r}\right).$$

We also have the following properties

THEOREM 10.5. *Let the matrix A be as in Theorem 10.2, then*

1. $A^+ = \sum_{i=1}^{r} \dfrac{v_i u_i^T}{\sigma_i}$.

2. The matrix AA^+ is the matrix of the orthogonal projection onto $rang(A)$.
3. A^+A is the matrix of the orthogonal projection onto $range(A^T)$.

Next we give an important result on low-rank approximation of a matrix.

THEOREM 10.6. *Let A be an $n \times m$ matrix of rank r. Then*

$$\min_{rank(X)=k<r} \|A - X\|_2 = \sigma_{k+1}$$

where

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > \sigma_{k+1} \geq \ldots \geq \sigma_n$$

are the singular values of A.
A minimzer X_* is given by

$$X_* = \sigma_1 u_1 v_1^T + \ldots + \sigma_k u_k v_k^T. \tag{10.1}$$

We also have

$$\min_{rank(X)=k<r} \|A - X\|_F = \sum_{i=k+1}^{n} \sigma_i^2,$$

The unique minimzer X_* is given by (10.1).

For a pair of matrices, $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times m}$, there exsit two orthonormal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{p\times p}$ and an invertible matrix X such that

$$U^T AX = C = diag(c_1, \ldots, c_q),\ c_i \geq 0$$

and

$$V^T BX = S = diag(s_1, \ldots, s_q),\ s_i \geq 0,$$

where $q = min(p, m)$. This factorization is called the Generalized Singular Value Decomposition (GSVD).

11. The QR decomposition. Let A be a real $n \times m$ matrix and assume that $n \geq m$. Then a QR factorization of the matrix A, consists in computing an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an upper $n \times m$ triangular matrix R such that

$$A = QR.$$

The well known transformations that can compute such a decomposition are the Householder transformation, the Givens and fast Givens transformations and the Gram-Schmidt process (the classical one and the modified version).

11.1. The Householder transformation. [5] Let $v \in \mathbb{R}^n$ be a nonzero vector. The associated Householder matrix [6] is defined by

$$H_v = I - 2\frac{vv^T}{v^T v}.$$

It can be easily shown that H_v is symmetric and orthogonal. Remark also that

$$H_v\, v = -v, \text{ and } H_v x = x \text{ if } x \in span\{v\}^\perp.$$

Let x be any nonzero vector in $\mathbb{R}^n$, then we would like to find a vector v such that

$$H_v x = \alpha e_1,$$

where e_1 is the first unit vector of $\mathbb{R}^n$. The vector v can be chosen as follows

$$v = x + sign(x_1)\|x\|_2 e_1.$$

This simple determination of v makes the Householder reflexions very useful. Notice that applying the Householder transformation H_v on a matrix A leads to the following expression

$$H_v A = (I - 2\frac{vv^T}{v^T v}) = A + uw^T$$

where $w = -2\frac{2}{v^T v} A^T v$. Let see now how to use Householder transformations to get a QR decomposition. Let us do that on a matrix A of dimension 5×3 whose columns are denoted by $a_1^{(i)}$: $A = A^{(1)} = [a_1^{(1)}, a_2^{(1)}, a_3^{(1)}]$

Step 1: we look for a vector $v^{(1)} \in \mathbb{R}^5$ and the corresponding Householder matrix H_1 such that

$$H_1 A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & \star & * \\ 0 & \star & * \\ 0 & \star & * \end{pmatrix},$$

and we set $v^{(1)} = a_1^{(1)} + sign(a_1)\|a_1^{(1)}\|_2 e_1$.

For the second step, we look for the Householder matrix $H_2 = diag(I_1, \widetilde{H}_2)$ such that

$$\widetilde{H}_2 \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix} = \begin{pmatrix} \times \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and then

$$H_2 H_1 A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix},$$

The 4×4 Householder matrix $\widetilde{H}_2$ is defined by

$$\widetilde{H}_2 = I_4 - 2\frac{v^{(2)}(v^{(2)})^T}{(v^{(2)})^T v^{(2)}},$$

where

$$v^{(2)} = a_2^{(2)} + sign((a_2^{(2)}))_1 \|a_2^{(2)}\|_2 e_1^{(2)} \in \mathbb{R}^4$$

and $a_2^{(2)}$ is the first column of the matrix $A^{(2)}$ obtained by deleting the first row and the first column of the matrix H_1A. In the same way, we define the Householder matrix $H_3 = diag(I_2, \widetilde{H}_3)$ such that

$$H_3H_2H_1A = R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Setting $Q = H_1H_2H_3$, the matrix Q is orthogonal and we have

$$A = QR.$$

We notice that the upper triangular part of A could be overwritten by the upper triangular matrix R, while the Householder vectors $v^{(j)}$ could be stored in the lower triangular part of the matrix A as follows

$$A = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ v_2^{(1)} & r_{22} & r_{13} \\ v_3^{(1)} & v_3^{(2)} & r_{33} \\ v_4^{(1)} & v_4^{(2)} & v_4^{(3)} \\ v_5^{(1)} & v_5^{(2)} & v_5^{(3)} \end{pmatrix}.$$

The Householder method requires $2m^2(n-m/3)$ arithmetic operations if the matrix Q is not required explicitly. If this matrix is needed, then the method requires $4(n^2m - nm^2 + m^3/3)$ flops.
Other transformations such as givens or fast Givens could also be used to compute the QR factorization of a matrix A, see [3]. The Householder transformation shows that the QR factorization exists.

11.2. Givens rotations. [2] Givens rotations, allows us to zero many element of a vector selectively. Givens matrices are rank-two corrections of the identity matrix and are defined as follows

$$G(i,k,\theta) = \begin{pmatrix} 1 & \dots & 0\dots & 0\dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & c\dots & s\dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & -s\dots & c\dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0\dots & 0\dots & 1 \end{pmatrix},$$

where $c = \cos(\theta)$, $s = \sin(theta)$ for some θ. The coefficient c is located at the (i,i) and the (k,k) positions while s is the $(i,,k)$-element of the matrix located in the $G(i,k,\theta)$. The $n \times n$ matrix $G(i,k,\theta)$ is orthonormal. If x is a vector in $\mathbb{R}^n$ and $y = G(i,k,\theta)^T x$, then we get

$$\begin{cases} y_i = cx_i - sx_k \\ y_k = sx_i + cx_k \\ y_j = x_j, \text{ for } j \neq i,k \end{cases}$$

If we want to force y_k, $k \neq i, j$, to zero, then we can set

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}}, \text{ and } s = -\frac{x_k}{\sqrt{x_i^2 + x_k^2}}.$$

Hence Givens rotations allows us to zero any component of any vector. Therefore, applying Givens rotation to the matrix A gives the QR decomposition where Q is the orthogonal matrix obtained as a product of Givens matrices.
Consider the QR decomposition of $n \times m$ matrix $A = QR$ (obtained with Householder or Givens transformations) and assume that $n \geq m$ and that A is of full rank. Let

$$Q = [Q_1, Q_2], \text{ and } R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix},$$

where $Q_1 \in \mathbb{R}^{n \times m}$, $Q_2 \in \mathbb{R}^{n \times n-m}$ have orthonormal columns and $R_1 \in \mathbb{R}^{m \times m}$ the square upper triangular matrix part of R with positive entries on its diagonal. Then

$$A = [Q_1, Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1.$$

The last decomposition is unique and is called the skinny QR factorization.

In the next subsection, we will see another well known process allowing us to obtain such a factorization.

11.3. The Gram-Schmidt process. Let the $\{u_1, \ldots, u_k\}$, k vectors of $\mathbb{R}^n$ assumed to be linearly independent. Then the Gram-Schmidt allows us to construct an orthonormal basis $\{q_1, \ldots, q_k\}$ of the space $F = span\{u_1, u_2, \ldots, u_k\}$. In the following, we give the modified version of the process which is more stable numerically.

- $r_{11} = \|u_1\|_2$, $q_1 = \dfrac{u_1}{r_{11}}$
- For $j = 2, \ldots, k$
  1. $\tilde{q} = u_j$
  2. for $i = 1, \ldots, j-1$
     (a) $r_{ij} = \langle \tilde{q}, q_i \rangle$
     (b) $\tilde{q} = \tilde{q} - r_{ij} q_i$
  3. endfor
  4. Compute $r_{jj} = \|\tilde{q}\|_2$.
  5. If $r_{jj} = 0$, stop, else
  6. $q_j = \tilde{q}/r_{jj}$.
- EndFor.

Setting $U = [u_1, \ldots, u_k]$, $Q_1 = [q_1, \ldots, q_k]$ and $R_1 = [r_{ij}]$ the $k \times k$ triangular matrix obtained from the modified Gram-Schmidt process, we get

$$u_j = \sum_{i=1}^{j} r_{ij} q_i, \; j = 1, \ldots, k,$$

and in a matrix form, we have

$$U = Q_1 R_1, \text{ with } Q_1^T Q_1 = I,$$

which is called the Gram-Schmidt QR decomposition of the matrix U also called the skinny QR factorization of U. The $n \times k$ matrix Q_1 has orthonormal columns and R_1 has positive diagonal entries. This skinny factorization is unique, see [3].

12. Application to least-squares problems. We consider here the following Least Squares (LS) problem. Find a vector $x$such that:

$$\min_{x\in\mathbb{R}^m} \|Ax-b\|_2, \tag{12.1}$$

where $A \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ and assume that $n \geq m$.
If we set $\phi(x) = \|Ax-b\|_2^2$, then ϕ is a differentiable function and the minimizers satisfy $\nabla\phi(x) = 0$ where $\nabla\phi$ denotes the gradient of ϕ.
Assume that A has a full rank m. Then there exists a unique solution x_{LS} of the LS problem (12.1) which is the unique solution of the symmetric positive definite linear system

$$A^T A x_{LS} = A^T b.$$

Let us see now how to solve the LS problem (12.1) by using the QR-decomposition. Assume that $A = QR$ where Q is an $n \times n$ orthonormal and R is an $n \times m$ upper triangular matrix. Then

$$\|Ax-b\|_2 = \|Q(Rx-\tilde{b})\|_2$$

where $\tilde{b} = Q^T b$. And then since Q is orthogonal, we get

$$\|Ax-b\|_2 = \|Rx-\tilde{b}\|_2.$$

Setting $R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$, where R_1 is the square upper triangular part of R and $\tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}$, we get

$$\|Ax-b\|_2^2 = \|R_1 x - \tilde{b}_1\|_2^2 + \|\tilde{b}_2\|_2^2$$

and then the minimum solves $R_1 x_{LS} = \tilde{b}_1$ and the corresponding residual $r_{LS} = b - Ax_{LS}$ has the following norm

$$\rho_{LS} = \|r_{LS}\|_2 = \|\tilde{b}_2\|_2.$$

Another way of computing the LS solution is the use of the SVD decomposition of the matrix A. For that, we have the following result.

THEOREM 12.1. *Assume that the matrix A has the SVD decomposition $A = U\Sigma V^T$. Then, the solution x_{LS} of the problem (12.1) is given as follows*

$$x_{LS} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

and

$$\rho_{LS}^2 = \|b - Ax_{LS}\|_2^2 = \sum_{i=r+1}^{n} (u_i^T b)^2,$$

where r is the rank of A.

Proof. We have $\|Ax - b\|_2^2 = \|U(\Sigma V^T x - U^T b\|_2^2$ and since U is orthonormal, we get $\|Ax - b\|_2^2 = \|\Sigma V^T x - U^T b\|_2^2$ which can be expanded as

$$\|\Sigma V^T x - U^T b\|_2^2 = \sum_{i=1}^{r} (\sigma_i z_i - u_i^T b)^2 + \sum_{i=r+1}^{n} (u_i^T b)^2.$$

Therefore, the minimum is obtained for $z_i = (u_i^t b/\sigma_i)$. □

REMARK 6. *Notice that the solution x_{LS} can also be written in term of the pseudo-inverse as*

$$x_{LS} = A^+ b = V\Sigma^+ U^T b = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

13. Gaussian elimination and the LU factorization.

13.1. Gaussian elimination. In this section, we present the classical Gaussian method for solving linear systems of equations

$$Ax = b, \tag{13.1}$$

where $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, $x = (x_1, \ldots, x_n)^T$ and $b = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$. At the k stage of the Gauss process, we solve the linear sytem

$$A^{(k)} x = b^{(k)},$$

where

$$A^{(k)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \ldots & \ldots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \ldots & \ldots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \ldots & \ldots & a_{3n}^{(3)} \\ \vdots & \vdots & & \ldots & \ldots & \vdots \\ 0 & & & a_{kk}^{(k)} & \ldots & a_{nk}^{(k)} \\ 0 & & & 0 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots & \ldots & \vdots \\ 0 & & & 0 & \ldots & 0 \end{pmatrix}.$$

If at step k, the pivot $a_{kk}^{(k)} \neq 0$, then in order to compute the new entries of the matrix $A^{(k+1)}$, we have the following relations

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - a_{ik}^{(k)} (a_{kk}^{(k)})^{-1} a_{kj}^{(k)}, \ k+1 \le i, j \le n,$$

and

$$b_i^{(k+1)} = b_i^{(k)} - a_{ik}^{(k)} (a_{kk}^{(k)})^{-1} b_k^{(k)}, \ i = k+1, \ldots, n.$$

Notice that if we set

$$S_{ij}^{(k)} = \begin{pmatrix} a_{ij}^{(k)} & a_{ik}^{(k)}, \\ a_{kj}^{(k)} & a_{kk}^{(k)} \end{pmatrix},$$

then $a_{ij}^{(k+1)}$ is the Schur complement of $a_{kk}^{(k)}$ in $S_{ij}^{(k)}$:

$$a_{ij}^{(k+1)} = (S_{ij}^{(k)} / a_{kk}^{(k)}).$$

14. The LU factorization. We are looking for an upper triangular matrix $U = [u_{ij}]$ and a unit lower triangular matrix $L = [l_{ij}]$ such that

$$A = LU.$$

Setting $A^{(1)} = A$, and assuming that $a_{11} \neq 0$, we define the first Gaussian transformation by

$$M_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -\dfrac{a_{21}^{(1)}}{a_{11}^{(1)}} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -\dfrac{a_{n1}^{(1)}}{a_{11}^{(1)}} & 0 & \dots & 1 \end{pmatrix} = I - \tau^{(1)} e_1^T,$$

with

$$\tau^{(1)} = (0, a_{21}^{(1)}/a_{11}^{(1)}, \dots, a_{n1}^{(1)}/a_{11}^{(1)})^T.$$

Therefore,

$$A^{(2)} = M_1 A^{(1)}.$$

The matrix $A^{(2)}$ can be expressed as follows

$$A^{(2)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & \dots & a_{nn}^{(2)} \end{pmatrix}.$$

Assuming that the second pivot $a_{22}^{(2)} \neq 0$, define the second Gaussian transformation as follows

$$M_2 = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ 0 & -\dfrac{a_{32}^{(2)}}{a_{22}^{(2)}} & 1 & \dots & 0 \\ 0 & \vdots & \vdots & \ddots & 0 \\ 0 & -\dfrac{a_{n2}^{(2)}}{a_{22}^{(2)}} & 0 & \dots & 1 \end{pmatrix} = I - \tau^{(2)} e_2^T,$$

with

$$\tau^{(2)} = (0, 0, a_{32}^{(2)}/a_{22}^{(2)}, \dots, a_{n2}^{(2)}/a_{22}^{(2)})^T.$$

Then, we have

$$A^{(3)} = M_2 A^{(2)} = M_2 M_1 A.$$

At the end of the process, we have constructed the Gauss matrices $M_1, M_2, \dots, M_{n-1}$ such that

$$M_{n-1} \dots M_2 M_1 A = U$$

where U is an upper triangular matrix. To define the matrix L, we first notice that

$$M_i^{-1} = I + \tau^{(i)} e_i^T,$$

where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$. Therefore, setting

$$L^{-1} = M_{n-1} \dots M_2 M_1,$$

we get

$$L = M_1^{-1} \dots M_{n-1}^{-1} = I + \sum_{k=1}^{n-1} \tau^{(k)} e_k^T,$$

and we get the LU factorization

$$A = LU.$$

The lower part of A is overwritten by the lower part of L and the upper part of A is overwritten by the upper part of U. For the existence of the LU factorization, we have the following theorem.

THEOREM 14.1. *The matrix $A \in \mathbb{R}^{n \times n}$ has an LU factorization if the determinants of the leading principal submatrices $A_k = A(1:k, 1:k)$ are such that $det(A_k) \neq 0$, for $k = 1, \dots, n-1$. In this case and if A is nonsingular this factorization is unique.*

Of course this factorization is possible if all the pivots $a_{kk}^{(k)}$ are nonzero. If it is not the case, then we have to use permutation matrices allowing us row permutation (partial pivoting) and the decomposition is expressed as

$$PA = LU$$

where P is a permutation matrix.

THEOREM 14.2. *(Factorization LDM^T) If all the leading principal submatrices of A are nonsingular, then there exit two unit lower triangular matrices L and M and a digonal matrix D such that*

$$A = LDM^T.$$

If A is symmetric then $M = L$ and

$$A = LDL^T.$$

In the particular case where A is symmetric and positive definite, we have the following result

THEOREM 14.3. *If the matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, then the factorization $A = LDL^T$ is unique and there exists a unique lower triangular matrix G ($G = LD^{1/2}$) with positif diagonal elements such that*

$$A = GG^T$$

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