Week 05: Linear Algebra Review

CS-537: Interactive Computer Graphics
Dr. Chloe LeGendre
Department of Computer Science

Some materials from the companion slides of Angel and Shreiner, “Interactive Computer Graphics, A Top-Down Approach with WebGL.”

Objectives

- Introduce elements of geometry
    - Scalars, Vectors, and Points
    - Most of this part should be a brief review from linear algebra
- Develop mathematical operations among them in a coordinate-free manner

Geometry and Geometric Primitives Recap

- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- We use a minimum set of primitives from which we can build more sophisticated objects (example: polygon mesh)
- We use three basic elements
  - Scalars
  - Vectors
  - Points

Coordinate-Free Geometry

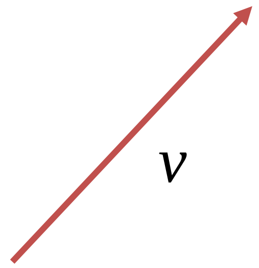
- When we learned simple geometry, most of us started with a Cartesian approach
  - Points were at locations in space **p**=(x,y,z)
  - We derived results by algebraic manipulations involving these coordinates
- This approach was nonphysical
  - Physically, points exist regardless of the location of an arbitrary coordinate system
  - Most geometric results are independent of the coordinate system
  - Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical

Scalars

- Need three basic elements in geometry
  - Scalars, Vectors, Points
- Scalars can be defined as members of sets that can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutativity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

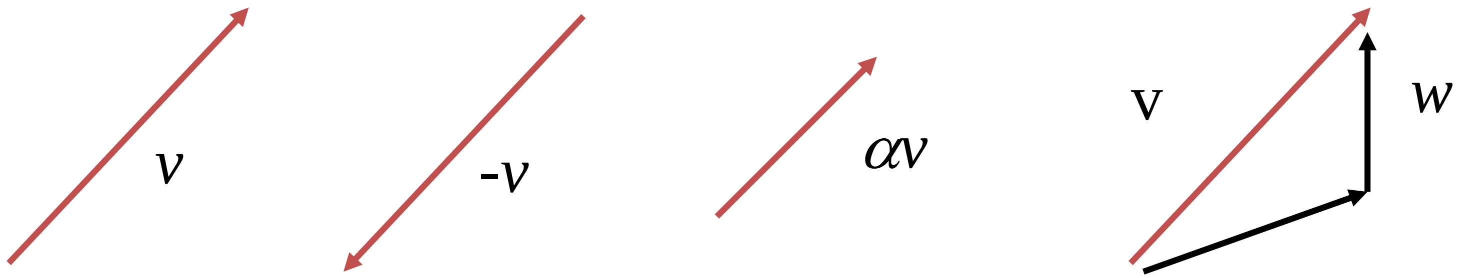
Vectors

- Physical definition: a vector is a quantity with two attributes:
  - Direction
  - Magnitude
- Examples include:
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



Vector Operations

- Every vector has an inverse
    - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
    - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
    - Use head-to-tail axiom



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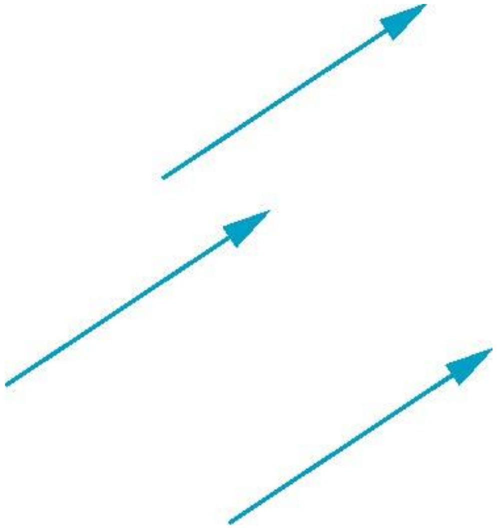
Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication $u = \alpha v$
  - Vector-vector addition: $w=u+v$
- Expressions such as the following make sense in a vector space

  $v=u+2w-3r$

Vectors Lack Position

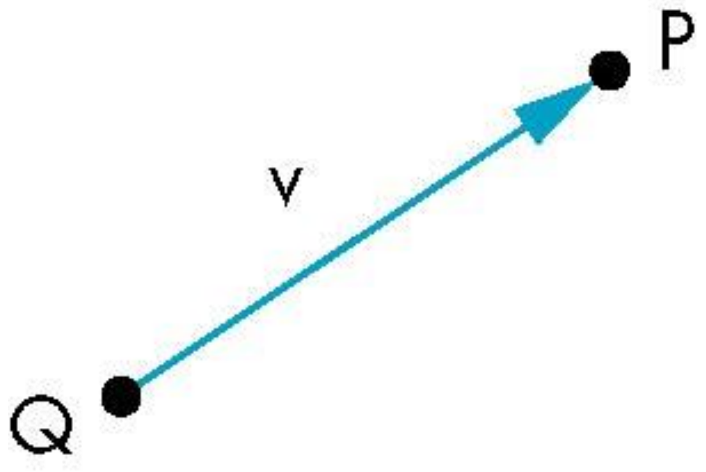
- These vectors are identical:
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



$$v = P - Q$$

$$P = Q + v$$

Affine Spaces

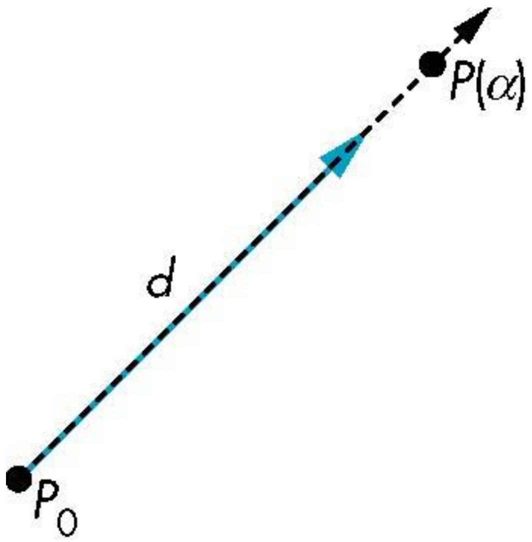
- Extension of vector space that also includes points
- Operations:
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define:

  $1 \cdot P = P$

  $0 \cdot P = 0$ (zero vector)

Lines

- Consider all points of the form
  - $P(\alpha)=P_0 + \alpha\ \mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector $\mathbf{d}$



Parametric Form

- The form of the equation for a line on the previous slide is called the "parametric form"
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms:
  - Explicit: $y = mx + h$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$

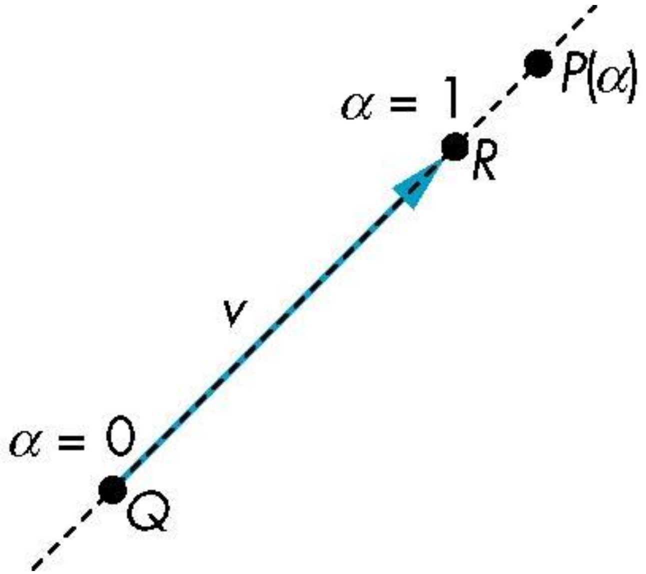
$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

Rays and Line Segments

- If α >= 0, then $P(\alpha)$ is the *ray* leaving P_0 in the direction $\mathbf{d}$
- If we use two points to define v, then

$$P(\alpha) = Q + \alpha\,(R-Q) = Q+\alpha v$$
$$= \alpha R + (1-\alpha)Q$$

- For $0<=\alpha<=1$ we get all the points on the *line segment* joining R and Q



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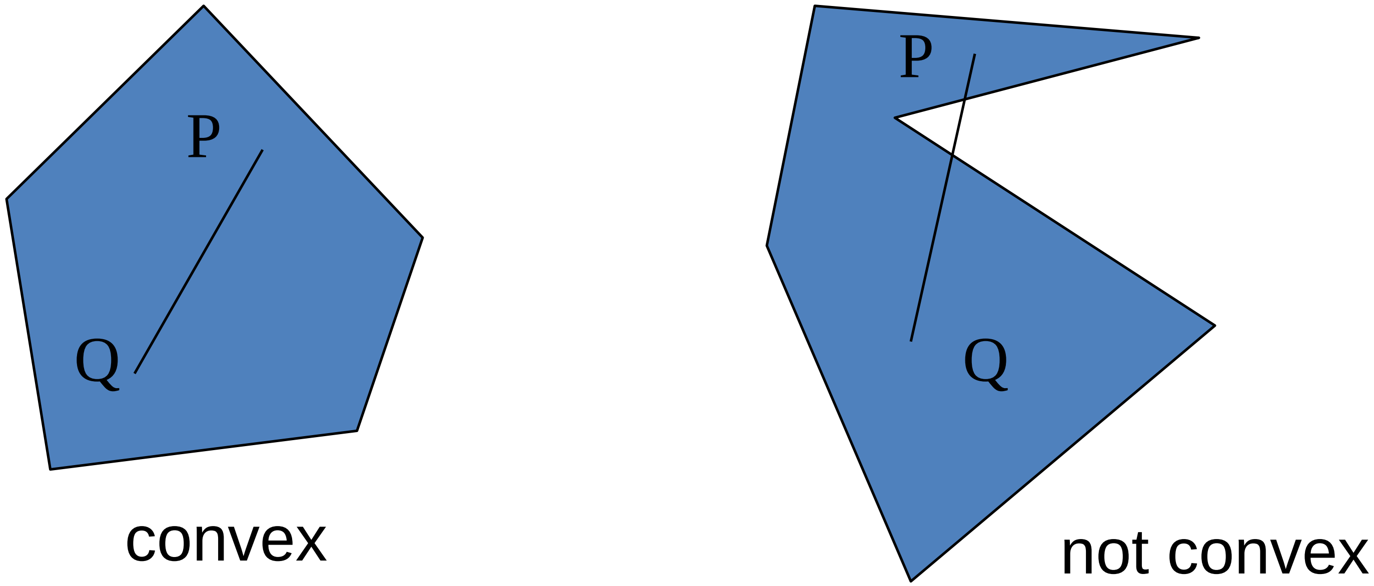
Convexity

- An object is *convex* if and only if for any two points in the object all points on the line segment between these points are also in the object
- Recall: WebGL requires triangles, which are always convex

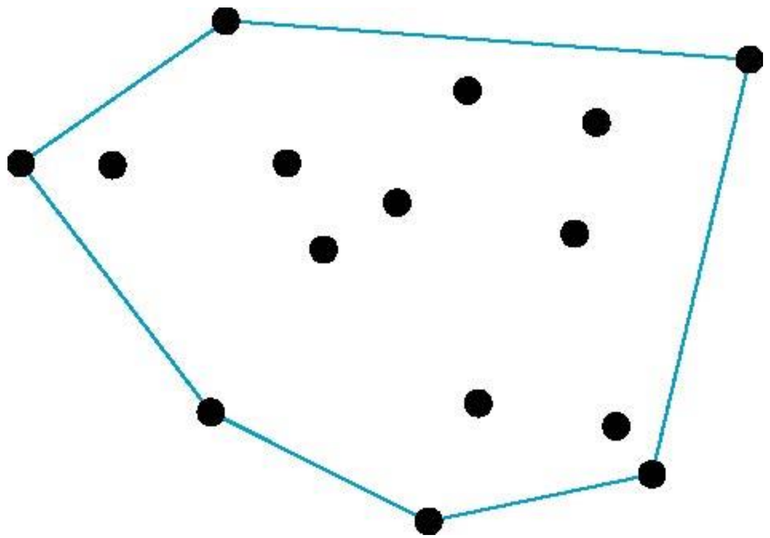


Affine Sums

- Consider the weighted "sum" across points:

  $P=\alpha_1 P_1+\alpha_2 P_2+\ldots\ldots+\alpha_n P_n$
- With the added condition that all weights sum to 1:

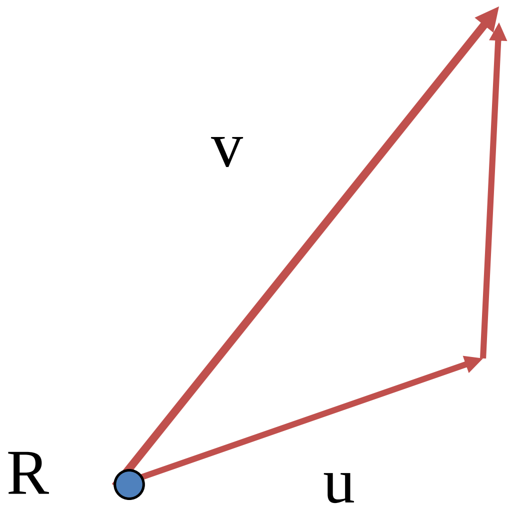
  $\alpha_1+\alpha_2+\ldots\ldots\alpha_n=1$
- This is called the affine sum of the points $P_1, P_2, \ldots\ldots P_n$
- If, in addition, $\alpha_i>=0$, we have the *convex hull* of $P_1, P_2, \ldots\ldots P_n$
- Convex Hull:
  - Smallest convex object containing $P_1, P_2, \ldots\ldots P_n$
  - Formed by "shrink wrapping" points

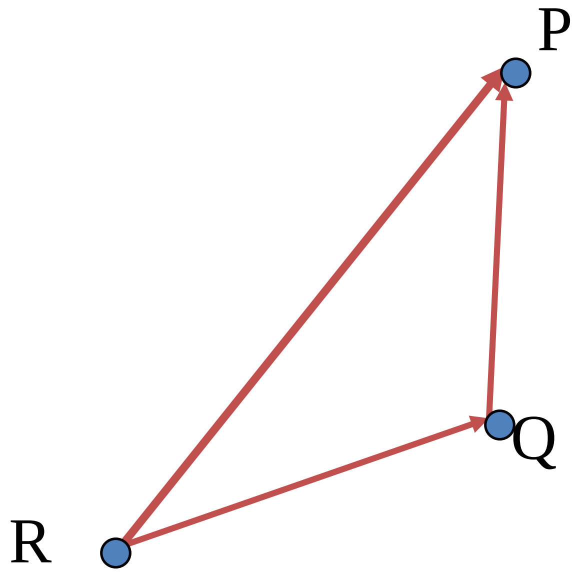


Convex hull of point set

Planes

- A plane can be defined by a point and two vectors or by three points



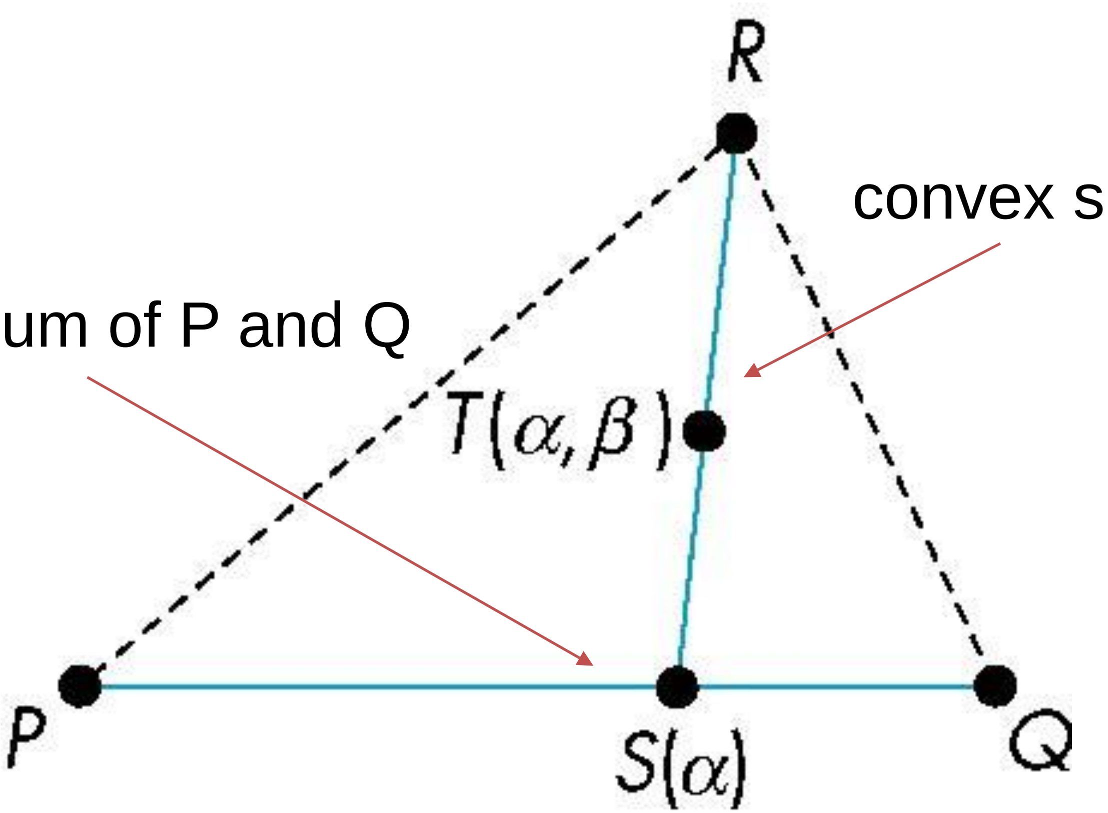


$$P(\alpha,\beta)=R+\alpha u+\beta v$$

$$P(\alpha,\beta)=R+\alpha(Q-R)+\beta(P-Q)$$

Triangles

- A plane can be defined by a point and two vectors or by three points



for 0<=a, b<=1, we get all points in triangle

Barycentric Coordinates

- A triangle is convex so any point inside can be represented as an affine sum

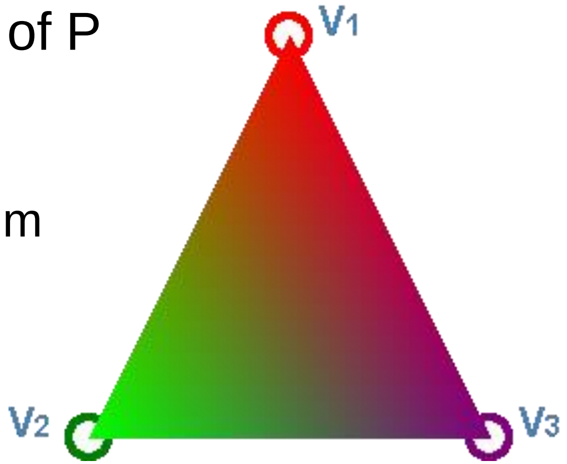
$P(\alpha_{1,}\ \alpha_{2,}\ \alpha_3)=\alpha_1 P+\alpha_2 Q+\alpha_3 R$

where:

$\alpha_1 + \alpha_2 + \alpha_3 = 1$

$\alpha_i >= 0$

- This representation is called the ***barycentric coordinate*** representation of P
- Can be used to interpolate values (like color) across a triangle
  - For a given point, solve for $\alpha_{1,}\alpha_{2,}\alpha_3$ and find the interior color as the affine sum



Normals

- In three dimensional spaces, every plane has a vector n perpendicular or orthogonal to it called the **normal vector**
- From the two-point vector form $P(\alpha,\beta)=P+\alpha u+\beta v$, we know we can use the cross product to find $n = u \times v$ and the equivalent form

$$(P(\alpha, \beta)-P) \cdot n = 0$$