Cardiac Output

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1 Cover Letter

Dear Simone Rossi, dear Robert Hunt,

This is a list of changes in our report:

1. **Include a plot of the interpolating polynomials used.**

Please see **Figure 1**, **Figure 2**, **Figure 3** and **Figure 4** that we have included in the Results section.

2. **There should be exactly one polynomial of order at most $n-1$ going through n data points, so interpolated values should agree with both methods. You can plug in a volume to get a pressure, but how would you find a pressure given a volume? Did you use linear interpolation to patch phases? How did you calculate the error?**

In order to determine the pressure at 90 mL, we created an interpolating function of the form P(V), which volume being the input parameter and pressure being the output parameter. In order to determine the volume at 110 mmHg, we created a new interpolating function of the form V(P).

In order to patch phases between the intervals when calculating the volume at 110 mmHg, we used a linear function to interpolate between two phases because of the piecewise nature of the data points. Using the last two points of each interval, we constructed the following linear functions:

Between Phase I & II:

Slope calculation:

$$\frac{V_2 - V_1}{P_2 - P_1} = \frac{115.1 - 119.4}{95.3 - 79.9} = -.279$$

y-intercept:

$$V = -.279P + b$$

We substituted 95.3 mmHg for the pressure and 115.1 mL for the volume and found a y-intercept of 141.6887. Thus, this gave us a linear interpolation equaiton of $V = -.279P + 141.6887$.

Between Phase II & III:

Slope calculation:

$$\frac{V_2 - V_1}{P_2 - P_1} = \frac{50.7 - 55.8}{90.6 - 112.9} = .229$$

y-intercept:

$$V = .229P + b$$

We substituted 90.6 mmHg for the pressure and 50.7 mL for the volume and found a y-intercept of 29.9526. Thus, this gave us a linear interpolation equaiton of $V = .229P + 29.9526$.

To calculate the error for the Newton interpolation method and Lagrange interpolation method, we used the formula

$$E_n(x) = \frac{f^{(n+1)}(\epsilon)}{(n+1)!}(x - x_0)\dots(x - x_n)$$

for $a < \epsilon < b$ on the interval from $[a, b]$.

3. **Could you clarify your discussion of integration? You plotted piecewise linear interpolations. Did you use trapezoid method with the intervals provided and the linear interpolation, or with your**

higher order interpolating polynomial? Same for Simpson's rule? What is the interval size, and how does the error scale with interval size and derivatives of the function you are approximating?

To calculate the error of integration from using the Trapezoid Method, we used the formula

$$-\frac{h^2(b-a)}{12}f"(\epsilon)$$

To calculate the error of integration from using Cavalieri-Simpson's Rule, we used the formula

$$\frac{1}{90}(\frac{b-a}{2})^5|f^{(4)}(\epsilon)|$$

We used the same size interval for Simpson's method. Generally, the Cavalieri-Simpson rule produces a more accurate result than the trapezoid rule because the Cavalieri-Simpson rule fits three points to a parabola, whereas the trapezoid rule fits two points to a line. **Table 7** displays the errors for both methods, and we observe that Cavalieri-Simpson's Rule has a much smaller error compared to the trapezoid method. Because the Cavalieri-Simpson rule has an order of 3, it is thought to be more accurate than the trapezoid rule. Therefore, we believe the values listed in **Table 6** above for the Cavalieri-Simpson rule are more accurate the results listed for the trapezoid method.

We used an interval length size of 5. We see that the error for both the Cavalieri-Simpson method and Trapezoid method decreases as you increase the number of intervals and decrease the step size. This is because, in the case of the Simpson method, parabolas are fit in between these intervals and are used to integrate this function. The more intervals we have, the more accurate our result will be. The same principle applies in the case of the Trapezoid method, except that a line is used instead of a parabola. Furthermore, the error also tends to decrease as you reach higher order derivatives. In the case of the Cavalieri-Simpson method, polynomials of order 3 or less can be interpolated exactly, since the fourth derivative of the function becomes 0 in the error formula. The trapezoid method produces exact results for only polynomials of order 1 or less, since the second derivative of the function becomes 0 in the error formula. This further demonstrates the benefits of using the Cavalieri-Simpson method.

4. **If you use Simpson's rule, you need intervals of the same size.**

Upon using the same interval size, we obtained the values displayed in **Table 6**.

2 Introduction

Blood in the heart always flows in the direction of lower pressure. A heartbeat consists of four phases: I) isovolumetric contraction, II) ejection, III) isovolumetric relaxation and IV) filling. During isovolumetric contraction, the valves are closed and no blood can get in or out of the ventricle; during this first phase, the ventricular muscle starts to contract rapidly, which rapidly increases the ventricular pressure. When the ventricular pressure exceeds the aortic valve pressure, we enter the ejection phase in which the valve opens and the blood flows out of the ventricle, thus reducing the volume of it. As the ventricle empties, its pressure decreases and the aortic valve closes so the blood cannot flow backward. We now are in the isovolumetric relaxation phase, in which the ventricle relaxes and both the valves are closed and the blood volume is kept constant. The ventricle relaxes and the pressure decreases. When the ventricular pressure becomes smaller than the atrial pressure, we enter the filling phase. Blood fills the ventricle once again and pressure increases and we enter Phase I again.

3 Statement of Problem

Based on the data presented in **Table 1**, we want to find out what the values of the pressure are when the volume is 90 ml and what the values of volume are when the pressure is 110 mmHg. In addition, we attempt to find the stroke work based on the data below by calculating the area under the pressure-volume curve.

Table 1: Data for pressure and volume during the four phases together with the time, starting at $t_0 = 0$.

Phase I						Phase II					
t_I[ms]	0	11	22	34	-	t_{II}[ms]	68	119	170	221	255
p_I[mmHg]	8.1	32.2	56.2	79.9	-	p_{II}[mmHg]	95.3	115.0	125.1	126.3	112.9
v_I[ml]	120.0	119.8	119.6	119.4	-	v_{II}[ml]	115.1	105.3	85.4	70.7	55.8
Phase III						Phase IV					
t_{III}[ms]	272	280	289	297	-	t_{IV}[ms]	331	425	510	578	646
p_{III}[mmHg]	90.6	70.0	50.1	30.6	-	p_{IV}[mmHg]	13.5	8.4	6.2	5.1	7.5
v_{III}[ml]	50.7	49.7	49.5	49.2	-	v_{IV}[ml]	50	64.3	79.7	94.1	109.0

4 Numerical Methods

4.1 Interpolation Methods

In order to find the values of the volume and pressure, we employed the Newton Interpolation Method and Lagrange Interpolation Method. In order to determine the pressure at 90 mL, we will create an interpolating polynomial for Phase II and Phase IV of the form P(V), a function of pressure in terms of volume. In order to determine the volume at a pressure of 110 mmHg, a new interpolating polynomial of the form V(P) (a function of volume in terms of pressure) will be created for between Phase I and Phase II and for between Phase II and Phase III. We will use both Newton's method and Lagrange's method to build these interpolating polynomials.

4.1.1 Newton's Method of Interpolating Polynomials

The Newton form of the interpolating polynomial is

$$P_{x_0 \ldots x_n}(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

with Newton's divided difference computed as:

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

We will treat each phase as an interval of values. To interpolate the pressure when volume was 90 mL, we look at where the volume was 90 mL once during Phase II and once during Phase IV. Using Newton's Method of Interpolating Polynomials, we obtained the following polynomials for each phase:

Phase II:

$$P(V) = 95.3 - 2.0102(x-115.1) - 0.050595(x-115.1)(x-105.3) - 0.0008622(x-115.1)(x-105.3)(x-85.4) - 0.00021638(x-115.1)(x-105.3)(x-85.4)(x-70.7)$$

Phase IV:

$$P(V) = 13.5 - 0.35664(x-50) + 0.0071982(x-50)(x-64.3) - 0.00011265(x-50)(x-64.3)(x-79.7) + 0.0000041366(x-50)(x-64.3)(x-79.7)(x-94.1)$$

Blood will always move in the direction of lower pressure, as pressure and volume share an inverse relationship. Therefore, the volume of the ventricle is always decreasing and the pressure inside of it always increases. For example, we know that in Phase II, or the ejection phase, that the volume is always decreasing by definition. Therefore, the pressure will strictly increase and will be 110 mmHg at some point between 68 seconds and 119 seconds, when the pressure goes from 95.3 mmHg to 115.0 mmHg. Knowing this, we can interpolate volume at 110 mmHg during this interval. The same principle applies between Phase I and Phase II and between Phase II and Phase III, as we exit the ejection phase and enter the isovolumetric relaxation phase.

When using the Lagrange Method, the values obtained for the volume when only looking at Phase II values produced a value that was not in between 115.1 mL and 105.3 mL or 55.8 mL and 50.7 mL. This was particularly due to the fact that a pressure of 110 mmHg occurred towards the endpoints of the Phase II interval, and the endpoints of intervals tend to be more prone to error. In order to mitigate this error, we used values from the end of Phase I and the beginning of Phase II in order to calculate a more reasonable value for the pressure at 110 mmHg the first time during Phase II, which we found to be at 110.4 mL, as recorded in Table 3. To calculate the volume the second time the pressure would drop to 110 mmHg, we used values at the end of the Phase II and at the beginning of Phase III. Since we know the volume is always decreasing and the pressure is always increasing, this will continue to happen between phases. Therefore we know that a pressure of 110 mmHg occurs between Phase II and Phase III. Upon doing this, we obtained a value of between 53 and 54 mL, which was consistent with our original thought of this value having to be between 55.8 mL and 50.7 mL. Based on the error values displayed in Table 4 for this problem, we can see that Newton Method of Interpolation produces less error than the Lagrange Method of Interpolation as well.

In order to patch phases between the intervals when calculating the volume at 110 mmHg, we used a linear function to interpolate between two phases because of the piecewise nature of the data points. Using the last two points of each interval, we constructed the following linear function:

Between Phase I & II:

Slope calculation:

$$\frac{V_2 - V_1}{P_2 - P_1} = \frac{115.1 - 119.4}{95.3 - 79.9} = -.279$$

y-intercept:

$$V = -.279P + b$$

We substituted 95.3 mmHg for the pressure and 115.1 mL for the volume and found a y-intercept of 141.6887. Thus, this gave us a linear interpolation equaiton of $V = -.279P + 141.6887$.

Between Phase II & III:

Slope calculation:

$$\frac{V_2 - V_1}{P_2 - P_1} = \frac{50.7 - 55.8}{90.6 - 112.9} = .229$$

y-intercept:

$$V = .229P + b$$

We substituted 90.6 mmHg for the pressure and 50.7 mL for the volume and found a y-intercept of 29.9526. Thus, this gave us a linear interpolation equaiton of $V = .229P + 29.9526$.

The pressure of 110 mmHg occurred twice: once between Phase I and Phase II, and once more between Phase II and Phase III. For finding the volume when the pressure was 110 mmHg, we used the following Newton Interpolating Polynomials:

Between Phase I & II:

$$P(V) = 119.6 - 0.00845(x-56.2) - 0.00693(x-56.2)(x-79.9) + 0.00001203(x-56.2)(x-79.9)(x-95.3) - 0.00001405(x-56.2)(x-79.9)(x-95.3)(x-115)$$

Between Phase II & III:

$$P(V) = 70.7 + 1.112(x-126.3) + 0.0247(x-126.3)(x-112.9) + 0.0003649(x-126.3)(x-112.9)(x-90.6) + 0.00000412(x-126.3)(x-112.9)(x-90.6)(x-70)$$

4.1.2 Lagrange Method of Interpolating Polynomials

We want to interpolate at the point x. The Lagrange form of the interpolating polynomial is

$$P(x) = y_0 b_0(x) + y_1 b_1(x) + \cdots + y_n b_n(x) = \sum_{i=0}^{n} y_i b_i(x).$$

The coefficients of the polynomial can be calculated using the divided difference:

$$b_i = \frac{(x - x_0)(x - x_1)\dots(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}$$

To find the pressure when the volume is 90 mL at Phase II and Phase IV, we calculated five $y_n b_n$ terms as follows:

Phase II:

$$L_1(x) = (95.3)\frac{(x - 105.3)(x - 85.4)(x - 70.7)(x - 55.8)}{766337.6952}$$

$$L_2(x) = (115)\frac{(x - 115.1)(x - 85.4)(x - 70.7)(x - 55.8)}{-334010.754}$$

$$L_3(x) = (125.1)\frac{(x - 115.1)(x - 105.3)(x - 70.7)(x - 55.8)}{257168.9736}$$

$$L_4(x) = (126.3)\frac{(x - 115.1)(x - 105.3)(x - 85.4)(x - 55.8)}{-336482.6472}$$

$$L_5(x) = (112.9)\frac{(x - 115.1)(x - 105.3)(x - 85.4)(x - 70.7)}{1294606.764}$$

Phase IV:

$$L_1(x) = (13.5)\frac{(x-64.3)(x-79.7)(x-94.1)(x-109)}{1105052.949}$$

$$L_2(x) = (8.4)\frac{(x-50)(x-79.7)(x-94.1)(x-109)}{-293346.2532}$$

$$L_3(x) = (6.2)\frac{(x-50)(x-64.3)(x-94.1)(x-109)}{192977.7696}$$

$$L_4(x) = (5.1)\frac{(x-50)(x-64.3)(x-79.7)(x-109)}{-281970.4608}$$

$$L_5(x) = (7.5)\frac{(x-50)(x-64.3)(x-79.7)(x-94.1)}{1151366.061}$$

To find the volume when the pressure is 110 mmHg, we also calculated five $y_n b_n$ terms as follows:
Between Phase I & II:

$$L_1(x) = (119.6)\frac{(x-79.9)(x-95.3)(x-115)(x-125.1)}{3754236.704}$$

$$L_2(x) = (119.4)\frac{(x-56.2)(x-95.3)(x-115)(x-125.1)}{-579048.0696}$$

$$L_3(x) = (115.1)\frac{(x-56.2)(x-79.9)(x-115)(x-125.1)}{353492.3084}$$

$$L_4(x) = (105.3)\frac{(x-56.2)(x-79.9)(x-115)(x-125.1)}{-410650.2036}$$

$$L_5(x) = (85.4)\frac{(x-56.2)(x-79.9)(x-95.3)(x-115)}{937335.9944}$$

Phase II & III:

$$L_1(x) = (70.7)\frac{(x-112.9)(x-90.6)(x-70)(x-50.1)}{2052278.903}$$

$$L_2(x) = (55.8)\frac{(x-126.3)(x-90.6)(x-70)(x-50.1)}{-805056.9384}$$

$$L_3(x) = (50.7)\frac{(x-126.3)(x-112.9)(x-70)(x-50.1)}{664194.573}$$

$$L_4(x) = (49.7)\frac{(x-126.3)(x-112.9)(x-90.6)(x-50.1)}{-990115.7838}$$

$$L_5(x) = (49.5)\frac{(x-126.3)(x-112.9)(x-90.6)(x-70)}{385670.892}$$

To calculate the error for the Newton interpolation method and Lagrange interpolation method, we used the formula

$$E_n(x) = \frac{f^{(n+1)}(\epsilon)}{(n+1)!}(x-x_0)\dots(x-x_n)$$

for $a < \epsilon < b$ on the interval from $[a, b]$.

4.2 Numerical Integration

We chose to use the trapezoid method and the Cavalieri-Simpson Rule to calculate the area underneat the pressure-volume curve in order to find the work.

4.2.1 Trapezoid Method

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

4.2.2 Cavalieri-Simpson Rule

$$\frac{(b-a)}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]$$

4.2.3 Calculating Integration Error

To calculate the error of integration from using the Trapezoid Method, we used the formula

$$-\frac{h^2(b-a)}{12}f"(\epsilon)$$

To calculate the error of integration from using Cavalieri-Simpson's Rule, we used the formula

$$\frac{1}{90}(\frac{b-a}{2})^5|f^{(4)}(\epsilon)|$$

5 Results

Figure 1: Newton interpolation polynomial and Lagrange interpolation polynomial for finding pressure at 90 mL in Phase II.

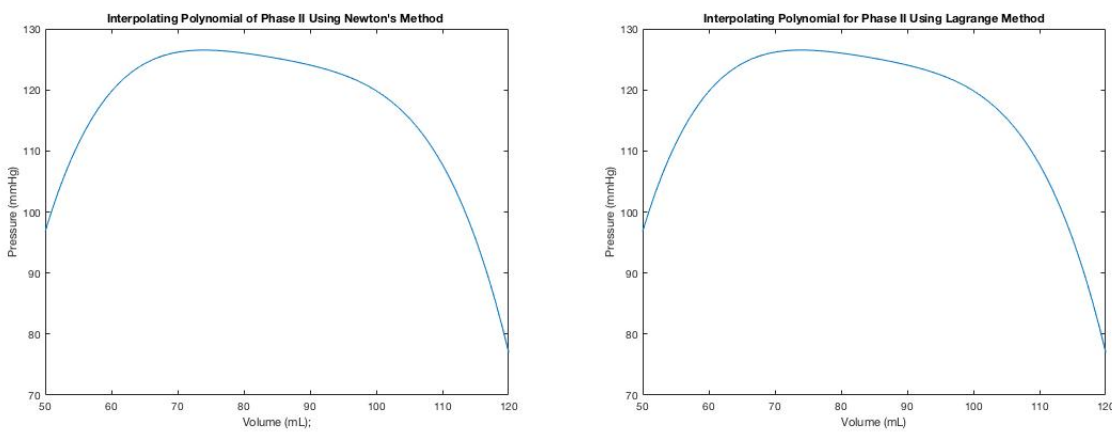

Figure 2: Newton interpolation polynomial and Lagrange interpolation polynomial for finding pressure at 90 mL in Phase IV.

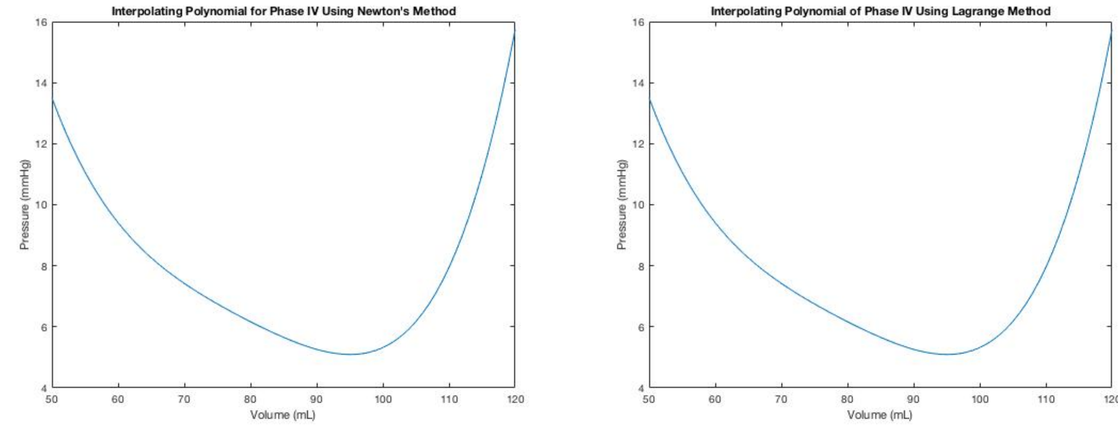

Table 2: Interpolated pressure when the volume is 90 mL using Newton and Lagrange Methods.

Method	Phase II	Phase IV
Newton's Method	124.0651831 mmHg	5.261786441 mmHg
Lagrange Method	124.0652118 mmHg	5.261679667 mmHg

Figure 3: Newton interpolating polynomial and Lagrange interpolating polynomial for finding volume at 110 mmHg between Phase I & II.

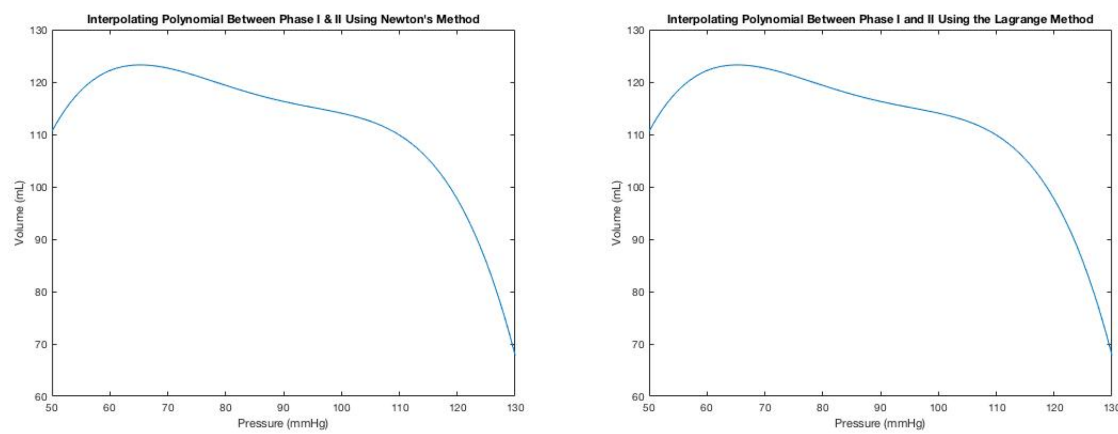

Table 7: Error obtained from performing integration to find stroke work using Trapezoid Method and Cavalieri-Simpson's Rule.

Trapezoid Method	Cavalieri-Simpson's Rule
2.28908×10^{-9}	$6.1903843 \times 10^{-19}$

Figure 4: Newton interpolating polynomial and Lagrange interpolating polynomial for finding volume at 110 mmHg between Phase II & III.

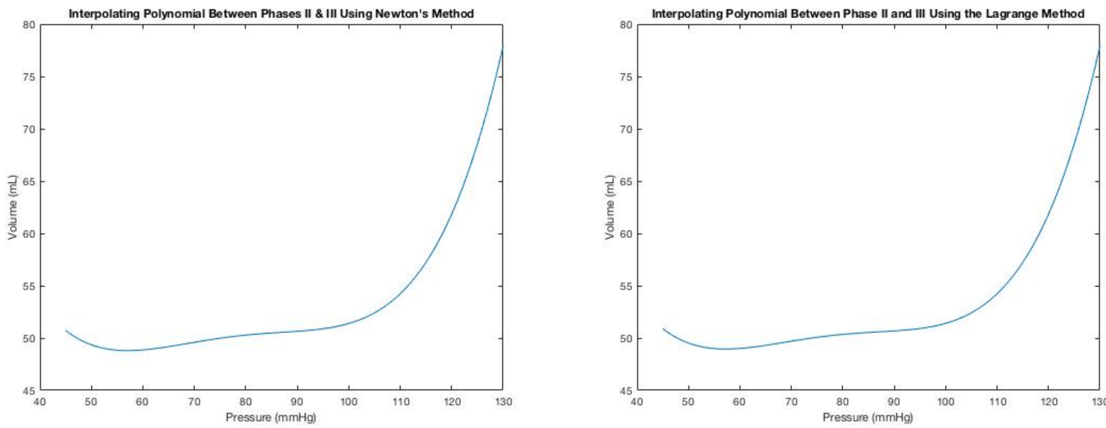

Table 3: Interpolated volumes when the pressure is 110 mmHg using Newton and Lagrange Methods.

Method	Between Phase I & II	Between Phase II & III
Newton's Method	108.3232208 mL	54.22772403 mL
Lagrange Method	109.8897733 mL	54.23017657 mL

Table 4: Error obtained from interpolating pressure in the function P(V) using Newton and Lagrange Methods.

Method	Phase II Error	Phase IV Error
Newton's Method	5.6205×10^{-14}	3.6592×10^{-18}
Lagrange Method	3.4564×10^{-10}	3.6371×10^{-12}

Table 5: Error obtained from interpolating volume in the function V(P) using Newton and Lagrange Methods.

Method	Error Between Phase I & II	Error Between Phase II & III
Newton's Method	$3.388121789017201 \times 10^{-21}$	0
Lagrange Method	3.908×10^{-13}	5.2015×10^{-7}

Table 6: Stroke work calculated using Trapezoid Method and Cavalieri-Simpson's Rule.

Trapezoid Method	Cavalieri-Simpson's Rule
4746.1	4795.5

6 Discussion/Conclusions

Upon calculating the pressure when the volume was 90 mL using both Newton and Lagrange interpolation methods, we found that the calculated pressures had an insignificant difference as seen in **Table 2** and **Table 3**. We see that the the polynomial curves for both the Newton and Lagrange interpolation methods also look very similar, as evinced in **Figure 1**, **Figure 2**, **Figure 3** and **Figure 4**.

Based on the error values displayed in **Table 4** and **Table 5**, we can see that Newton interpolation method produces a smaller error than the Lagrange interpolation method. Lagrange interpolation can be more useful compared to Newton's method when using the same data points over and over again. However, this method can have a high computational cost. Limitations to the Lagrange interpolation occur when additional data points are added or removed to improve the appearance of the curve. This was apparent when calculating the volume at a pressure of 110 mmHg. In contrast, Newton interpolation allows us to calculate the coefficients of the polynomial relatively quickly and the evaluation is much more stable because there is a single dominant term given for an x. The advantage of Newton interpolation is the use of nested multiplication and the ease of adding more data points for higher-order interpolating polynomials. We believe that the Newton interpolation method is more accurate than the Lagrange interpolation method because the error calculated upon using the Newton interpolation method is smaller than the error calculated for the Lagrange interpolation method.

The error of interpolation was much smaller than the error obtained from the integration methods. We used the same size interval for Simpson's method. Generally, the Cavalieri-Simpson rule produces a more accurate result than the trapezoid rule because the Cavalieri-Simpson rule fits three points to a parabola, whereas the trapezoid rule fits two points to a line. **Table 7** displays the errors for both methods, and we observe that Cavalieri-Simpson's Rule has a much smaller error compared to the trapezoid method. Because the Cavalieri-Simpson rule has an order of 3, it is thought to be more accurate than the trapezoid rule. Therefore, we believe the values listed in **Table 6** above for the Cavalieri-Simpson rule are more accurate the results listed for the trapezoid method.

We used an interval length size of 5. We see that the error for both the Cavalieri-Simpson method and Trapezoid method decreases as you increase the number of intervals and decrease the step size. This is because, in the case of the Simpson method, parabolas are fit in between these intervals and are used to integrate this function. The more intervals we have, the more accurate our result will be. The same principle applies in the case of the Trapezoid method, except that a line is used instead of a parabola. Furthermore, the error also tends to decrease as you reach higher order derivatives. In the case of the Cavalieri-Simpson method, polynomials of order 3 or less can be interpolated exactly, since the fourth derivative of the function becomes 0 in the error formula. The trapezoid method produces exact results for only polynomials of order 1 or less, since the second derivative of the function becomes 0 in the error formula. This further demonstrates the benefits of using the Cavalieri-Simpson method.