MTH 686 Project

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1 Model Assumptions and Least Squares Estimation

- **Model 1 :** $y(t) = \alpha_0 + \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + \varepsilon(t)$

This model assumes that the response $y(t)$ can be represented as a sum of exponential functions, each with unique coefficients α_i and exponential rates β_i. This functional form is suitable for data that exhibit growth or decay patterns. Given the non-linear nature of the model parameters, we used the **Gauss-Newton** method to iteratively estimate the optimal values of α and β.

Optimized Parameters	**Values**
α_0	1.93
α_1	2.91
α_2	-0.33
β_1	-0.45
β_2	0.000

Table 1: Optimized Parameters for Model 1

- **Model 2 :** $y(t) = \frac{\alpha_0 + \alpha_1 t}{\beta_0 + \beta_1 t} + \varepsilon(t)$

Similar to Model 1, we used the Gauss-Newton method to estimate parameters α and β because of the model's non-linear form. Convergence reached after 54 iterations.

Optimized Parameters	**Values**
α_0	-2.759543
α_1	-1.415336
β_0	-0.252336
β_1	-0.958082

Table 2: Optimized Parameters for Model 2

- **Model 3 :** $y(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 + \varepsilon(t)$

The third model is a polynomial of degree four, providing a flexible functional form that can adapt to various patterns within the data. Because this model is linear in its parameters β, we can directly compute the least squares estimates using a closed-form solution.

Optimized Parameters	**Values**
β_0	3.1426
β_1	-0.2681
β_2	0.0149
β_3	-0.0003
β_4	0.0000

Table 3: Optimized Parameters for Model 3

2 How did you find the least squares estimators? What kind of initial guesses you have chosen?

2.1 Model 1: Gauss-Newton Method

Initial Guesses for Model 1
Since Gauss-Newton requires good initial guesses for convergence:

- We set α_0 as an approximate mean of $y(t)$ (3) to match the data's general level.
- α_1 and α_2 were set to small positive values based on observed data patterns that suggested an increasing trend.
- For β_1 and β_2, we started with negative values (e.g., between -0.5 and 0) to capture moderate growth or decay patterns.

2.2 Model 2: Ordinary Least Square Formula

Initial Guesses for Model 2

- α_0 is set to 1 and α_2 to -1 so that it decays linearly.
- Both β_0 and β_1 are set to 1 for the same reason.

2.3 Model 3: Gauss-Newton Method

- This model is linear in its parameters $\beta_0, \beta_1, \ldots, \beta_4$, which allowed us to use the ordinary least squares (OLS) formula directly:

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

where X is the design matrix with columns $1, t, t^2, t^3, t^4$.

3 Find the best fit model

- To determine the best-fitting model, we compared three models based on their Residual Sum of Squares (RSS) values. Lower RSS indicates a better fit, as it signifies smaller deviations between predicted and actual values.

Models	**RSS Values**
Model 1	0.031640
Model 2	0.035911
Model 3	0.044045

Table 4: RSS values of 3 Models

- Based on the RSS values, Model 1, which is the exponential model, exhibits the lowest residual sum of squares, suggesting it provides the most accurate fit to the data.

4 Estimate of σ^2

To estimate σ^2, we typically use the Mean Squared Error (MSE), which is calculated as follows:

$$\hat{\sigma}^2 = \frac{\text{RSS}}{n - p}$$

- where :

  RSS: Residual Sum of Squares

  n: Number of observations

  p: Number of parameters in the model (including the intercept)

Models	**estimate of σ^2**
Model 1	0.00057
Model 2	0.00064
Model 3	0.00080

Table 5: Estimate of values of σ^2

5 Confidence Intervals based on the Fisher information matrix.

- 95% Confidence Intervals for the Parameters:
  - α_0: [1.512330, 1.630249]
  - α_1: [-241.191729, 249.278808]
  - β_1: [-4.149947,3.394209]
  - α_2: [-247.328068, 244.928123]
  - β_2: [-8.162164, 7.598999]