Probability and Statistics
Problem Set II

1. Are the following functions cumulative distribution function?

   (a) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$

   (b) $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x, -\infty < x < \infty.$

   (c) $F(x) = \begin{cases} 0 & \text{if } x < -5 \\ x & \text{if } -5 \leq x \leq 0.5 \\ 1 & \text{if } x > 0.5 \end{cases}$

   Hint: Use the property of cumulative distribution function
   Answer: (a) Yes (b) Yes and (c) No

2. Let X be a random variable having the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2}{3} & \text{if } 0 \leq x < 1 \\ \frac{7-6c}{6} & \text{if } 1 \leq x < 2 \\ \frac{4c^2-9c+6}{4} & \text{if } 2 \leq x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$$

   where c is a real constant.

   (a) Find the value of constant c.

   (b) Find the p.m.f of X.

   (c) Find $P(1 < X < 2), P(2 \leq X < 3), P(0 < X \leq 1), P(1 \leq X \leq 2), P(X \geq 3)$ and $P(X = 2.5)$.

   (d) Find the conditional probabilities $P(\{X = 1\}|\{1 \leq X \leq 2\}), P(\{1 \leq X < 2\}|\{X > 1\})$, and $P(\{1 \leq X \leq 2\}|\{X = 1\})$.

   Answer: (a) $\frac{1}{4}$, (b) the p.m.f is

$$p(x) = \begin{cases} \frac{2}{3} & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x = 1 \\ \frac{1}{12} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

   (c) $P(1 < X < 2) = 0, P(2 \leq X < 3) = \frac{1}{12}, P(0 < X \leq 1) = \frac{1}{4}, P(1 \leq X \leq 2) = \frac{1}{3}, P(X \geq 3) = 0$ and $P(X = 2.5) = 0$.
   (d) $P(\{X = 1\}|\{1 \leq X \leq 2\}) = \frac{3}{4}, P(\{1 \leq X < 2\}|\{X > 1\}) = 0$, and $P(\{1 \leq X \leq 2\}|\{X = 1\}) = 1$.

3. Let us select five cards at random and without replacement from an ordinary deck of playing cards.

   (a) Find the p.m.f. of X, the number of hearts in the five cards.

(b) Determine $P(\{X \leq 1\})$.

Answer: (a) the p.m.f is

$$p(x) = \begin{cases} \dfrac{\dbinom{13}{x}\dbinom{39}{5-x}}{\dbinom{52}{5}} & \text{if } x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P(\{X \leq 1\}) = p(0) + p(1)$

4. Let X be a random variable having the p.m.f.

$$p(x) = \begin{cases} \frac{c}{(2x-1)(2x+1)} & \text{if } x \in \{1, 2, 3, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

where c is a real constant.

(a) Find the value of constant c.

(b) Find the cumulative distribution function of X.

(c) For the positive integers m and n such that $m < n$, evaluate $P(X < m+1), P(X \geq m), P(m \leq X < n)$ and $P(m < X \leq n)$.

(d) Find the conditional probabilities $P(\{X > 1\}|\{1 \leq X < 4\})$ and $P(\{1 < X < 6\}|\{X \geq 3\})$.

Answer: (a) $c = 2$ (b)the c.d.f is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - \frac{1}{2[x]+1} & \text{if } x \geq 1 \end{cases}$$

(c) $P(X < m+1) = 1 - \frac{1}{2m+1}, P(X \geq m) = \frac{1}{2m-1}, P(m \leq X < n) = \frac{2(n-m)}{(2m-1)(2n-1)}$ and $P(m < X \leq n) = \frac{2(n-m)}{(2m+1)(2n+1)}$.
(d) $P(\{X > 1\}|\{1 \leq X < 4\}) = \frac{2}{9}$ and $P(\{1 < X < 6\}|\{X \geq 3\}) = \frac{6}{11}$.

5. Let X be a random variable having the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{2} & \text{if } 0 \leq x < 1 \\ \frac{x}{2} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

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(a) Show that X is of continuous type and find p.d.f of X.

(b) Find $P(1 < X < 2), P(1 \leq X < 2), P(1 < X \leq 2), P(1 \leq X \leq 2), P(X \geq 1)$ and $P(X = 1)$.

Answer: (a) X is of continuous type with p.d.f

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P(1 < X < 2) = \frac{1}{2}, P(1 \le X < 2) = \frac{1}{2}, P(1 < X \le 2) = \frac{1}{2}, P(1 \le X \le 2) = \frac{1}{2}, P(X \ge 1) = \frac{1}{2}$ and $P(X = 1) = 0$.

6. A bag contains ten balls. Among them six are red and four are white. Three balls are drawn at random and not replaced. Find the probability mass function for the number of red balls drawn.
Answer: the p.m.f is

$$p(x) = \begin{cases} \dfrac{\binom{6}{x}\binom{4}{3-x}}{\binom{10}{3}} & \text{if } x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

7. Let X be a random variable with p.d.f

$$f(x) = \begin{cases} c(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

where c is a real constant. Then

(a) Find the value of c.
(b) Find the cumulative distribution function of X.

Answer: (a) $c = 3/8$ (b)the c.d.f

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4}(3x^2 - x^3) & \text{if } 0 \le x < 2 \\ 1 & \text{if} x \ge 2 \end{cases}$$

8. Let X be a random variable with p.m.f

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}, \ r = 0, 1, 2, \cdots, n, \ 0 \le p \le 1.$$

Find the p.m.f of the random variables (a) $Y = aX + b$ and (b) $Y = X^2$.
Answer: (a) the p.m.f

$$\text{For } a = 0, \ p(y) = \begin{cases} 1 & \text{if } y = b \\ 0 & \text{if } y \ne b \end{cases}$$

$$\text{For } a \ne 0, \ p(y) = \begin{cases} \binom{n}{\frac{y-b}{a}} p^{\frac{y-b}{a}} (1-p)^{n-\frac{y-b}{a}} & \text{if } y \in \{b, a+b, 2a+b, \cdots, na+b\} \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$p(y) = \begin{cases} \binom{n}{\sqrt{y}} p^{\sqrt{y}}(1-p)^{n-\sqrt{y}} & \text{if } y \in \{0, 1, 4, \cdots, n^2\} \\ 0 & \text{otherwise} \end{cases}$$

9. A random variable X has a p.d.f $f(x)$ given by ce^{-x} in the interval $0 < x < \infty$ and zero elsewhere. Find the value of the constant c and hence calculate the probability that X lies in the interval $1 < X \leq 2$.

   Answer: $c = 1$ and $P(1 < X \leq 2) = e^{-1} - e^{-2}$.

10. Let X have range $[0, 3]$ and density $f_X(x) = kx^2$. Let $Y = X^3$.
    (a) Find k and the cumulative distribution function of X. (b) Compute $E[Y]$.
    (c) Compute $Var(Y)$.
    (d) Find the probability density function $f_Y(y)$ for Y .
    Answer: (a) $k = 1/9$, (b) 13.5, (c) 60.75 (d) $\frac{1}{27}$ on $[0, 27]$.

11. Let X be a random variable with p.m.f.

$$p(x) = \begin{cases} \frac{1}{7} & \text{if } x \in \{-2, -1, 0, 1\} \\ \frac{3}{14} & \text{if } x \in \{2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the m.g.f. of X.

   (b) Find the p.m.f. of $Z = X^2$ and its distribution function.

   Answer: (a) The m.g.f. of X is $M_X(t) = \frac{1}{7}(e^{-2t} + e^{-t} + e^t + 1) + \frac{3}{14}(e^{2t} + e^{3t})$ (b) The p.m.f. of Z is

$$p_Z(z) = \begin{cases} \frac{1}{7} & \text{if } z = 0 \\ \frac{2}{7} & \text{if } z = 1 \\ \frac{5}{14} & \text{if } z = 4 \\ \frac{3}{14} & \text{if } z = 9 \\ 0 & \text{otherwise} \end{cases}$$

   and the distribution function of Z is

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{7} & \text{if } 0 \leq z < 1 \\ \frac{3}{7} & \text{if } 1 \leq z < 4 \\ \frac{11}{14} & \text{if } 4 \leq z < 9 \\ 1 & z \geq 9 \end{cases}$$

12. Let X be a random variable with p.m.f.

$$p(x) = \begin{cases} \frac{1}{3}(\frac{2}{3})^x & \text{if } x \in \{0, 1, 2, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function of $Z = \frac{X}{X+1}$ and hence find p.m.f. of Z.
Answer: The distribution function of Z is

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 - (\frac{2}{3})^{i+1} & \text{if } \frac{i}{i+1} \leq z < \frac{i+1}{i+2}, i \in \{0, 1, 2, \cdots\} \end{cases}$$

and p.m.f. of Z is

$$f_Z(z) = \begin{cases} \frac{1}{3}(\frac{2}{3})^{\frac{z}{1-z}} & \text{if } z \in \{0, \frac{1}{2}, \frac{1}{3}, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

13. Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function $Z = X^2$ and hence find its p.d.f.
Answer: The distribution function of Z is

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 - e^{-\sqrt{z}} & \text{if } z \geq 0 \end{cases}$$

and p.d.f. of Z is

$$f_Z(z) = \begin{cases} \frac{e^{-\sqrt{z}}}{2\sqrt{z}} & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

14. Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} 6x(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function $Z = X^2(3 - 2X)$ and hence find its p.d.f.
Answer: The distribution function of Z is

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ z & \text{if } 0 \leq z < 1 \\ 1 & \text{if } z \geq 1 \end{cases}$$

and p.d.f. of Z is

$$f_Z(z) = \begin{cases} 1 & \text{if } 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

15. If m.g.f. of a random variable X is $M_X(t) = \frac{e^t - e^{-2t}}{3t}$, for $t \neq 0$, then find a p.d.f. of X.

16. Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 < x \leq 1 \\ \frac{1}{2} & \text{if } 1 < x \leq 2 \\ \frac{3-x}{2} & \text{if } 2 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of $Y = X^2 - 5X + 3$.
Answer: $-\frac{11}{6}$

17. Let X be a random variable having the p.m.f.

$$p(x) = \begin{cases} \frac{3}{\pi^2 x^2} & \text{if } x \in \{\pm 1, \pm 2, \pm 3, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

Then show that expectation of X is not finite.

18. Let X be a random variable having a p.d.f.

$$f(x) = \frac{e^{-|x|}}{2}, \text{ for } x \in \mathbb{R}$$

Then show that expectation of X is finite and find its value.
Answer: 0

19. For any random variable X having the mean μ and finite second moment, show that $E((X-\mu)^2) \leq E((X-c)^2)\ \forall\, c \in \mathbb{R}$.

20. Let X be a continuous random variable with p.d.f. $f_X(x)$ that is symmetric about $\mu \in \mathbb{R}$, i.e., $f_X(\mu + x) = f_X(\mu - x)$, for all $x \in \mathbb{R}$. If $E(X)$ finite, then show that $E(X) = \mu$.

21. Let X be a random variable such that $P(X \leq 0) = 0$ and $\mu = E(X)$ is finite. Show that $P(X \geq 2\mu) \leq 0.5$. (Use Markov inequality)

22. If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$. (Use Chebyshev inequality)
Answer: $\frac{21}{25}$

$P(|X-u| < a) > [1 - \frac{\sigma^2}{a^2}]$

Note: p.d.f - probability density function, p.m.f - probability mass function, & c.d.f - cumulative distribution function