Sequential Bargaining with Multiple Buyers*

Yihang Zhou[†], David Sibley[‡]

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Abstract

A seller bargains with two buyers to make a deal with each of them, using an alternating-offer protocol ("AO"). The bargaining begins with one buyer, with the second entering at a future date. The seller has a concave utility function defined over total payments from buyers, so the two bargains affect each other. The absolute risk aversion properties of the utility function and the arrival date determine how the two bargains affect each other and whether there is a first or second-mover advantage. Although agreements in our model are reached on different dates, the usual limit payoffs for AO do not approach those of the sequential Nash bargaining solution ("NBS"). Instead, the limit payoff of our sequential AO model is the simultaneous NBS, which is the same as the limit of the simultaneous AO model. We also compare seller utility with sequential bargaining to simultaneous bargaining with both buyers. We find that the seller prefers simultaneous bargaining given the choice.

Keywords: Sequential bargaining, multilateral bargaining, concave utility

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[†]Department of Economics, Nankai University, yhzhou96@163.com

[‡]Department of Economics, University of Texas at Austin, sibley@austin.utexas.edu

1 Introduction

The literature on alternating offer ('AO') bargaining models beginning with Rubinstein (1982) has focused on interactions between a single buyer and a single seller. Within this setup, there have been hundreds of variations. However, despite the fact that actual bargains between buyers and sellers usually take place within a broader economic setting, little has been said about how AO behavior would affect market outcomes.

We want to use the AO bargaining approach to say something about how bargaining affects market outcomes. To do so, we need to account for the fact that the bargain made by one buyer with a seller may affect the profits of another buyer who will make her own bargain with the seller. It is clear that bargaining theory must consider the interdependence between bargains if it is to be useful in elucidating market behavior. To do this, we have extended the AO bargaining model to account for different, but interdependent, bargains.

The starting point of our paper is that we study the interactions between bargains. We analyze a bargaining model with one seller and two buyers. Different from Horn and Wolinsky (1988) who use Nash bargaining solution, we based ours on the solution concept of Rubinstein (1982).

We differ from much of the Rubinstein-based literature, in that we assume two buyers and a concave utility function for the seller. This leads to interdependence between two sequential bargains. For the sake of simplicity, we assume that buyers have linear utility.

In our model, the seller and one of the buyers are in the game at the beginning, but another buyer arrives at a known future date. The seller bargains for a contract with each of the two buyers. For most of this paper, a contract generates a gross flow payoff of 1 to the buyer each period, and buyers and the seller bargain over the flow prices paid to the seller each period.

Before the arrival of the second buyer, there is a two-player AO bargaining game between the seller and the first buyer. If they reach an agreement before the second buyer arrives, there will be another two-player AO bargaining game between the seller and the second buyer. If the first buyer does not have an agreement when the second buyer arrives, the seller will bargain with the two buyers simultaneously in an AO bargaining protocol, for each bargain. In that case, the seller makes offers to both buyers simultaneously in odd periods and two buyers make offers to the seller simultaneously in even periods.

As an application, we consider the situation of two health insurance providers who want to include a particular hospital in their provider networks. To do so, insurers need to pay a fixed transfer to the hospital. The hospital will then provide treatment to patients who buy insurance from one of the two insurers. In this example, the hospital is the seller, and the insurers are the buyers. The hospital can be included in one, none, or both networks and will bargain with the two insurers to decide on the transfers. Two negotiations can start on different dates. Alternatively, the hospital may be renewing contracts with two insurers, and those contracts have different termination dates.

To motivate the assumption of the hospital's concave utility in total payments, we imagine that the hospital uses these payments to fund projects requiring investment money, such as outpatient clinics, imaging centers, etc. Ranking these investment opportunities by profitability, the hospital will start from the project with the highest return and work its way down, as long as it has the funds to do so. Thus, the marginal payoff from successive monetary payments will decrease. This will generate a concave seller utility function defined over total payments to the seller.

Given the concave seller utility in our model, if the seller already has a payment from the first bargain, she is less eager for the extra payment from a second bargain. This fact leads to interactions between the bargain with the first buyer and the bargain with the second buyer. If we assume linear utility for the seller, then the two bargains do not affect each other. The outcome of our model would simply be two separate Rubinstein bargains.

If the first buyer has already reached an agreement with the seller when the second buyer arrives, the second flow price is affected by the first flow price. The specific form of the utility function determines just how the second flow price is affected.

As we will see, we can think of seller utility as if it were over a special kind of lottery. Risk preferences in this implied lottery drive the relationship between bargains. We find the following.

- When the seller has decreasing absolute risk aversion ("DARA"), the second price increases with the first price.[1]

- When the seller has increasing absolute risk aversion ("IARA"), the second price decreases with the first price.

If the first buyer does not reach an agreement with the seller until the second buyer arrives, the seller will propose prices to both buyers simultaneously when it is her turn and the buyers will propose simultaneously to the seller in their turns. The equilibrium in this continuation game is symmetric, where buyers pay equally.

Based on the results above, we construct an equilibrium for the whole game, where the two agreements are reached immediately after each buyer arrives. In this equilibrium, though two buyers are identical except for the arrival dates, they end up paying different prices[2]. The shape of the utility function and, also, the arrival date T determine whether there is a first-mover advantage or second-mover advantage in sequential bargaining.

As a benchmark, we compare the two-buyer equilibrium to the one-seller-one-buyer Rubinstein bargaining equilibrium with the same seller utility function, which serves as the bargaining unaffected by the other buyer. We find that the first buyer can benefit from the presence of a second buyer in some situations, and be harmed in others. For example, with the square root seller utility (a DARA utility), the first buyer can benefit from the fact that there is a subsequent bargaining process with the second buyer.

We also consider certain limiting results of our model. First, we consider a simple limiting result, where time remains discrete, but the discount factor approaches one. In this exercise, the limit outcomes in each bargain are the same as in simultaneous Nash bargaining. However, even with the discount factor going to one, the solution to our sequential AO bargaining model does not go to *sequential* Nash bargaining solution ("NBS"). In other words, the relationship between the two bargains is different from that generated by sequential Nash bargaining. Instead, the limit is the same as that generated by simultaneous Nash bargaining, which is also the limit of simultaneous AO bargaining. The discrete difference in information between sequential and simultaneous AO bargaining is no longer important

[1]A utility $u(.)$ displays DARA, if and only if $-\frac{u''(x)}{u'(x)}$ decreases with x. With this property, the player becomes less risk-averse when she is wealthier. IARA has an opposite statement.

[2]This is also true in sequential Nash bargaining.

as players become patient. Second, we make the intervals between two proposals arbitrarily small, leaving the arrival time of the second buyer unchanged. Though this limiting result is asymmetric, it is not a sequential NBS.

Next, we allow the seller to choose between sequential bargaining with the two buyers and simultaneous bargaining with them. We find that the seller prefers simultaneous bargaining to sequential, given the choice.

Finally, we extend the analysis to a vertical market setting. The two buyers are downstream duopolists. Each bargains with a single upstream seller of a critical input. One buyer enters first, the other second. We find that the length of time until entry affects the price negotiated between the seller and the incumbent, which does not appear in the traditional vertical model using take-it-or-leave-it offers.

Our results have implications for both theoretical and empirical work using Nash Bargaining models. On the theoretical side, Collard-Wexler, Gowrisankaran and Lee (2019) have shown that the Nash-in-Nash approach to bargaining has a strong microfoundation in alternating offers bargaining theory. However, there are limits to theories of microfoundations for the Nash-in-Nash approach.

We show that this microfoundation result does not extend to a Nash-in-Nash model based on sequential Nash bargaining. The seminal paper in this area is that of Horn and Wolinsky (1988). A number of other papers apply its insights to various problems in applied bargaining; see Grennan (2013). We find that even as δ goes to 1, the limiting sequential AO result does not go to the sequential Nash Bargaining result[3]. Therefore, a Nash-in-Nash model based on sequential Nash bargaining cannot be said to have the same microfoundation as the one based on simultaneous Nash bargaining.

Our results have implications for empirical work, too. Current examples include bargaining between insurers and providers. Examples are: Gowrisankaran, Nevo and Town (2015), Capps, Dranove and Satterthwaite (2003), and Grennan (2013). The papers implement the simultaneous Nash bargaining model. They use data from a number of such negotiations to do this.

Typically, these negotiations take place at different times, so we cannot rule out the

[3]See Proposition 5 and 6 in Section 3.5.

possibility that one such negotiation was affected by others. If so, there may be a causal effect of the outcome of one bargain on the outcomes of future bargains.

Such effects will not show up in typical empirical Nash bargaining models. These models usually assume passive beliefs. This has great benefits in simplifying the bargaining problem. As Gowrisankaran, Nevo and Town (2015) show, assuming passive beliefs makes the bargaining between a hospital and one insurer independent of other bargains that the hospital may have completed.

Hence, any causal effects from one bargain to some future bargains are ruled out by assumption. Clearly, the passive beliefs assumption in Nash bargaining models is likely to lead to bias due to excluded variables. However, our model suggests that even with passive beliefs, bargains can interact with each other in the alternating-offer bargaining model with concave seller utility.

A second empirical implication is that models based on simultaneous Nash bargaining solutions are misspecified. Because the Nash bargaining model is static, when considering two players arriving sequentially, it cannot accommodate a change in the entry date of the second player. In our model, if the entry date were to change, the equilibrium outcome would also change. If we use the analytical outcomes of our model as data, the Nash bargaining parameters fitted for one set of data under a certain entry date would no longer fit the new data with other entry dates.

The organization of the paper is as follows. Section 2 provides the setup. Section 3 introduces the main message about the interaction between bargains. Section 4 discusses simultaneous bargaining results and bargaining strategy results. Section 5 extends the model to a vertical market structure with a downstream market. Finally, Section 6 concludes the paper.

Related literature. Though there is a large literature on bargaining, most papers study single-episode bargaining problems, where two players bargain to split one pie. They have studied this problem with many variations, such as incomplete information (Fudenberg and Tirole, 1983; Cramton, 1992), and different bargaining protocols (Baron and Ferejohn, 1989; Cho, 1990).

There are only a few papers studying multiple-episode bargaining problems. For example, Horn and Wolinsky (1988) studied a monopoly supplier of an input bargaining with two firms to sell inputs to both firms, while those firms compete in a downstream output market. In their model, the input prices are determined by sequential NBS between the seller and the two downstream firms. Their bargains interact because there is downstream competition between firms.

A recent paper, Abreu and Manea (2023), studies the situation where a seller sells products to a group of buyers using the random-proposer bargaining protocol. Players have linear utilities, and the interaction between the bargains with different buyers comes from scarce seller production capacity. Because the seller does not have enough output to satisfy the demand of all buyers, there will be competition among buyers for output. Hence, bargains are interdependent. The buyers do not compete with each other in a downstream setting.

Unlike the above two papers, we use the solution concept of Rubinstein (1982) to study the multiple-episode bargaining problem. The interaction across bargaining episodes does not come from the interactions between buyers as in Abreu and Manea (2023) and Horn and Wolinsky (1988). Instead, the interactions come from the concave utility function of the seller. The two bargains do not require buyers to compete with each other for their bargains to be interdependent. As noted above, the concavity may reflect differing investment options available to the seller.

The use of a concave utility function in bargaining has also been studied in the literature. Hoel (1986) studies nonlinear utility in the bargaining problem and finds that the limit price of AO bargaining goes to the Nash bargaining price as players become patient. Sobel (1981) considers a concave, increasing von Neumann-Morgenstern utility function for players and establishes the relationship between a class of bargaining solutions, like those of Nash and Raiffa-Kalai-Smorodinsky. Crawford and Varian (1979) finds that even if players are allowed to misrepresent their utilities as a weakly concave function in the Nash bargaining problem, they would report their utility as linear utility functions. Volij and Winter (2002) study risk aversion in the bargaining game, which is a property of the concave utility function. White (2008) studies the effect of prudence in a bargaining problem with risk. Adding to the literature, our model studies how risk aversion affects the interaction between two bargains instead of within one bargain.

2 The Model

We study a bargaining model with three players: one seller, and two buyers (B_1, B_2). The seller wants to make two contracts and each buyer demands one contract. The seller bargains with each buyer for one contract of infinite duration.

Time is discrete, and the horizon is infinite ($t = 1, 2, ...$). The seller and B_1 arrive at $t = 1$ and start bargaining, while B_2 arrives at date $T > 1$. Here we assume T to be odd to ensure that the seller is still the first proposer when B_2 arrives. Departing from the model in Rubinstein (1982), we have a second buyer arriving at date T and starting to bargain with the seller thereafter.

When the seller and the buyer reach an agreement, the contract between them takes effect immediately, and lasts permanently. The buyer gets a flow gross payoff of 1 from the contract each period, and she also pays a flow price of x to the seller each period, which is determined by bargaining. Thus, the flows of monetary payoffs are x to the seller and $1 - x$ to the buyer. For the most parts, we assume that the seller has a concave utility function $u(.)$ over the total payment in a period. The utility $u(.)$ is (1) strictly increasing, (2) twice differentiable, and (3) strictly concave. And we also normalize that $u(0) = 0$. Moreover, since the lowest payment to the seller is 0 and the highest is 2, $u(x)$'s domain is $[0, 2]$. On the contrary, buyers have a linear utility $v(x) = x$.[4] Suppose that the flow prices in two bargains are x_1 and x_2 respectively. When only one bargain is made, the seller's utility for each period is $u(x_1)$; when both bargains are made, the seller's utility for each period is $u(x_1 + x_2)$. As for buyers, they have utility flows of $1 - x_1$ and $1 - x_2$. Before the agreements are reached, players get 0 payoff from bargaining. Moreover, their flow payoffs are subject to a discount factor $\delta \in (0, 1)$, and we will normalize flow payoffs to $(1 - \delta)u(x)$ and $(1 - \delta)(1 - x)$.

[4] Our results can be generalized to a strictly increasing and strictly concave buyer utility $v(x)$. By letting $\hat{v} = v(x)$, we can write the seller's payoff as $u(v^{-1}(\hat{v}))$, and the buyer's payoff as $\hat{v}$. Notice that if $v(.)$ is concave, the function $u(v^{-1}(\hat{v}))$ is a decreasing and concave function in $\hat{v}$, so we can transform the problem of two concave utility functions into the problem of one concave utility function and one linear utility function.

2.1 The Bargaining Protocol

Players use alternating-offer (AO) bargaining to determine the flow payment in each contract, as in Rubinstein (1982). In odd periods, the seller proposes a flow price to every active buyer, and each active buyer decides to accept or reject; in even periods, each active buyer proposes to the seller, and the seller decides to accept or reject. Once an offer is accepted, the contract takes effect immediately, and the buyer becomes inactive.

Before T, there is only one buyer in the game, which is a one-to-one alternating offer bargaining. After T, there are two possibilities. If B_1 has left the game, the bargain between the seller and B_2 is the same as Rubinstein bargaining, apart from the concavity of the seller's payoff. If B_1 has not left the game, then the seller bargains simultaneously with the two buyers. In that case, the seller makes offers to both buyers simultaneously in her turns and two buyers propose offers to the seller simultaneously in their turns.

2.2 Strategies and the Equilibrium Concept

We assume that the seller observes the whole history of the game. This includes all past offers, acceptances, and whether buyers are active in the game. A buyer can observe the whole history of her own bargaining. However, the buyer cannot observe the detailed offers made in the other bargaining process and can only observe the final result of the other bargain after it is completed—the agreement date and the agreement price.[5] For example, if one agreement is reached in period t at price p_t, starting from period $t + 1$, the other buyer knows t and p_t. Before an agreement is reached, the other buyer cannot observe the offers and acceptances in this bargaining.

In an odd period, the seller proposes a flow price $x \in [0, 1]$ to each active buyer in the game, conditional on the whole history. In even periods, she sees the flow prices offered by each active buyer, as well as the past history, and decides whether to accept or reject.

As for the buyers' strategy, in an even period, an active buyer will propose flow prices $y \in [0, 1]$ to the seller based on the history that she observes. In an odd period, an active buyer will decide whether or not to accept an offer from the seller, given the offer from the

[5]If the other buyer has not made an agreement, the buyer knows that she is still in the game.

seller and the history observable to that buyer.

In a given period, if there are two contracts that have been made, and B_1 pays a flow price x_1 and B_2 pays a flow price x_2, then the normalized flow payoff of the seller is $(1-\delta)u(x_1+x_2)$, while two buyers have the flow payoff $(1-\delta)(1-x_1)$ and $(1-\delta)(1-x_2)$. Here $1-\delta$ is scaling the payoff.

We use perfect Bayesian equilibrium (PBE) as the equilibrium concept. In this equilibrium, at each information set, players maximize the discounted sum of normalized flow payoffs. Also, we apply the refinement of *passive beliefs*, which is a common assumption in empirical works (Gowrisankaran, Nevo and Town, 2015; Schulman and Sibley, 2023).

This refinement is important when the seller bargains with two buyers simultaneously. Here *passive beliefs* refer to B_i's belief about what is happening with B_j in the current period when B_i receives an out-of-equilibrium offer from the seller, which affects her continuation payoff. With passive beliefs, B_i will believe the seller is playing the on-path strategy to B_j in the current period, even if B_i observes an out-of-equilibrium offer from the seller. Similarly, when a buyer observes the out-of-equilibrium accepting or rejecting of the seller, she still believes that the seller and the other buyer are playing on-path actions in this period.

3 Equilibrium Behavior in Sequential AO Bargaining

To solve for the equilibrium of the whole game, we first need to specify what happens in the bargaining between the seller and B_2 if the seller already has an agreement with B_1 before the date T. We also need to specify what will happen if there is no agreement before the date T.

3.1 Bargaining after Having an Agreement with B_1

In this section, we analyze what would happen in the second bargain, if the seller and B_1 have already reached an agreement before B_2 arrives.

The equilibrium of the continuation game can be constructed in the same way as in Rubinstein (1982). Let the seller propose the price x_2 in odd periods, and B_2 proposes the price y in even periods. These offers satisfy the following conditions in equilibrium:

$$\begin{cases} 1 - x_2 = \delta(1 - y) \\ u(x_1 + y) = (1 - \delta)u(x_1) + \delta u(x_1 + x_2) \end{cases} \tag{1}$$

The first equation requires that the seller's proposal makes B_2 indifferent between accepting and rejecting. The second equation requires that B_2's proposal makes the seller indifferent between accepting and rejecting. From (1), we can solve for x_2 as a function of x_1, denoted as $\hat{x}_2(x_1)$. This is the price that the second bargain will arrive at, given the price reached in the first bargain.

If the seller has an agreement with B_1 at price x_1, the unique equilibrium outcome of the continuation game starting at T is that the seller proposes a price following a function $\hat{x}_2(x_1)$ and B_2 accepts it at date T, where $\hat{x}_2(x_1)$ is determined by (1). This statement can be proved using an argument similar to Shaked and Sutton (1984). See Appendix A.1.

Given a payment in the first bargain, this will affect the outcome of the second bargaining. The direction of the effect of x_1 on $\hat{x}_2(x_1)$ depends on the specific form of the utility $u(.)$. We address this in Proposition 1.

Proposition 1

- *If $u(.)$ has decreasing absolute risk aversion (DARA) , $\hat{x}_2(.)$ is increasing*
- *If $u(.)$ has increasing absolute risk aversion (IARA) , $\hat{x}_2(.)$ is decreasing*

To see this, we first rewrite (1) as

$$u\Big(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}\Big) = \delta u\big(x_1 + \hat{x}_2(x_1)\big) + (1 - \delta)u(x_1) \tag{2}$$

The RHS of (2) is the seller's payoff from rejecting B_2's offer, which can be regarded as the expected payoff of an implicit lottery. This lottery has the probability of δ that the seller gets $x_1 + \hat{x}_2(x_1)$, and a probability of $1 - \delta$ that she gets x_1. Thus, the LHS, which is the buyer's proposal, can be thought of as the certainty equivalent of the implied lottery. Figure 1 illustrates the expected payoff of the lottery (point A) and its certainty equivalent (point

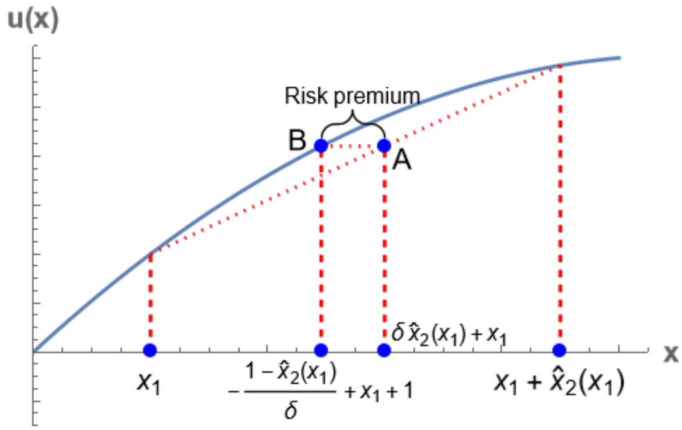


Figure 1: How risk aversion affects the second price

B), as well as the risk premium. Corresponding to point A is the expected payoff $x_1+\delta\hat{x}_2(x_1)$. At point B, the certainty equivalent payoff of the seller is given by $-\frac{1-\hat{x}_2(x_1)}{\delta}+x_1+1$

If the seller is more risk-averse, she has a larger risk premium and thus a smaller certainty equivalent. As a result, B_2 will propose less to her, which leads to a smaller flow price $\hat{x}_2(x_1)$.

Consider a utility function $u(.)$ with DARA. If x_1 is larger, then the seller is more wealthy. According to DARA, the seller is less risk averse with a larger x_1. Thus, as x_1 goes up, the certainty equivalent of the lottery, which is also the buyer's proposal, becomes larger, which leads to a larger $\hat{x}_2(x_1)$. Similarly, with IARA, $\hat{x}_2(x_1)$ decreases with x_1.

Though there is no uncertainty involved in our model, the risk aversion of the seller still plays a key role here. This is similar to Roth (1985), he also finds that even without uncertainty, the "strategic risk" inherent in bargaining makes players with more risk aversion worse off.

Thus far, we have regarded δ only as a discount factor. However, it can also be interpreted as implying a breakdown probability. Having a discount factor of δ is the same as having a breakdown probability $1-\delta$, because both of them make the rejection payoff be $(1-\delta)u(x_1)+\delta u\big(x_1+\hat{x}_2(x_1)\big)$. In this interpretation, a seller who is less risk-averse is less afraid of the possible breakdown and can ask more from the buyer.

3.2 Simultaneous Alternating Offer Bargaining with Two Buyers

In this section, we ask what would happen if there is no agreement made before the date T. We assume that this would lead to simultaneous bargaining at that point.

The two buyers can make or receive different offers. However, in the equilibrium of this continuation game, the outcomes are symmetric: buyers pay equally to the seller.

Suppose that, in equilibrium, the seller proposes x^* to both buyers and both buyers accept. If a buyer rejects, she believes that the other buyer will accept the offer x^*, so the rejecting buyer will bargain with a seller who has a contract at the price x^* starting next period. As a result, in the next period, the rejecting buyer proposes $1 - \frac{1-\hat{x}_2(x^*)}{\delta}$ in her turn. Then, x^* should make the buyer indifferent between accepting and proposing in the next period. This implies that x^* is a fixed point:

$$1 - x^* = \delta(1 - (1 - \frac{1 - \hat{x}_2(x^*)}{\delta})) \Leftrightarrow x^* = \hat{x}_2(x^*)$$

Proposition 2 *If B_1 does not reach an agreement before date T, the equilibrium outcome in the continuation game starting at date T is: the seller proposes x^* to both buyers at date T and the two buyers accept the offers, where x^* solves $x = \hat{x}_2(x)$.*

Proposition 2 also says that there are no asymmetric results in this continuation game. The detailed proof for Proposition 2 is in Appendix A.4. Notice that though any equilibrium must have the condition $x = \hat{x}_2(x)$, uniqueness is not guaranteed. This is because $\hat{x}_2(x)$ may have more than one fixed point.

With IARA, $\hat{x}_2(x)$ is decreasing according to Proposition 1. In this case, $\hat{x}_2(x) = x$ has only one solution.

However, in other cases, $\hat{x}_2(x) = x$ may have more than one solution. One sufficient condition to guarantee the uniqueness of the fixed point is that the function $\hat{x}_2'(x)$ is monotone.

Notice that the possible multiplicity of fixed points here does not lead to a delayed agreement, i.e., the agreement that is not made immediately at date T. With multiple equilibria, one possible way to construct a delay in equilibrium is that in the current period, players are playing a bad equilibrium for buyers, but in the next period, they play a better equilibrium for buyers, so buyers may want to reject the current offer. However, in such a candidate equilibrium where the two buyers reject on date T, it is always beneficial for the seller to deviate to an offer that makes buyers indifferent between accepting and rejecting (and thus accepting). By doing this, the seller is in the situation of the first proposer, while letting buyers reject makes buyers the first proposer. Being the first proposer grants the seller an advantage because there is a discount cost in delay that would induce buyers to accept the seller's higher price.

3.3 Equilibrium Results

With these results, we can construct the equilibrium of our model. At date $T-1$, the seller knows that if she rejects, then, in the next period, she will get involved in simultaneous bargaining with two buyers. She will get a flow payoff $u(2x^*)$ from that. Considering this, B_1 will do backward induction by making the seller indifferent between accepting and rejecting. That is, B_1 will make an offer of p_{T-1} such that $(1-\delta)u(p_{T-1}) + \delta u(p_{T-1} + \hat{x}_2(p_{T-1})) = \delta u(2x^*)$. Further backward induction gives the seller's proposal at date $t = 1$, which will be accepted by B_1 immediately. Denote the proposed flow price at date $t = 1$ in equilibrium as x_1^*. Then, the second bargain generates an equilibrium flow price of $x_2^* = \hat{x}_2(x_1^*)$, and the agreement is done at date T.

Because our equilibrium concept requires sequential rationality, the equilibrium of the game must satisfy the backward induction described above, and the first agreement will be reached at date 1. Moreover, in Appendix A.1, we prove that after having the first agreement of price x_1^* at date 1, the seller proposing price $\hat{x}_2(x_1^*)$ and B_2 accepting immediately is the unique equilibrium outcome of the continuation game starting at date T. Thus, we have the following proposition.

Proposition 3 *The equilibrium outcome of the game is that two agreements are reached immediately at date* 1 *and date* T.

To study the uniqueness of the equilibrium, we need the following lemma.

Lemma 1 $\hat{x}_2'(x) > -\delta$.

See Appendix A.3 for proof. According to Lemma 1, we know $x + \hat{x}_2(x)$ is increasing, so each step of backward induction has a unique outcome.[6] The process of backward induction does not by itself lead to multiple outcomes, so whether or not we have a unique equilibrium in the game mostly depends on whether or not we have a unique equilibrium in a simultaneous bargaining process starting at date T.

The simultaneous bargaining process with two active buyers has a unique result when $\hat{x}_2(x)$ has a unique fixed point. The result of backward induction is unique, so our x_1^* is unique in this case. Moreover, the outcome is unique in the continuation game after the seller and B_1 reach an agreement of x_1^*. Therefore, the outcome that B_1 pays a flow price of x_1^* and B_2 pays a flow price of $x_2^* = \hat{x}_2(x_1^*)$ is the unique equilibrium outcome in our model when $\hat{x}_2(x)$ has a unique fixed point.

3.4 How Bargains Affect Each Other

To see how these two episodes of bargaining affect each other, we compare the equilibrium flow prices in our model (x_1^*, x_2^*) to equilibrium prices in the one-seller-one-buyer Rubinstein bargaining model, with the same utility function. If the results are different, then the two bargains must be affecting each other. We refer to the one-seller-one-buyer Rubinstein bargaining with the same utility function as *unaffected bargaining* and denote its flow price outcome as x_U. In the application of the hospital and insurers, this comparison compares the situation of the insurer being the only buyer of the hospital's services, with the outcome with a second buyer.

When $u(.)$ is concave, because x_1^* and x_2^* are generically unequal to the unaffected bargaining price x_U, the existence of one bargain must affect the other bargain. How the first flow price affects the second flow price depends on the form of $u(.)$. Players in the first bargain will change their actions strategically due to the existence of such an effect. However, if $u(.)$ is linear, there is no interaction between two bargains. Therefore, the interaction

[6] Let p_t be the price at date T. For each step of backward induction, the condition is either $1 - p_t = \delta(1 - p_{t+1})$ (the seller is the proposer) or $(1 - \delta^{T-t})u(p_t) + \delta^{T-t}u(p_t + \hat{x}_2(p_t)) = (\delta - \delta^{T-t})u(p_{t+1}) + \delta^{T-t}u(p_{t+1} + \hat{x}_2(p_{t+1}))$ (B_1 is the proposer). Clearly, $1 - x$ is monotone in x. But we have $x + \hat{x}_2(x)$ increasing in x, which means $(1 - \delta^{T-t})u(x) + \delta^{T-t}u(x + \hat{x}_2(x))$ increases in x. Thus, each step of backward induction will give a unique outcome.

between the two bargains comes entirely from the concavity (risk aversion) of $u(.)$.

The relationships between x_1^*, x_2^*, and x_U are ambiguous. They depend on the specific form of the utility function $u(.)$ and on T.

If $u(.)$ has DARA, then $x_2^* > x_U$. If $u(.)$ has IARA, then $x_2^* < x_U$. To see this, notice that the outcome from unaffected bargaining is the same as the outcome in the second bargaining process when the first price is 0 in our model, i.e., $x_U = \hat{x}_2(0)$. The seller is less wealthy with 0 first price, so with DARA, the seller is more risk averse in this situation, and the price she can get is lower. For IARA, the seller becomes less risk-averse with a zero first price.

As for the relationship between x_1^* and x_U, to assume DARA does not drive the result. For example, assume a utility function with DARA, $u(x) = \sqrt{x}$, and assume also that $\delta = 0.9$, we have $x_1^* > x_U$ with $T = 3$, and $x_1^* < x_U$ with $T = 9$.

If we assume IARA for the seller, we have $x_1^* < x_U$. See Appendix A.6.

Summarizing the results above, we have the proposition:

Proposition 4

- *If* $u(.)$ *has IARA,* x_1^*, $x_2^* < x_U$.
- *If* $u(.)$ *has DARA,* $x_2^* > x_U$, *but the relationship between* x_1^* *and* x_U *is ambiguous.*

Compared to the situation where there is only one buyer, B_1 is better off if she has another buyer in the game if the seller has IARA. With the more realistic assumption of DARA, being the only buyer is better than being the second one in the sequential bargaining model. However, when compared with being the first one in sequential bargaining, it is ambiguous whether being the only buyer is better off or worse off. The first buyer knows that her payment also changes the seller's payment in the second bargaining process. This interaction in the bargaining can make the first buyer pay more or pay less than if she were the only buyer.

The important role that the arrival date T plays here emphasizes the limit of the Nash bargaining model in empirical use. For example, when we consider an incumbent and an entrant, if we want to know how the arrival dates affect prices, the Nash bargaining model cannot help with this. However, our model can shed light on issues like this.

3.4.1 Forces Affecting Bargaining Outcomes

In this section, we examine the economic forces underlying the setting of the payments made by the first and second buyers. The setting of the first price is complicated. This is because it is affected by two forces. One force is the absolute risk aversion of the seller. The second force is the seller's marginal utility from the second price, which affects the marginal utility of the first price as well. We have discussed how the first force affects the second price x_2^*. However, the first price x_1^* is affected by both forces.

To see why x_1^* is affected by both forces, recall that x_1^* is determined by backward induction. Let the seller's proposal in periods 1 and 3 be x^3, and the buyer's proposal in period 2 be y^2. Using backward induction from period 2 to period 1, we have:

$$1 - x^1 = \delta(1 - y^2)$$

Backward induction from period 3 to period 2 yields:

$$(1-\delta^{T-2})u(y^2) + \delta^{T-2}u\Big(y^2 + \hat{x}_2(y^2)\Big) = (\delta - \delta^{T-2})u(x^3) + \delta^{T-2}u\Big(x^3 + \hat{x}_2(x^3)\Big)$$

Thus, we can link periods 3 and 1. Doing so, we have the following.

$$\begin{aligned} u(1 - \frac{1-x^1}{\delta}) + \delta^{T-2}\Big[u\Big(1 - \frac{1-x^1}{\delta} + \hat{x}_2(1 - \frac{1-x^1}{\delta})\Big) - u(1 - \frac{1-x^1}{\delta})\Big] = \\ \underbrace{\delta u(x^3)}_{\text{Certainty equivalent}} + \delta^{T-2} \underbrace{\Big[u\Big(x^3 + \hat{x}_2(x^3)\Big) - u(x^3)\Big]}_{\text{Marginal utility from second payment}} \end{aligned} \tag{3}$$

The expression for x^1 illustrates two forces. The first terms on both sides of (3) represent the first force, and the second terms on both sides of (3) are the marginal utilities of the price in the second bargaining, representing the second force.

Notice that the first term on the RHS of (3) can be regarded as the expected payoff of a lottery, which gives x^3 with probability δ, and 0 with probability $1 - \delta$. Consequently, the first term on the LHS is the certainty equivalent of this lottery. Thus, when the seller is less risk averse, she will have a lower risk premium and a higher certainty equivalent, which makes x^1 higher.

The second term of LHS is the marginal utility of the second price when the first price is $1-\frac{1-x^1}{\delta}$. The second term of RHS is the marginal utility of the second price when the first price is x^3. Thus, the marginal utility of the second term is affecting the result. Adding the marginal utility effect to the certainty equivalent effect will determine the value of x^1.

For example, let us first define $\bar{x}^1$ as the value of x^1 that equates only the certainty equivalence terms, i.e., $u(1-\frac{1-\bar{x}^1}{\delta}) = \delta u(x^3)$. This is the price determined solely by the first force. Now account for the marginal utility effects of the second prices, (adding the second terms of both sides back into the equation). This gives the following:

$$u(1-\frac{1-\bar{x}^1}{\delta}) + \delta^{T-2}\Big[u\Big(1-\frac{1-\bar{x}^1}{\delta}+\hat{x}_2(1-\frac{1-\bar{x}^1}{\delta})\Big) - u(1-\frac{1-\bar{x}^1}{\delta})\Big] \tag{4}$$

and

$$\delta u(x^3) + \delta^{T-2}\Big[u\Big(x^3+\hat{x}_2(x^3)\Big) - u(x^3)\Big]. \tag{5}$$

If the marginal utility of the second price in (4) is smaller than in (5), that means $\bar{x}^1$ does not solve the equation (3)—it makes LHS less than RHS. Thus, the value of x^1 that equates (3) must be larger than $\bar{x}^1$. In this situation, the second force tends to increase x^1. Similarly, if the marginal utility of the second payment in (4) is larger than in (5), the second force makes x^1 smaller than $\bar{x}^1$. This is why the net effect is ambiguous.

As for whether the marginal utility of the second price is larger in (4) or (5), this depends both on the second price and the seller's marginal utility $u'(.)$.

First notice that for $\bar{x}^1$ s.t. $u(1-\frac{1-\bar{x}^1}{\delta}) = \delta u(x^3)$, we have $1-\frac{1-\bar{x}^1}{\delta} < x^3$, since $u(.)$ is an increasing function. With DARA, $\hat{x}_2(.)$ is increasing, so $\hat{x}_2(x^3) > \hat{x}_2(1-\frac{1-\bar{x}^1}{\delta})$, which tends to make the marginal utility term in (5) larger. However, a higher x^3 not only makes the second price higher but also makes $u'(x^3)$ lower due to concavity. Thus, the second force has an ambiguous effect with DARA. As for IARA, a smaller value of $1-\frac{1-\bar{x}^1}{\delta}$ makes both $\hat{x}_2(1-\frac{1-\bar{x}^1}{\delta})$ and $u'(1-\frac{1-\bar{x}^1}{\delta})$ larger, so the second force here makes x^1 smaller.

A more direct explanation of the second force is that when the marginal utility from the second price is higher, the seller will require a lower first price from B_1 since the seller is satisfied more easily.

These forces can explain the relationship between x_1^*, x_2^*, and x_U. x_2^* is only affected by the first force, so DARA, which leads to a less risk-averse seller in bargain 2, will make $x_2^* > x_U$.

But, with DARA, x_1^* is affected by both forces, and the second force may lead to a larger or smaller x_1^*, which causes ambiguous results in the comparison between x_1^* and x_U.

3.5 Limit Results

It has long been known that in certain limiting cases, the AO result converges to the Nash bargaining solution ("NBS") result. In this spirit, we look at two types of limit results. The first is to let δ go to 1. The second begins by fixing T, and then allowing for an interval of size $\Delta = \frac{T}{2n+1}$ between offers. We then look at the limiting result where the interval between proposals Δ goes to zero as n goes to infinity, with T remaining unchanged. At this limit, there is an infinite number of periods before time T. In this case, we write the discount cost of a one-period delay as $e^{-r\Delta}$, where r is the discount factor.

In the first case noted above, fixing the number of periods before the arrival of B_2 and letting δ go to 1, we get the somewhat counter-intuitive result that though two agreements are reached at different dates, when we let the discount factor δ go to 1, the limit result is a simultaneous NBS, which solves (see Appendix A.5for proof)

$$\begin{cases} x = \arg\max_s (u(A+s) - u(A))(1-s) \\ x = A \end{cases}. \tag{6}$$

Of course, the sequential NBS is different from the simultaneous NBS. With two Nash bargaining processes done sequentially, in the second Nash bargaining, the seller's outcome from the first Nash bargaining will be the disagreement point. Let the seller have a payment of x_1 from bargaining with the first buyer. Then the second Nash bargaining problem is

$$\max_s (u(x_1+s) - u(x_1))(1-s),$$

which gives an outcome depending on x_1, denoted as $\hat{x}^N(x_1)$.

Anticipating this, the first Nash bargaining problem is

$$\max_s u(s + \hat{x}^N(s))(1 - s),$$

where the disagreement point of the seller is getting nothing in the bargaining.

When the seller has concave utility, $\hat{x}^N(x_1)$ depends on x_1 and two bargains have different outcomes. For example, with $u(x) = \sqrt{x}$, the outcomes in two bargains are 0.120 and 0.424. The situation is different from the symmetric outcomes of simultaneous NBS.[7] This observation, together with the following proposition, implies that as δ goes to 1, the AO equilibrium outcome does not converge to the sequential NBS outcome.

Proposition 5 *As $\delta \to 1$, the equilibrium payoffs of the model go to the symmetric simultaneous NBS – buyers end up paying the flow price z^* which solves (6).*

To see this, we consider the continuation game in which two buyers are actively bargaining after the date T. As simultaneous AO bargaining between one seller and two buyers, its limit result is the Nash bargaining solution with one seller bargaining with two buyers simultaneously. This limit outcome is symmetric. Furthermore, as $\delta \to 1$, the backward induction beginning at date T does not change the price offered. As a result, the limit result becomes the symmetric simultaneous NBS. See Appendix A.5 for the detailed proof.

Notice that in the proof of this proposition in Appendix A.5, we have also proved that the simultaneous AO bargaining outcome goes to the simultaneous NBS as $\delta \to 1$, so Proposition 5 also says that the limit result of sequential AO bargaining is the same as simultaneous AO bargaining.

To explain this result, the first point is that when players become patient enough, reaching an agreement at the exact same time or with a difference of a single period will make a negligible difference. However, there is a discrete difference in information between the two cases that does not disappear as players become more patient. In sequential bargaining, when B_2 bargains with the seller, he knows the deal made by B_1, but in simultaneous bargaining, no buyers know the other one's deal when bargaining. This difference exists even if players

[7] Even if we consider another sequential NBS where the disagreement point of the seller in the first bargain is the payoff in the sequential NBS, the outcome will still be different from the sequential NBS.

are fully patient, so we need some extra explanation of why sequential and simultaneous bargaining are the same at the limit. This is because when δ goes to 1, there is no cost in rejecting offers. Thus, in sequential bargaining, the offers on the dates before T will be exactly the same and are equal to the offer that will be made if B_1 is still bargaining with the seller at date T[8], which is the simultaneous bargaining result. Sequential bargaining can transfer to simultaneous bargaining costlessly when players are fully patient.

We then look at the second type of limit, where we fix T and send the intervals between proposals to 0. Recall that to solve for the equilibrium, we must first consider the continuation game after T with no previous agreement and then do backward induction. For the continuation game with simultaneous AO bargaining, the limit result is a simultaneous NBS, which solves (6).

Denote the solution of (6) by z^*. Let p_t be the proposal for some $t < T$. From Appendix A.5, we have that the backward induction at the limit implies that p_t must satisfy:

$$p_T = z^*$$

and

$$ru(p_t) = (2p_t' + r(1-p_t))\Big[(1-e^{-r(T-t)})u'(p_t) + e^{-r(T-t)}u'\big(p_t + \lim_{\Delta\to 0}\hat{x}_2(p_t)\big)\big(1 + \lim_{\Delta\to 0}\hat{x}_2'(p_t)\big)\Big],$$

where $\lim_{\Delta\to 0}\hat{x}_2(p_t)$ is the solution to the Nash bargaining problem $\max_s(u(p_t+s)-u(p_t))(1-s)$. See Appendix A.5 for the proof.

Proposition 6 *As the interval between proposals $\Delta = \frac{T}{2n+1}$ approaches zero and the entry date T remains unchanged, the limit result is asymmetric but different from the sequential NBS.*

z^* satisfies $\lim_{\Delta\to 0}\hat{x}_2(z^*) = z^*$ because it is the solution to (6). Thus, if z^* is the first price, the outcome will be symmetric, i.e., prices resulting from the two bargains are the same. But the first price is decided by the backward induction. If we send the conditions decided by the backward induction to the limit, the first price at the limit will be different

[8]That is, the backward induction process does not change the price proposed when $\delta \to 1$.

from z^*, so there is an asymmetric result, instead. Moreover, it is not the usual sequential NBS, because this limit result changes with T.

The two kinds of limiting results say that models based on sequential Nash bargaining do not have the microfoundation of sequential AO bargaining, which is different from simultaneous Nash bargaining.

4 Discussion

4.1 Simultaneous Bargaining vs. Sequential Bargaining

Thus far, we have analyzed multi-agent bargaining in a sequential bargaining framework. In this section, we analyze another type of interdependent bargaining, in which the seller engages in alternating-offers bargaining with both buyers simultaneously[9]. We refer to this as "two-track" simultaneous bargaining.

So far, we have assumed that T is a fixed date, determined by market conditions. However, the seller may be able to vary T. In this section, we compare sequential bargaining to two-track simultaneous bargaining. We can consider this comparison as the seller determining the size of T. In a sense, this gives us "optimal" sequential bargaining, from the standpoint of the seller.

One difference between the two approaches is that with sequential bargaining, x_1 is known when the seller bargains with B_2. This is not so with two-track simultaneous bargaining. One disadvantage of sequential bargaining is that, if the second bargain fails, there is nothing that the seller can do about improving the bargain made with the first buyer. This is because we have assumed that contracts are permanent, once signed[10].

In this section, we will analyze what happens if the seller can control T, the date of the second entry. If so, would the seller prefer to bargain sequentially, with entry after T? Or, would the seller prefer to bargain simultaneously with each buyer? In two-track simultaneous bargaining, each bargaining process follows the Rubenstein protocol, but is independent of

[9]Actually, the result of this bargaining has been solved when we solve the equilibrium for sequential bargaining, see Section 3.2.

[10]Even if we allowed for re-negotiation, B_1 might not agree to it. Recall that with IARA and (sometimes) DARA, B_1 is better off than he would be in a single Rubenstein bargaining episode.

the other. The seller's utility depends on the outcome of both bargains.

Clearly, simultaneous bargaining has one advantage for the seller—her payoffs are not lagged and discounted. Actually, we do find that in every example we have solved, the seller gets a higher utility with simultaneous bargaining than with sequential bargaining. However, the result is not straightforward, because sequential bargaining sometimes can give the seller a higher total payment from buyers. For example, with $u(x) = 4x - x^2$, x_1^* gets closer to x_U as T goes up. As an IARA utility function, $x_U > x^*$ with $u(x) = 4x - x^2$, so x_1^* can be larger than x^* here. Moreover, x_2^* decreases as x_1^* goes up in this case, but it does not change much, which can make $x_1^* + x_2^* > 2x^*$[11]. Thus, when comparing sequential bargaining and simultaneous bargaining, we may have two effects with different directions: (1) sequential bargaining can lead to a higher total payment from buyers, which benefits the seller; (2) the payments in sequential bargaining are delayed, which hurts the seller.

In comparing these two types of bargaining, we need to choose T. Based on many simulations with varying values of T, we assume that the seller will choose $T = 3$. This is the lowest that T can be while allowing for sequential effects in the bargaining. Setting T above 3 reduces the seller's utility in every example that we have solved.

We set up simultaneous bargaining as if the seller has the capacity to conduct two alternating offers bargaining processes at once, without loss of efficiency. A buyer only participates in one bargaining process. The seller, however, is aware of both, and the presence of one bargaining process will affect the offers that the seller makes in the other.

Given that bargaining has been concluded with B_1 with a price x_1, the equilibrium condition for B_2 is the same as (1) in Section 3.1. We use the same notation ($\hat{x}_2(x_1)$) for the second price giving the first price x_1 again.

Assuming symmetric outcomes, we can solve for the equilibrium prices in two-track simultaneous bargaining. The equilibrium price for the simultaneous bargaining solves the equation $x = \hat{x}_2(x)$. Table 1 contains the results for two seller utility functions: $u_1(x) = 4x - x^2$ and $u_2(x) = \sqrt{x}$. In this table, we show the prices, x_1^* and x_2^*, paid by B_1 and B_2 with sequential bargaining. We also show x^*[12], the price reached with both buyers with two-track

[11] As an example, $x_1^* + x_2^* > 2x^*$ when $u(x) = 4x - x^2$, $\delta = 0.9$, and $T = 29$.

[12] Here we use the same notation as in Section 3.2

simultaneous bargaining. Note that the discount factor, δ, affects the outcomes in both types of bargaining. In both cases, δ plays the same role as in any Rubinstein bargaining problem. In the sequential case, it also discounts future seller utility.

Equilibrium payments to the seller are shown in Table 1. There are consistent patterns across the different utility functions. First, x^* exceeds either x_1 or x_2. Second, simultaneous prices x^* fall as the discount factor approaches 1.

Table 2 shows numerical results of the seller's utility for different values of δ. In all cases, the seller does better with two-track simultaneous bargaining than with sequential bargaining.

Table 1: Equilibrium prices in sequential bargaining and simultaneous bargaining

	$u_1(x) = 4x - x^2$			$u_2(x) = \sqrt{x}$		
	x_1^*	x_2^*	x^*	x_1^*	x_2^*	x^*
$\delta = 0.99$	0.45364	0.45581	0.45574	0.45649	0.46300	0.46336
$\delta = 0.95$	0.45689	0.46701	0.46668	0.44290	0.47343	0.47509
$\delta = 0.9$	0.46325	0.48168	0.48108	0.43137	0.48749	0.49044
$\delta = 0.85$	0.47213	0.49715	0.49636	0.42586	0.50275	0.50660

Table 2: Equilibrium seller utilities in sequential bargaining and simultaneous bargaining

	$u_1(x) = 4x - x^2$		$u_2(x) = \sqrt{x}$	
	Sequential	Simultaneous	Sequential	Simultaneous
$\delta = 0.99$	2.7868	2.8151	0.95326	0.96266
$\delta = 0.95$	2.7228	2.8623	0.92881	0.97478
$\delta = 0.9$	2.6496	2.9229	0.90123	0.99040
$\delta = 0.85$	2.5846	2.9854	0.87733	1.0066

These examples all show that, given the choice, the better strategy for the seller is to bargain with both buyers simultaneously, rather than sequentially. This is potentially an intending result, if it can be shown to be generally true.

We have been able to establish two theoretical results that bear on this point. First, when δ goes to 1, the two approaches get the same seller utility. In the proof for Proposition 5 in Appendix A.5, we show that both simultaneous AO bargaining and sequential AO bargaining converge to the symmetric simultaneous NBS as $\delta \to 1$. Second, if the seller utility function displays IARA and $T = 3$, then we have the result that the seller will prefer simultaneous to sequential bargaining. We state this as follows.

Proposition 7 *With IARA and $T = 3$, the seller always does better with simultaneous bargaining than with sequential bargaining.*

In the proof for Lemma 2 in Appendix A.6, we showed that with IARA utility and $T = 3$, we have $x_1^* < x^*$. This result, together with the fact that $(1 - \delta^T)u(x) + \delta^T u(x + \hat{x}_2(x))$ is increasing with x, says that the seller does better with simultaneous bargaining.

However, in other cases, the comparison between x_1^* and x^* is ambiguous. Therefore, we do not directly know about the comparison of utilities in all cases, purely on theoretical grounds.

A different approach is to use a combination of numerical approximations and Monte Carlo simulations. We write seller utility as:

$$\ln u(x) = \alpha_0 + \alpha_1 \ln x + 0.5\alpha_2(\ln x)^2 \tag{7}$$

As is well known, this is the translog function. It is equivalent to a second-order Taylor series approximation, at the point of $\ln x = 0$, of any function whose derivatives exist at that point[13]. It has been widely used as an approximation to a cost or production function[14].

We think of the translog function as a second-order Taylor expansion of any function around the extension point $\ln x = 0$ ($x = 1$), as in Boisvert (1982). Our utility functions are defined on x inside $[0, 2]$ (since we have two buyers each has a value of 1), which is around $x = 1$, so this is appropriate. Of course, we will still suffer from some loss in precision when x is away from 1, especially when x goes below 1.

Given the translog assumption, we treat the coefficients of the translog function as random variables, with values given by a uniform random number generator.[15] Each draw gives us a different seller utility function. Draws may lead to convex or concave seller utilities, including both DARA and IARA. If we get consistent results, this may be the next best thing to having stronger theoretical results.

[13]See Boisvert (1982), p.13, cites Allen (1938), 456-458 for this result.
[14]See Grant (1993), 235-240.
[15]α_0 is drawn among $[0, 1]$, α_1 is drawn among $[0, 2]$, and α_2 is drawn among $[0, 0.001]$.

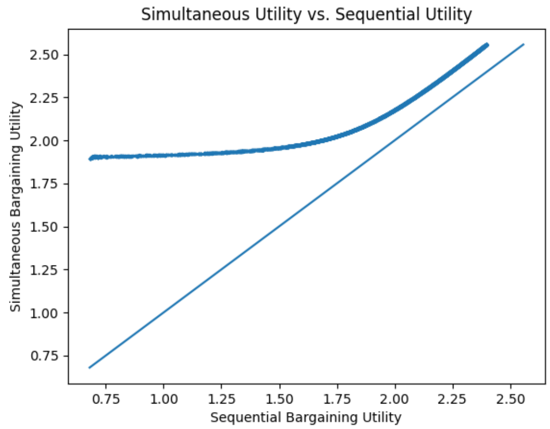


Figure 2: Seller's utility from sequential and simultaneous bargaining

In the simulations, we set $T = 3$. We do this to maximize the chance that sequential bargaining will do better than simultaneous bargaining. $T = 3$ is the lowest that T can be and still incorporates the timing assumption of our sequential bargaining model.

Figure 2 shows the results of 10,000 random draws. The resulting utility functions can be either concave or convex. In this figure, seller utility from sequential bargaining (with $T = 3$) is on the horizontal axis and seller utility from simultaneous bargaining is on the vertical axis. The solid line is the 45-degree line. All the simulation results lead to payoffs above the 45-degree line. Clearly, whatever the form of the utility function, discounted seller utility is higher with simultaneous bargaining than with sequential bargaining.

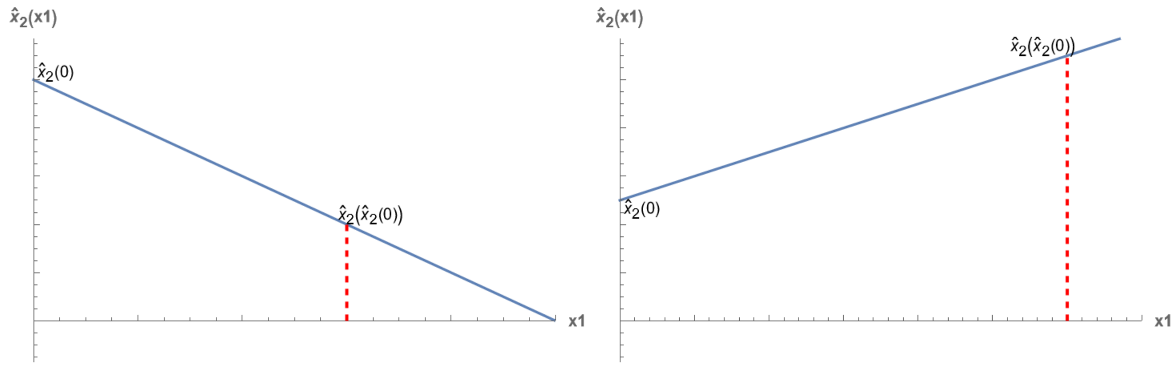
Considering the variety of utility functions generated by the Monte Carlo simulations, this result is quite powerful. Even with convex utilities, simultaneous bargaining beats sequential bargaining, when the seller can choose between them.

4.2 Bargaining Strategy Results

In our model, two identical buyers end up at different flow prices. It is natural to ask which buyer will pay less. The specific form of $u(.)$ and T determine which buyer pays a lower flow price. The results will be ambiguous. For example, see the following proposition.

Proposition 8

- *Given a utility function $u(x)$ with DARA, $\exists\ \bar{T}_1$ such that $x_1^* < x_2^*$ when $T > \bar{T}_1$.*



(a) Decreasing $\hat{x}_2(.)$

(b) Increasing $\hat{x}_2(.)$

Figure 3: Which buyer has advantage

- *Given a utility function $u(x)$ with IARA, $\exists\ \bar{T}_2$ such that $x_1^* > x_2^*$ when $T > \bar{T}_2$.*

As T goes to infinity, the outcome of the first bargain gets closer and closer to the outcome of unaffected bargaining. This is because the effect of the second bargain in the far future is negligible. As a result, the first flow price will be close to $x_U = \hat{x}_2(0)$. This means that the second flow price must be close to $\hat{x}_2(\hat{x}_2(0))$. The shape of $\hat{x}_2(.)$ will affect the relationship between two flow prices. For example, as shown in Figure 3 (a), if $\hat{x}_2(.)$ is decreasing (IARA), the first flow price of large T ($x_1^* \approx \hat{x}_2(0)$) is higher than the second flow price ($x_2^* \approx \hat{x}_2(\hat{x}_2(0))$). Figure 3 (b) says that the first price of large T is lower than the second price when $\hat{x}_2(x)$ is increasing (DARA).

We see that in our bargaining setup, buyers do not always have a first-mover advantage. The seller's attitude towards risk will affect which buyer has the advantage. If the seller has IARA and $\hat{x}_2(.)$ is decreasing, there is a second-mover advantage with large T. If the seller has DARA and $\hat{x}_2(.)$ is increasing, there is a first-mover advantage for buyers with large T.

The two forces affecting the bargaining outcome in Section 3.4.1 can also explain the relationship between x_1^* and x_2^*. In our example of large T, the second force is negligible, since the second terms on both sides of (3) go to zero as T increases. Therefore, the certainty equivalent effect dominates when T is large. This says that when the seller is less risk averse, she has a larger certainty equivalent and thus a larger flow price from bargaining. With DARA in Figure 3 (b), the seller is wealthier after the first bargain. Thus, the seller is less risk-averse in bargaining with the second buyer, making x_2^* larger than x_1^*.

Another interesting point arises when the two buyers are heterogeneous in their valuations of the seller's product. Assume that one buyer values the contract as 2 each period and one values the contract as 1 each period. Which buyer would the seller prefer to bargain with first? This is a question asked frequently in reality. For example, a hospital may need to bargain with a small insurance provider and a large insurance provider. The hospital may want to bargain with the large insurer first so she can get more patients quickly. Or she can bargain with the small insurer first to have greater bargaining power when bargaining with the large provider. Schulman and Sibley (2023) discuss a similar problem under downstream interaction of insurance providers using the sequential NBS. They find that it is better to bargain first with the small provider, so that more competition in the downstream market reduces the bargaining leverage of the large insurer.

In our model, having the small buyer coming first does sometimes induce higher total payments from buyers. In such cases, however, having the small buyer first makes a larger subsequent payment from the large buyer later. A later payment is subject to discounting, of course. For example, when $\delta = 0.9$, $u(x) = \sqrt{x}$, $T = 3$, and two buyers value the contract 1 and 2 respectively, the total payment each period ($x_1^* + x_2^*$) when the small one comes first is larger, but the present value of dealing with the large buyer first is higher.

This might appear to contradict the NBS result of Schulman and Sibley (2023). However. the two results do not actually conflict. In the NBS method, there is no discounting. In our model, sometimes having the small one first can make the total payment larger, which is consistent with their result. However, this benefit of having the smaller one first can be offset by the discount cost concerns.

5 Extension

5.1 Downstream Competition

In the analysis thus far, buyers do not interact with each other except via bargaining. In this part, we add the feature that the two buyers engage in Cournot competition in a downstream market. They each bargain separately with the seller over the price of the critical input.

5.1.1 Setup

As before, there are three players in the game, one seller and two buyers (B_1, B_2). Buyers buy a critical input from the seller. Assume that 1 unit of input can produce 1 unit of output. The price of that input to each of the two buyers is determined by AO bargaining between each buyer and the seller. Once concluded successfully, the terms of a bargain are permanent. After completing a bargain with the seller, B_1 enters the downstream market. B_2 arrives at date T (T is odd and $T > 1$). Once B_2 arrives and concludes a bargain with the seller, it competes with B_1 in the downstream market, assuming Cournot competition. All bargaining uses the same bargaining protocol as the main model. We still assume the seller has the utility function $u(x)$. The utility functions of buyers are their Cournot profits.

Put in the context of Industrial Organization, the timing of our game allows us to explore the interaction between monopoly pricing and entry. Vertical models such as ours are usually analyzed under one of two different assumptions regarding input pricing. Traditionally, sellers in the input market have been assumed to make take-it-or-leave-it ("TIOLI") offers to buyers of inputs. Recently, empirical models have used the simultaneous NBS to determine input prices. Our contribution is to use AO bargaining in an otherwise standard vertical model. In this spirit, we will sometimes infer to B_1 as the incumbent and to B_2 as the entrant.

The traditional TIOLI model and NBS model cannot account for the change in entry date T. Suppose the results in Table 3-5 in Section 5.1.2 are data, it would be easy to fit the bargaining parameters of a standard model to the data, for any given arrival date T. However, if T were to change, equilibrium prices would also change, as shown in the tables. Therefore, the bargaining parameters fitted for one set of prices would no longer fit the new data.

Before date T, there is a one-seller-one-buyer AO bargaining process between the seller and B_1. After date T, if the seller has already reached an agreement with B_1, then there is simply a one-seller-one-buyer AO bargaining between the seller and B_2, after which B_2 enters the downstream market. If the seller has not reached an agreement with B_1, then the seller bargains with two buyers simultaneously. In the seller's turns, she proposes two offers to buyers simultaneously. In buyers' turns, the two buyers propose offers to the seller simultaneously.

We assume that the Cournot market has a demand $p = \gamma - \beta(q_1 + q_2)$ each period, where p is the price of the output, q_1, q_2 are the quantities of output by B_1 and B_2 respectively, and γ, β are parameters. When B_1 has a purchasing contract with the seller with an input price c_1 and B_2 does not have a purchasing contract, B_1 is the monopolist in the downstream market. Then B_1 faces with the following problem in each period:

$$\max_q (\gamma - \beta q - c_1) q$$

Clearly, a monopoly buyer with an input price c_1, will choose the quantity of the output $q_m(c_1) = \frac{\gamma - c_1}{2\beta}$ each period, so her payoff in a period $v_m(c_1)$ is

$$v_m(c_1) = \big(\gamma - \beta q_m(c_1) - c_1\big) q_m(c_1) = \frac{(\gamma - c_1)^2}{4\beta} = \beta q_m(c_1)^2,$$

and the seller's utility of this period is $u\big(c_1 q_m(c_1)\big)$.

If both B_1 and B_2 compete in the downstream market, paying input prices c_1 and c_2 respectively, then Cournot competition decides their reduced form outputs in one period as $q_1(c_1, c_2)$ and $q_2(c_1, c_2)$:

$$q_1(c_1, c_2) = \frac{\gamma - 2c_1 + c_2}{3\beta}; \; q_2(c_1, c_2) = \frac{\gamma - 2c_2 + c_1}{3\beta}.$$

Then buyers' payoffs in this period $v_1(c_1, c_2)$ and $v_2(c_1, c_2)$ are:

$$v_1(c_1, c_2) = \frac{(\gamma - 2c_1 + c_2)^2}{9\beta} = \beta q_1(c_1, c_2)^2; \; v_2(c_1, c_2) = \frac{(\gamma - 2c_2 + c_1)^2}{9\beta} = \beta q_2(c_1, c_2)^2.$$

The seller's payoff in this period is $u\big(c_1 q_1(c_1, c_2) + c_2 q_2(c_1, c_2)\big)$.

The strategy definition and equilibrium concept are the same as in Section 2.

5.1.2 Equilibrium Results

Using the same method as in Section 3.3, we can solve for an equilibrium. Let $\hat{x}^d(x)$ denote the proposal made by the seller to B_2, assuming that the seller has previously negotiated a payment of x with B_1. If the seller has already reached an agreement on a flow price, x, with B_1 before date T, there is a one-seller-one-buyer AO bargaining between the seller and B_2 after date T, then the equilibrium condition for the continuation game after date T is

$$\begin{cases} v_2(x, c_x) = \delta v_2(x, c_y) \\ u\big[xq_1(x, c_x) + c_x q_2(x, c_x)\big] = (1-\delta)u\big[xq_m(x)\big] + \delta u\big[xq_1(x, c_y) + c_y q_2(x, c_y)\big] \end{cases}$$

where c_x is the input cost proposed by the seller, and c_y is the input cost proposed by B_2. The solution to c_x is the second price $\hat{x}^d(x)$.

In this model, the price in bargain 2 is more likely to be increasing in the price in bargain 1, as compared to the main model. This is because now the input price paid by the incumbent B_1 directly affects the entrant's payoff after entry. If B_1 has a higher production cost, B_2 faces less pressure from the competition and is more willing to accept a high input price.

On the other hand, if the seller has not reached an agreement with B_1 at T, then the game becomes the simultaneous bargaining between the seller and two buyers. The seller will propose the same price x^* to two buyers and x^* satisfies $x = \hat{x}^d(x)$.

After knowing the result of the simultaneous bargaining after date T, we can use backward induction to calculate all offers before date T. In each period, the proposer will make an offer that leaves the other player indifferent between accepting and rejecting. In this way, we have the seller's proposal x_1^d at date 1, and B_1 will accept that offer immediately. As a result, the seller will provide an offer of $x_2^d = \hat{x}^d(x_1^d)$ at date T, and B_2 accepts it immediately.

To see the story behind the algebra above, in Table 3-5 we present the numerical examples with demand $q = 2 - q_1 - q_2$, $\delta = 0.9$, two different seller utility functions, and four different values of T. Each T corresponds to the length of time from the start of the game until entry occurs. The pre-entry situation before date T is presented in Table 3, which includes the profits of B_1 each period, the input prices, and the market prices of output. After entry,

Table 3: B_1's profits each period and prices of output before date T

	$u(x) = 5x - x^2$			$u(x) = \sqrt{x}$		
	Profit of B_1	Output price	Input price	Profit of B_1	Output price	Input price
$T = 5$	0.51591	1.2817	0.56346	0.59309	1.2299	0.45975
$T = 7$	0.51479	1.2825	0.56502	0.60802	1.2202	0.44048
$T = 9$	0.51638	1.2814	0.56281	0.61917	1.2131	0.42626
$T = 11$	0.51922	1.2794	0.55886	0.62758	1.2078	0.41560

Table 4: Post-entry input prices

	$u(x) = 5x - x^2$		$u(x) = \sqrt{x}$	
	First input price	Second input price	First input price	Second input price
$T = 5$	0.56346	0.54808	0.45975	0.49134
$T = 7$	0.56502	0.54885	0.44048	0.48149
$T = 9$	0.56281	0.54775	0.42626	0.47420
$T = 11$	0.55886	0.54578	0.41560	0.46874

Table 5: Buyers' profits each period and prices of output after date T

	$u(x) = 5x - x^2$			$u(x) = \sqrt{x}$		
	Profit of B_1	Profit of B_2	Output price	Profit of B_1	Profit of B_2	Output price
$T = 5$	0.22441	0.239219	1.0372	0.27452	0.24242	0.98370
$T = 7$	0.22367	0.239220	1.0380	0.28463	0.24256	0.97399
$T = 9$	0.22472	0.239219	1.0368	0.29221	0.24267	0.96682
$T = 11$	0.22659	0.239216	1.0349	0.29795	0.24276	0.96145

input prices of two buyers are presented in Table 4. The post-entry equilibrium output prices, plus buyers' profits in each period after date T are presented in Table 5.

From Table 3, two points of interest emerge. For the DARA utility function $u(x) = \sqrt{x}$, first, the input price is lower, the more distant is the entry date. Second, as T increases, the pre-entry monopoly price falls. The decline in the output prices as T increases is, of course, induced by the reduction in the input prices. The patterns are different with the IARA utility function $u(x) = 5x - x^2$.

Compared with standard vertical models, one of these results is unusual. In a vertical model with complete information, the pre-entry monopoly price is not affected by how distant entry may be. This is because the standard model assumes TIOLI input pricing.

In the AO setup, the seller's offer to the first buyer is affected by its influence on the seller's offer to the second buyer. With the DARA utility $u(x) = \sqrt{x}$, for example, a higher input price to B_1 leads to a higher input price to B_2. With discounting, the further out is the entry date, the less important is this motive.

Table 4 describes input prices after entry occurs. Those paid by the first buyer do not change, by assumption. With the IARA utility function $u(x) = 5x - x^2$, the entrant pays a lower price than does the incumbent. With the DARA utility function $u(x) = \sqrt{x}$, the reverse is true. That is, there is a first-mover advantage in this case.

Table 5 shows post-entry profits for each buyer, as well as output prices. The equilibrium quantities in this table result from the input prices in Table 4. With the IARA example, the entrant pays less for the input than the incumbent. Therefore, its profits are slightly larger. In the case of the square root DARA utility function, the reverse is true.

These features of the equilibrium behaviors above would not emerge in a TIOLI setup. In the sequential NBS, there are no discount rates or entry dates, so the results in the tables could not even be attempted. Rather, Tables 3-5 reflect the unique influence of AO bargaining on market outcomes.

6 Concluding Remarks

In this paper, we have extended the Rubenstein AO bargaining framework to settings in which different bargaining processes are interdependent. In order to cause one bargain to affect the outcome of another bargain, we have assumed that the seller's utility function is concave in total payments made to the seller.

Propositions 3 describe the equilibrium in which a seller bargains with one buyer before bargaining with another. The equilibrium can exhibit a number of patterns, depending on the seller's utility function.

For example, a higher price in the first bargain can increase or decrease the price in the second bargain. The second of two sequential bargains to be concluded can lead to either a higher or lower price than the first. Two buyers pay different prices, even though the buyers are identical except for the arrival date.

The variety of results stems entirely from assuming that the seller has a nonlinear utility function. If it is linear, then our results coincide with those of Rubenstein. One bargain will not affect the other with linear seller utility.

The usual relationship between AO and NBS exists in sequential bargaining but in a limited way. The limit of the two-stage sequential AO problem is not the same as the sequential version NBS. Instead, the sequential AO problem shares the same limit as the simultaneous AO problem: the limit is the simultaneous NBS. Therefore, a Nash-in-Nash model based on sequential Nash bargaining does not have a solid microfoundation on the alternating-offer bargaining model as the simultaneous Nash bargaining model does.

Empirical papers usually use the simultaneous Nash bargaining model to estimate key parameters. With the common assumption of passive beliefs, different bargains are independent in the simultaneous Nash bargaining model. However, in our model, different bargains affect each other even under passive beliefs. Moreover, our model allows us to study the effect of different entry dates of the second buyer, which cannot be tried in the static model of Nash bargaining.

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A Appendix

A.1 Bargaining after Having an Agreement with B_1

Here $\hat{x}_2(x_1)$ is determined by (1).

It is easy to check there is no deviation for players because proposals are making the other player indifferent between accepting and rejecting.

Next, we prove that the equilibrium outcome is unique.

Suppose in an equilibrium, when the seller proposes, the minimal and maximal payoffs are m_s, M_s respectively; when the buyer proposes, the minimal and maximal payoffs are m_b, M_b respectively.

When the seller proposes, the buyer gets at most M_b in the next period, so the proposal by the seller is at least $1 - \delta M_b$. As a result,

$$m_s \geqslant u(x_1 + 1 - \delta M_b). \tag{8}$$

Similarly,

$$M_s \leqslant u(x_1 + 1 - \delta m_b). \tag{9}$$

When the buyer proposes, the seller gets at most $(1-\delta)u(x_1)+\delta M_s$ by rejecting. Because the buyer's proposal is $1 - u_b$, where u_b is the buyer's payoff, we have

$$u(x_1 + 1 - m_b) \leqslant (1 - \delta)u(x_1) + \delta M_s. \tag{10}$$

Similarly,

$$u(x_1 + 1 - M_b) \geqslant (1 - \delta)u(x_1) + \delta m_s. \tag{11}$$

By (8), (9), (10), and (11) we get

$$u(x_1 + 1 - m_b) \leqslant (1 - \delta)u(x_1) + \delta u(x_1 + 1 - \delta m_b), \tag{12}$$

$$u(x_1 + 1 - M_b) \geqslant (1 - \delta)u(x_1) + \delta u(x_1 + 1 - \delta M_b). \tag{13}$$

Let $f(x) = u(x_1 + 1 - x) - (1 - \delta)u(x_1) - \delta u(x_1 + 1 - \delta x)$.

But

$$\frac{df(x)}{dx} = -u'(x_1 + 1 - x) + \delta^2 u'(x_1 + 1 - \delta x).$$

Because $u'(x) > 0$ and $u''(x) < 0$, $\frac{df(x)}{dx} < 0$. Moreover, $f(0) = u(x_1 + 1) - (1 - \delta)u(x_1) - \delta u(x_1 + 1) > 0$; $f(1) = u(x_1) - (1 - \delta)u(x_1) - \delta u(x_1 + 1 - \delta) < 0$. So, there is a unique solution m^* s.t. $f(x) = 0$ in $(0, 1)$.

Then, (12) and (13) become $m_b \geqslant m^*$ and $M_b \leqslant m^*$, which says $m_b = M_b = m^*$.

Plug $m_b = M_b = m^*$ into (8) and (9) we get $m_s = M_s = u(x_1 + 1 - \delta m^*)$.

So, the equilibrium outcome is unique. By letting $\hat{x}_2(x_1) = 1 - \delta m^*$ and $y = 1 - m^*$, we can see that in this unique equilibrium outcome, the seller's proposal is our $\hat{x}_2(x_1)$, satisfying (1). And clearly, the solution $\hat{x}_2(x_1) \in (1 - \delta, 1)$.

A.2 Proof of Proposition 1

By (1), we get

$$u(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) - u(x_1) = \delta(u(x_1 + \hat{x}_2(x_1)) - u(x_1)). \tag{14}$$

Because (14) is satisfied by any $x_1 \in (1 - \delta, 1)$, we can take derivative w.r.t. x_1 on both sides:

$$u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})(1 + \frac{\hat{x}_2'(x_1)}{\delta}) - u'(x_1) = \delta(u'(x_1 + \hat{x}_2(x))(1 + \hat{x}_2'(x_1)) - u'(x_1))$$

$$\Leftrightarrow \hat{x}_2'(x_1) = \frac{\delta u'(x_1 + \hat{x}_2(x)) + (1 - \delta)u'(x_1) - u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\frac{u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))}.$$

Because $\frac{u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x)) > 0$ by $u''(x) < 0$ and $\delta \in (0, 1)$, we have

$$\hat{x}_2'(x_1) \geqslant 0 \Leftrightarrow \delta u'(x_1 + \hat{x}_2(x)) + (1 - \delta)u'(x_1) - u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) \geqslant 0.$$

To see the relationship between risk aversion and $\hat{x}_2(.)$, pick $x_1 < x_1'$.

The price $\hat{x}_2(x_1)$ satisfies

$$u(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) = \delta u(x_1 + \hat{x}_2(x_1)) + (1 - \delta)u(x_1).$$

Firstly, we assume DARA for the seller.

We keep the lottery of getting $\hat{x}_2(x_1)$ with probability δ and getting 0 with probability $(1 - \delta)$ unchanged, and rise the wealth from x_1 to x_1'. Then the expected utility from the lottery is $\delta u(x_1' + \hat{x}_2(x_1)) + (1 - \delta)u(x_1')$. According to DARA, the new risk premium π is smaller than the previous risk premium, which is $\frac{1-\delta}{\delta}\Big[1 - (1 + \delta)\hat{x}_2(x_1)\Big]$. As a result, the new certainty equivalent is

$$x_1' + \delta\hat{x}_2(x_1) - \pi > x_1' + \delta\hat{x}_2(x_1) - \frac{1 - \delta}{\delta}\Big[1 - (1 + \delta)\hat{x}_2(x_1)\Big] = x_1' + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}.$$

Thus, we have the price $\hat{x}_2(x_1)$ satisfies

$$u(x_1' + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) < \delta u(x_1' + \hat{x}_2(x_1)) + (1 - \delta)u(x_1'). \tag{15}$$

Notice that $\hat{x}_2(x_1')$ satisfies

$$u(x_1' + 1 - \frac{1 - \hat{x}_2(x_1')}{\delta}) = \delta u(x_1' + \hat{x}_2(x_1')) + (1 - \delta)u(x_1'). \tag{16}$$

Consider the function $t(x) = u(x_1' + 1 - \frac{1-x}{\delta}) - \delta u(x_1' + x)$. We have $t'(x) = \frac{1}{\delta}u'(x_1' + 1 - \frac{1-x}{\delta}) - \delta u'(x_1' + x)$.

We know that $x_1' + 1 - \frac{1-x}{\delta} < x_1' + x$ for $x \in (0, 1)$ and $u'(.)$ is decreasing, so $t'(x) > 0$.

According to (15) and (16), we know $t(\hat{x}_2(x_1)) < t(\hat{x}_2(x_1'))$. Thus, we have $\hat{x}_2(x_1') > \hat{x}_2(x_1)$, i.e., $\hat{x}_2(.)$ is increasing.

A similar process goes for IARA.

A.3 Proof for Lemma 1

From Appendix A.2, we have

$$\hat{x}_2'(x_1) = \frac{\delta u'(x_1 + \hat{x}_2(x)) + (1-\delta)u'(x_1) - u'(x_1 + 1 - \frac{1-\hat{x}_2(x_1)}{\delta})}{\frac{u'(x_1+1-\frac{1-\hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))}.$$

Thus, we have

$$\begin{aligned}
\hat{x}_2'(x_1) &> \frac{\delta u'(x_1 + \hat{x}_2(x)) - u'(x_1 + 1 - \frac{1-\hat{x}_2(x_1)}{\delta})}{\frac{u'(x_1+1-\frac{1-\hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))} \\
&= -\frac{u'(x_1 + 1 - \frac{1-\hat{x}_2(x_1)}{\delta}) - \delta u'(x_1 + \hat{x}_2(x))}{\frac{u'(x_1+1-\frac{1-\hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))} \\
&> -\frac{u'(x_1 + 1 - \frac{1-\hat{x}_2(x_1)}{\delta}) - \delta^2 u'(x_1 + \hat{x}_2(x))}{\frac{u'(x_1+1-\frac{1-\hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))} = -\delta.
\end{aligned}$$

A.4 Simultaneous Alternating Offer Bargaining with Two Buyers

A sufficient condition for the seller's proposal in equilibrium is

$$1 - x = \delta(1 - (1 - \frac{1-\hat{x}_2(x)}{\delta})). \tag{17}$$

The equation is that the seller's proposal makes the buyer indifferent between accept and reject, where the RHS is the buyer's payoff after rejecting – the buyer believes that the other buyer will accept the offer x according to passive beliefs.

(17) gives $x = \hat{x}_2(x)$. Notice that there must be a solution for this equation in $[1-\delta, 1]$, because $1 - \delta \leqslant \hat{x}_2(1-\delta)$ and $1 \geqslant \hat{x}_2(1)$.

Claim 1 *In an equilibrium of this continuation game, the seller will not provide two offers that induce one acceptance and one rejection.*

Proof. Suppose there is an equilibrium such that the seller provides two offers that induce one acceptance and one rejection at date t, denote the price of the acceptance offer as s_1.

The unique equilibrium starting at date $t+1$ is the buyer proposes $1 - \frac{1-\hat{x}_2(s_1)}{\delta}$, and the seller proposes $\hat{x}_2(s_1)$. Clearly, the seller's on-path payoff at date t will be $(1-\delta)(1+$

$\delta)u(s_1) + \delta^2 u(s_1 + 1 - \frac{1-\hat{x}_2(s_1)}{\delta})$.

However, if the seller deviates from the rejection offer at date t to the price $\hat{x}_2(s_1)$, the buyer will accept this offer. Thus, this off-path payoff of the seller at date t is $u(s_1 + \hat{x}_2(s_1))$, which is higher than the on-path payoff. As a result, there is a profitable deviation for the seller, which is a contradiction. ■

Also, there is no equilibrium where both buyers reject at date T, because in this case, the seller is better off by deviating to offers that make buyers indifferent and thus accept at date T.

So, we only need to consider cases where the seller provides offers such that both buyers will accept.

In this case, by a similar reason in the Rubinstein bargaining, the equation (17) characterizes an equilibrium proposal.

The equation is that the seller's proposal makes the buyer indifferent between accept and reject, where the RHS is the buyer's payoff after rejecting – the buyer believes that the other buyer will accept this offer x.

The equation directly gives $x = \hat{x}_2(x)$. Notice that there must be an equilibrium for this equation in $[1-\delta, 1]$, because $1 - \delta \leqslant \hat{x}_2(1-\delta)$ and $1 \geqslant \hat{x}_2(1)$.

The above presents a symmetric equilibrium. And the symmetric equilibrium is unique if and only if $\hat{x}_2(x) = x$ has a unique solution.

We now exclude the asymmetric equilibrium, i.e., the seller proposes differently to two buyers.

Suppose the asymmetric equilibrium exists, and the seller proposes x_1 and x_2 to two buyers, where $x_1 \neq x_2$. Then for the same reason as above, they satisfy

$$\begin{cases} x_1 = \hat{x}_2(x_2) \\ x_2 = \hat{x}_2(x_1) \end{cases}. \tag{18}$$

Suppose there is an asymmetric equilibrium where the seller proposes x_1 and x_2 to two buyers (w.l.o.g. $x_1 < x_2$), then we must have (x_1, x_2) and (x_2, x_1) on the graph of $\hat{x}_2(x)$.

But according to Lemma 1, $\hat{x}_2'(x) > -\delta$, so we have $\hat{x}_2(x_2) > \hat{x}_2(x_1) - \delta(x_2 - x_1) >$

$x_2 - (x_2 - x_1) = x_1$, contradicting $\hat{x}_2(x_2) = x_1$. Thus, there is no asymmetric equilibrium.

Now we can conclude that the equilibrium determined by (17) is unique when $\hat{x}_2(x)$ has a unique fixed point.

A.5 Limit Results

Let first prove the easier case, where $\delta \to 1$ and the number of periods before arrival T remains unchanged, i.e., Proposition 5.

Proof. Consider the continuation game starting at T where the seller does not reach an agreement with B_1. Recall the equilibrium condition for this continuation game (17), for the seller's offer at the limit $\delta \to 1$, it solves

$$x = \lim_{\delta \to 1} \hat{x}_2(x).$$

Notice that $\hat{x}_2(x)$ satisfies (14) at any $\delta \in (0, 1)$, so we can take derivative of both sides of (14) w.r.t. δ:

$$u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) \frac{\frac{d\hat{x}_2(x_1)}{d\delta}\delta + (1 - \hat{x}_2(x_1))}{\delta^2} = u(x_1 + \hat{x}_2(x_1)) - u(x_1) + \delta u'(x_1 + \hat{x}_2(x_1)) \frac{d\hat{x}_2(x_1)}{d\delta}.$$

Send δ to 1, we get:

$$u'(x_1 + \lim_{\delta \to 1} \hat{x}_2(x_1))(1 - \lim_{\delta \to 1} \hat{x}_2(x_1)) = u(x_1 + \lim_{\delta \to 1} \hat{x}_2(x_1)) - u(x_1). \tag{19}$$

Notice that $\lim_{\delta \to 1} \hat{x}_2(x_1)$ satisfying (19) means:

$$\lim_{\delta \to 1} \hat{x}_2(x_1) = \arg\max_s (u(x_1 + s) - u(x_1))(1 - s).$$

So, the solution to $x = \lim_{\delta \to 1} \hat{x}_2(x)$ solves:

$$\begin{cases} x = \arg\max_s (u(A + s) - u(A))(1 - s) \\ x = A \end{cases}.$$

Denote the solution as z^*. The limit outcome of the continuation game starting at T where the seller does not reach an agreement with B_1 is that the seller proposes z^* to both buyers, and buyers accept the offers, which is the same as the simultaneous NBS outcome.

As for the backward induction at date $t < T$, denote the offered prices at date t and $t+1$ are x_t, x_{t+1} respectively. If the seller is the proposer, x_t and x_{t+1} satisfy:

$$1 - x_t = \delta(1 - x_{t+1}). \tag{20}$$

If B_1 is the proposer, x_t and x_{t+1} satisfy:

$$(1-\delta^{T-t})u(x_t)+\delta^{T-t}u(x_t+\hat{x}_2(x_t)) = \delta((1-\delta^{T-t-1})u(x_{t+1})+\delta^{T-t-1}u(x_{t+1}+\hat{x}_2(x_{t+1}))). \tag{21}$$

As δ goes to 1, both (20) and (21) give us $x_t = x_{t+1}$.

Thus, at the limit of $\delta \to 1$, we have that the seller proposes a price of z^* at date 1, and proposes a price of $\lim_{\delta\to 1} \hat{x}_2(z^*) = z^*$ at date T. Both the offers are accepted immediately.

Now we see that the limit payoff of the model is the simultaneous NBS, where both buyers pay z^* to the seller:

$$(1 - \delta^T)u(x_1) + \delta^T u(x_1 + \hat{x}_2(x_1)) \to u(z^* + \lim_{\delta\to 1} \hat{x}_2(z^*)) = u(2z^*),\ as\ \delta \to 1.$$

■

Then we look at the limit where the intervals go to zero while the time before arrival remains unchanged. In this process, the number of periods before B_2 arrives goes to infinite.

The equilibrium outcome of the simultaneous AO bargaining after time T, by the same reason as Proposition 5, is the solution to (6).

We still need to figure out what the backward induction does at the limit.

Let p_t be the proposal made at time t. If p_t is the proposal made by the seller, then it is making the buyer indifferent:

$$1 - p_t = e^{-r\Delta}(1 - p_{t+\Delta}). \tag{22}$$

And $p_{t+\Delta}$, the buyer's proposal, is making the seller indifferent:

$$\begin{aligned}(1-e^{-r(T-t-\Delta)})u(p_{t+\Delta}) &+ e^{-r(T-t-\Delta)}u\big(p_{t+\Delta}+\hat{x}_2(p_{t+\Delta})\big)\\ &= (e^{-r\Delta}-e^{-r(T-t-\Delta)})u(p_{t+2\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+2\Delta}+\hat{x}_2(p_{t+2\Delta})\big).\end{aligned} \tag{23}$$

We can rewrite (23) as

$$\begin{aligned}&(1-e^{-r\Delta})u(p_{t+2\Delta})\\ &\quad= \Big[(1-e^{-r(T-t-\Delta)})u(p_{t+2\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+2\Delta}+\hat{x}_2(p_{t+2\Delta})\big)\Big]\\ &\quad- \Big[(1-e^{-r(T-t-\Delta)})u(p_{t+\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+\Delta}+\hat{x}_2(p_{t+\Delta})\big)\Big].\end{aligned} \tag{24}$$

The RHS of (24) can be written as:

$$(p_{t+2\Delta}-p_{t+\Delta})\Big[(1-e^{-r(T-t-\Delta)})u'(p_{t+\epsilon}) + e^{-r(T-t-\Delta)}u'\big(p_{t+\epsilon}+\hat{x}_2(p_{t+\epsilon})\big)\big(1+\hat{x}_2'(p_{t+\epsilon})\big)\Big],$$

where $p_{t+\epsilon}$ is between $p_{t+\Delta}$ and $p_{t+2\Delta}$.

Notice that by (22), $p_{t+2\Delta}-p_{t+\Delta} = p_{t+2\Delta}-1+e^{r\Delta}(1-p_t) = p_{t+2\Delta}-p_t+(e^{r\Delta}-1)(1-p_t)$

Thus, (24) is

$$\begin{aligned}&(1-e^{-r\Delta})u(p_{t+2\Delta})\\ &\quad= \Big[p_{t+2\Delta}-p_t+(e^{r\Delta}-1)(1-p_t)\Big]\\ &\quad\cdot \Big[(1-e^{-r(T-t-\Delta)})u'(p_{t+\epsilon}) + e^{-r(T-t-\Delta)}u'\big(p_{t+\epsilon}+\hat{x}_2(p_{t+\epsilon})\big)\big(1+\hat{x}_2'(p_{t+\epsilon})\big)\Big].\end{aligned} \tag{25}$$

Divide both sides of (25) by Δ and send Δ to 0 we get

$$ru(p_t) = (2p_t'+r(1-p_t))\Big[(1-e^{-r(T-t)})u'(p_t) + e^{-r(T-t)}u'\big(p_t+\lim_{\Delta\to 0}\hat{x}_2(p_t)\big)\big(1+\lim_{\Delta\to 0}\hat{x}_2'(p_t)\big)\Big]. \tag{26}$$

This equation and $p_T = z^*$ lead to an asymmetric result after considering backward induction.

A.6 Proof for $x_1^* < x_U$ with IARA

According to Proposition 1, when the seller has IARA, $\hat{x}_2(x)$ is decreasing.

We know the outcome of the continuation game after T where the seller does not have an agreement with B_1 is the flow price x^*. Denoting the proposed flow price in date t as x_t, we have the following lemmas.

Lemma 2 *If the seller has IARA, $x_{T-2} < x_U$.*

Proof. Because T is odd, the backward induction gives us $x_{T-1} = 1 - \frac{1-x_{T-2}}{\delta}$. So, we need to show $1 - \frac{1-x_U}{\delta} > x_{T-1}$ to prove the lemma.

Notice that $x^* = \hat{x}_2(x^*)$, $x_U = \hat{x}_2(0)$ and $\hat{x}_2(x)$ decreasing, we have $x_U > x^*$. Thus, we only need to show $1 - \frac{1-x^*}{\delta} > x_{T-1}$ to prove the lemma.

By the backward induction, x_{T-1} satisfies

$$(1-\delta)u(x_{T-1}) + \delta u(x_{T-1} + \hat{x}_2(x_{T-1})) = \delta u(2x^*). \tag{27}$$

Lemma 1 says $\hat{x}_2'(x_{T-1}) > -\delta$, so $1 + \hat{x}_2(x_{T-1})$ is increasing in x_{T-1}. Thus, the LHS of (27) is also increasing in x_{T-1}, so we only need to show

$$(1-\delta)u(1 - \frac{1-x^*}{\delta}) + \delta u(1 - \frac{1-x^*}{\delta} + \hat{x}_2(1 - \frac{1-x^*}{\delta})) > \delta u(2x^*) \tag{28}$$

to prove $1 - \frac{1-x^*}{\delta} > x_{T-1}$.

According to $x^* = \hat{x}_2(x^*)$, x^* satisfies

$$u(x^* + 1 - \frac{1-x^*}{\delta}) = (1-\delta)u(x^*) + \delta u(2x^*).$$

So (28) is equivalent to

$$(1-\delta)[u(1 - \frac{1-x^*}{\delta}) + u(x^*)] + \delta u(1 - \frac{1-x^*}{\delta} + \hat{x}_2(1 - \frac{1-x^*}{\delta})) > u(x^* + 1 - \frac{1-x^*}{\delta}). \tag{29}$$

Because $\hat{x}_2(x)$ is decreasing, $\hat{x}_2(1 - \frac{1-x^*}{\delta}) > \hat{x}_2(x^*) = x^* \Rightarrow u(1 - \frac{1-x^*}{\delta} + \hat{x}_2(1 - \frac{1-x^*}{\delta})) > u(1 - \frac{1-x^*}{\delta} + x^*)$.

By the concavity of $u(x)$ and $u(0) = 0$, we have that for any $x \in (0, 1)$ and $t \in (0, 1)$:

$$u(tx) = u(tx + (1 - t) \cdot 0) > tu(x) + (1 - t)u(0) = tu(x). \tag{30}$$

Then using (30), we have that for any $x, y \in (0, 1)$:[16]

$$u(x)+u(y) = u(\frac{x}{x+y}(x+y))+u(\frac{y}{x+y}(x+y)) > \frac{x}{x+y}u(x+y)+\frac{y}{x+y}u(x+y) = u(x+y). \tag{31}$$

(31) says that $u(1 - \frac{1-x^*}{\delta}) + u(x^*) > u(1 - \frac{1-x^*}{\delta} + x^*)$.[17]

Thus, (29) holds. ■

Lemma 3 *For any odd number $\tau < T$:*

$$(1-\delta^\tau)u(1-\frac{1-x_U}{\delta})+\delta^\tau u(1-\frac{1-x_U}{\delta}+\hat{x}_2(1-\frac{1-x_U}{\delta})) \geqslant (\delta-\delta^\tau)u(x_U)+\delta^\tau u(x_U+\hat{x}_2(x_U)). \tag{32}$$

Proof. According to the equilibrium condition of the unaffected bargaining, (32) is equivalent to

$$(1 - \delta^\tau)\delta u(x_U) + \delta^\tau u(1 - \frac{1-x_U}{\delta} + \hat{x}_2(1 - \frac{1-x_U}{\delta})) \geqslant (\delta - \delta^\tau)u(x_U) + \delta^\tau u(x_U + \hat{x}_2(x_U))$$

$$\Leftrightarrow \delta u(x_U)(\delta^{\tau-1} - \delta^\tau) + \delta^\tau u(1 - \frac{1-x_U}{\delta} + \hat{x}_2(1 - \frac{1-x_U}{\delta})) \geqslant \delta^\tau u(x_U + \hat{x}_2(x_U))$$

$$\Leftrightarrow (1 - \delta)u(x_U) + u(1 - \frac{1-x_U}{\delta} + \hat{x}_2(1 - \frac{1-x_U}{\delta})) \geqslant u(x_U + \hat{x}_2(x_U))$$

$$\Leftrightarrow u(1 - \frac{1-x_U}{\delta} + \hat{x}_2(1 - \frac{1-x_U}{\delta})) - u(1 - \frac{1-x_U}{\delta}) \geqslant u(x_U + \hat{x}_2(x_U)) - u(x_U)$$

$$\Leftrightarrow u(x_U) - u(1 - \frac{1-x_U}{\delta}) \geqslant u(x_U + \hat{x}_2(x_U)) - u(1 - \frac{1-x_U}{\delta} + \hat{x}_2(1 - \frac{1-x_U}{\delta})). \tag{33}$$

By the concavity of $u(x)$,

$$RHS\ of\ (33) < [\frac{1-\delta}{\delta}(1 - x_U) + \hat{x}_2(x_U) - \hat{x}_2(1 - \frac{1-x_U}{\delta})]u'(1 - \frac{1-x_U}{\delta} + \hat{x}_2(1 - \frac{1-x_U}{\delta})),$$

[16] $u(x)$ is defined on $[0, 2]$, so all terms in (30) and (31) are defined.

[17] The proof in Appendix A.1 says that $x^* \in (1 - \delta, 1)$

$$LHS \ of \ (33) > \frac{1-\delta}{\delta}(1-x_U)u'(x_U).$$

But $\frac{1-\delta}{\delta}(1-x_U) < 1-\delta < \hat{x}_2(1-\frac{1-x_U}{\delta}) \Rightarrow x_U < 1-\frac{1-x_U}{\delta} + \hat{x}_2(1-\frac{1-x_U}{\delta})$

$$\Rightarrow u'(x_U) > u'(1-\frac{1-x_U}{\delta} + \hat{x}_2(1-\frac{1-x_U}{\delta})) > 0.$$

Moreover, because $\hat{x}_2(x)$ is decreasing,

$$\frac{1-\delta}{\delta}(1-x_U) + \hat{x}_2(x_U) - \hat{x}_2(1-\frac{1-x_U}{\delta}) < \frac{1-\delta}{\delta}(1-x_U).$$

Thus, (33) holds. ■

Lemma 4 *If the seller has IARA, for some odd number $t < T$, if $x_t < x_U$, then $x_{t-2} < x_U$.*

Proof. Let us look at an $x_t < x_U$ where t is odd and $t < T$. The backward induction implies

$$(1-\delta^{T-t+1})u(x_{t-1}) + \delta^{T-t+1}u(x_{t-1} + \hat{x}_2(x_{t-1})) = (\delta-\delta^{T-t+1})u(x_t) + \delta^{T-t+1}u(x_t + p_2 * (x_t)).$$

Due to $x_t < x_U$ and Lemma 3, we have

$$\begin{aligned}(\delta-\delta^{T-t+1})&u(x_t) + \delta^{T-t+1}u(x_t + \hat{x}_2(x_t)) \\ &<(\delta-\delta^{T-t+1})u(x_U) + \delta^{T-t+1}u(x_U + \hat{x}_2(x_U)) \\ &\leqslant(1-\delta^{T-t+1})u(1-\frac{1-x_U}{\delta}) + \delta^{T-t+1}u(1-\frac{1-x_U}{\delta} + \hat{x}_2(1-\frac{1-x_U}{\delta})).\end{aligned}$$

Thus, $x_{t-1} < 1-\frac{1-x_U}{\delta}$. But $x_{t-2} = 1-\delta(1-x_{t-1})$ by backward induction, so $x_{t-2} < x_U$. ■

According to Lemma 2 and Lemma 4, if the seller utility has IARA, for any odd number $t < T$, we have $x_t < x_U$, which means $x_1 < x_U$.

Moreover, because $\hat{x}_2(x)$ is decreasing with an IARA seller, $\hat{x}_2(x_1) < \hat{x}_2(0) = x_U$. As a result, both flow prices in sequential bargaining are smaller than the flow price in the unaffected AO benchmark. Thus, the seller is worse off than with the unaffected bargaining.