Density Frankl–Rödl on the Sphere

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Abstract

We establish a density variant of the Frankl–Rödl theorem on the sphere $\mathbb{S}^{n-1}$, which concerns avoiding pairs of vectors with a specific distance, or equivalently, a prescribed inner product. In particular, we establish lower bounds on the probability that a randomly chosen pair of such vectors lies entirely within a measurable subset $A \subseteq \mathbb{S}^{n-1}$ of sufficiently large measure. Additionally, we prove a density version of spherical avoidance problems, which generalize from pairwise avoidance to broader configurations with prescribed pairwise inner products. Our framework encompasses a class of configurations we call *inductive configurations*, which include simplices with any prescribed inner product $-1 < r < 1$. As a consequence of our density statement, we show that all inductive configurations are sphere Ramsey.

Keywords: Frankl–Rödl, Sphere Ramsey, Sphere Avoidance, Reverse Hypercontractivity, Forbidden Angles

1 Introduction

The Frankl–Rödl theorem [FR87] is a foundational result in extremal combinatorics and theoretical computer science. It states that for any fixed $0 < \gamma < 1$, assuming $(1-\gamma)n$ is even, if a set $A \subseteq \{-1,1\}^n$ contains no pair of distinct points at Hamming distance exactly $(1-\gamma)n$, then the fractional size of A must be exponentially small. That is, there exists a constant $\epsilon = \epsilon(\gamma) < 1$ such that

$$|A|/2^n \le \epsilon^n.$$

The corresponding Frankl–Rödl graph FR_γ^n is defined on vertex set $\{-1,1\}^n$ with edges between points at Hamming distance $(1-\gamma)n$, with the Frankl-Rödl theorem bounding the independence number of this graph. This theorem has found broad applications in theoretical computer science, particularly in the analysis of hardness of approximation. The graph FR_γ^n has been used to construct integrality gap instances for problems such as 3-Coloring [KG98, KMS98, Cha02, AG11, KOTZ14] and Vertex-Cover [KG98, Cha02, ABLT06, GMT08, GMPT10, KOTZ14].

The Frankl–Rödl theorem extends naturally from pairs of points to general forbidden configurations. A configuration of k vertices is specified by a set of pairwise distances (or equivalently, inner products). Frankl and Rödl [FR87] showed that for any fixed k, any subset $A \subseteq \{-1,1\}^{4n}$ that avoids r pairwise orthogonal vectors must also have exponentially small fractional

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size. This result has been applied to show an $\Omega(\log n)$ integrality gap for the SDP relaxation of Min-Multicut [ACMM05].

While the original theorem asserts that for any $A \subseteq \{-1,1\}^n$ of size $|A|/2^n \geq \epsilon(\gamma)^n$,

$$\mathop{\mathbf{Pr}}_{\substack{x,y \in \{-1,1\}^n \\ x \cdot y = (2\gamma-1)n}} (x \in A, y \in A) > 0,$$

it is natural to ask whether one can give a quantitative lower bound on this probability in terms of the density $|A|/2^n$. Benabbas, Hatami, and Magen [BHM12] answered this by proving a density version of the Frankl–Rödl theorem: for any $0 < \alpha < 1$, if $|A|/2^n \geq \alpha$, then

$$\mathop{\mathbf{Pr}}_{\substack{x,y \in \{-1,1\}^n \\ x \cdot y = (2\gamma-1)n}} (x \in A, y \in A) \geq 2\,(\alpha/2)^{\frac{2}{1-|2\gamma-1|}} - o(1).$$

This result is based on the reverse hypercontractivity of the Bonami–Beckner semigroup [MOR$^+$06], and was used to prove integrality gaps for Vertex-Cover. Later Kauers et. al [KOTZ14] showed that the proof can be conducted in the sum-of-squares (SOS) proof system. In particular, they showed that for any $0 < \gamma \leq 1/4$, the SOS/Lasserre SDP hierarchy at degree $4\left\lceil \frac{1}{4\gamma} \right\rceil$ certifies that the maximum independent set in FR_γ^n has fractional size $o(1)$. This implies that a degree-4 algorithm from the SOS hierarchy can certify that the FR_γ^n SDP integrality gap instances for 3-Coloring have chromatic number $\omega(1)$, and that a degree-$\lceil 1/\gamma \rceil$ SOS algorithm can certify the FR_γ^n SDP integrality gap instances for Min-Vertex-Cover has minimum vertex cover $> 1 - o(1)$.

In the continuous setting, the sphere $\mathbb{S}^{n-1}$ provides a natural high-dimensional geometric analogue. This motivates the study of spherical avoidance problems: how dense can a subset $A \subseteq \mathbb{S}^{n-1}$ be while avoiding a fixed configuration of pairwise inner products? Let $\Delta_k(n,r)$ denote the k-simplex in $\mathbb{S}^{n-1}$ with pairwise inner product r. Witsenhausen's problem [Wit74] asks for the maximum density of a measurable set $A \subseteq \mathbb{S}^{n-1}$ avoiding orthogonal pairs, i.e., $\Delta_2(n,0)$. Frankl and Wilson [FW81] gave the first exponentially decreasing upper bound $\sigma(A) \leq (1+o(1))(1.13)^{-n}$, where σ denotes the uniform surface measure on $\mathbb{S}^{n-1}$. Kalai [KW09] conjectured that the extremal set consists of two opposite caps of geodesic radius $\pi/4$; this *Double Cap Conjecture* implies new lower bounds for the measurable chromatic number of $\mathbb{R}^n$ [DP16].

Regev and Klartag [RK11] established a density version of the Frankl–Rödl theorem on the sphere for pairs of orthogonal vectors:

$$\mathop{\mathbf{Pr}}_{\substack{x,y \in \mathbb{S}^{n-1} \\ x \cdot y = 0}} (x \in A,\ y \in A) \geq 0.9\,\sigma(A)^2,$$

valid for any measurable set $A \subseteq \mathbb{S}^{n-1}$ with $\sigma(A) \geq C\exp(-cn^{1/3})$, where C and c are universal constants. This result was a key component in their proof of an $\Omega(n^{1/3})$ lower bound on the classical communication complexity of the Vector in Subspace Problem (VSP). As a major consequence, they resolved a long-standing open question posed by Raz [Raz99], demonstrating that quantum one-way communication is indeed exponentially stronger than classical two-way communication.

For configurations $\Delta_k(n,r)$ with $r > 0$, Castro-Silva et al. [CSdOFSV22] proved that for any $k \geq 2$, there exists $c = c(k,r) < 1$ such that any $A \subseteq \mathbb{S}^{n-1}$ avoiding $\Delta_k(n,r)$ satisfies $\sigma(A) \leq (c+o(1))^n$. A configuration is sphere Ramsey if any c-coloring of of the sphere, there exists a monochromatic congruent copy of the configuration. Matoušek and Rödl [MR95] showed that any configuration P with circumradius less than 1 is sphere Ramsey.

This sphere Ramsey theorem was recently used by Brakensiek, Guruswami, and Sandeep [BGS23] to demonstrate integrality gaps for the basic SDP relaxation of certain promise CSPs: namely Boolean symmetric promise CSPs defined by a single predicate pair that lack Majority or Alternate-Threshold (AT) polymorphisms of all odd arities. Via Raghavendra's general connection tightly linking SDP integrality gaps to Unique-Games hardness [Rag08], this enabled BGS to conclude that such promise CSPs do not admit a robust satisfiability algorithm (in the sense of Zwick [Zwi98]) under the Unique Games conjecture. Complemeting this hardness result, BGS gave a robust satisfiability algorithm for Boolean promise CSPs that admit Majority polymorphisms of all odd arities or AT polymorphisms of all odd arities—the algorithm applied in the general promise CSP setting that could have multiple predicate pairs. Together these results led to a dichotomy theorem with respect to robust satisfiability for Boolean symmetric promise CSPs defined by a single predicate. Towards extending their hardness result to Boolean symmetric PCSPs that could include multiple predicate pairs, BGS posed the following problem: can one show a density version of spherical avoidance problems for the configuration $(x_1, \cdots, x_b, -x_{b+1}, \cdots, -x_k)$ where $(x_1, \cdots, x_b, x_{b+1}, \cdots, x_k) \in \Delta_k(n, r)$. Namely, obtain a non-trivial lower bound on the quantity

$$\mathop{\mathbf{Pr}}_{(x_1,\cdots,x_b,x_{b+1},x_k)\in\Delta_k(n,r)} \left(x_1 \in A, \cdots, x_b \in A, -x_{b+1} \in A, \cdots, -x_k \in A\right).$$

in terms of $\sigma(A)$. While we do not directly resolve the exact configuration posed by BGS, our result applies to a broad class of configurations that includes closely related structures.

In this paper, we show the following result:

Theorem 1.1. *Fix k, r such that $-\frac{1}{k-1} < r < 1$, there exists constant $C = C(k, r)$, $\epsilon = \epsilon(k, r)$ such that*

$$\mathop{\mathbf{Pr}}_{x_1,\ldots,x_k\in\Delta_k(n,r)}(x_i \in A \ \forall i) \geq \Omega_{k,r}\left(\sigma(A)^C\right)$$

for all measurable $A \subseteq \mathbb{S}^{n-1}$ with $\sigma(A) \geq \omega_{k,r}(n^{-\epsilon})$.

Our result generalizes the work of Castro-Silva et al. to simplices with any $-\frac{1}{k-1} < r < 1$, and more broadly, to a class of *inductive configurations* defined in Section 4 (see Theorem 4.6). As a consequence, we show that all inductive configurations are sphere Ramsey.

Inspired by the techniques of Benabbas et al. [BHM12], who used reverse hypercontractivity of the Bonami–Beckner semigroup to show the density variant of Frankl–Rödl on $\{-1, 1\}^n$, we develop analogous tools on the sphere. Specifically, we prove reverse hypercontractivity of the operator

$$A_t f(x) := \mathop{\mathbb{E}}_{\substack{y\in\mathbb{S}^{n-1}\\ x\cdot y=e^{-t}}}[f(y)],$$

which enables us to derive density versions of spherical avoidance for inductive configurations. In Section 2, we analyze the eigen-decomposition of A_t; in Section 3, we relate A_t to the Poisson Markov semigroup P_t and establish reverse hypercontractive inequalities. Finally, in Section 4, we prove our main results on density versions of the Frankl–Rödl theorem on the sphere and density spherical Ramsey statements for inductive configurations.

2 Eigen-decomposition via Spherical Harmonics

In order to analyze the eigen-decomposition of A_t, we first introduce some fundamental notions from the theory of spherical harmonics. This decomposition of A_t plays a central role in our analysis of reverse-hypercontractive. For a comprehensive background, we refer to the standard references [Mül06, SW71] and notes such as [Gal09].

2.1 Spherical Harmonics

For each integer $n \geq 2$, let $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ denote the unit n-sphere, defined by

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}.$$

Given $p > 0$ and a measurable function $f : \mathbb{S}^{n-1} \to \mathbb{R}$, we define its L^p norm by

$$\|f\|_{L^p(\mathbb{S}^{n-1})} := \left(\int_{\mathbb{S}^{n-1}} |f(x)|^p \, d\sigma(x) \right)^{1/p},$$

where σ denotes the uniform surface measure on $\mathbb{S}^{n-1}$, normalized such that $\int_{\mathbb{S}^{n-1}} d\sigma(x) = 1$. It follows that for $p \geq q > 0$, we have $L^p(\mathbb{S}^{n-1}) \subseteq L^q(\mathbb{S}^{n-1})$.

We focus in particular on $L^2(\mathbb{S}^{n-1})$, the space of square-integrable functions. It admits an orthonormal basis consisting of spherical harmonics $\{S_{k,\ell}\}$, indexed by integers $k \geq 0$ (the degree) and $1 \leq \ell \leq c_{k,n}$, where

$$c_{k,n} = \binom{k+n-2}{k} + \binom{k+n-3}{k-1}.$$

Each $S_{k,\ell}$ is the restriction to $\mathbb{S}^{n-1}$ of a harmonic, homogeneous polynomial of degree k on $\mathbb{R}^n$. We denote by $\mathcal{S}_k \subseteq L^2(\mathbb{S}^{n-1})$ the finite-dimensional subspace spanned by spherical harmonics of degree k:

$$\mathcal{S}_k := \operatorname{span}\{S_{k,1}, \ldots, S_{k,c_{k,n}}\}.$$

Notably, $\mathcal{S}_0$ consists of the constant functions. Rotations $U \in \mathrm{SO}(n)$ act on $f \in L^2(\mathbb{S}^{n-1})$ via

$$(Uf)(x) := f(U^{-1}x).$$

Each space $\mathcal{S}_k$ is invariant under the action of $\mathrm{SO}(n)$, and furnishes an irreducible representation, i.e. any subspace $V \subseteq \mathcal{S}_k$ that is invariant under rotations must either be $\{0\}$ or $\mathcal{S}_k$. Moreover, for $n \geq 3$, the spaces $\mathcal{S}_k$ for different k have different dimensions therefore are inequivalent as representations. Throughout the remainder of this paper, we assume $n \geq 3$.

For $t \geq 0$, define an operator $A_t : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ by

$$(A_t f)(x) := \int_{\mathbb{S}^{n-1}} f(y) \, d\sigma_{x,e^{-t}}(y) = \mathop{\mathbb{E}}_{\substack{y \in \mathbb{S}^{n-1} \\ x \cdot y = e^{-t}}} [f(y)],$$

where $\sigma_{x,r}$ is the uniform probability measure on the $(n-2)$-subsphere $S_{x,r} := \{y \in \mathbb{S}^{n-1} : x \cdot y = r\}$.

The operator A_t is easily seen to commute with rotations via

$$(A_t U f)(x) = \mathop{\mathbb{E}}_{\substack{y \in \mathbb{S}^{n-1} \\ x \cdot y = e^{-t}}} [f(U^{-1}y)] = \mathop{\mathbb{E}}_{\substack{z \in \mathbb{S}^{n-1} \\ x \cdot Uz = e^{-t}}} [f(z)] = \mathop{\mathbb{E}}_{\substack{z \in \mathbb{S}^{n-1} \\ U^{-1}x \cdot z = e^{-t}}} [f(z)] = (U A_t f)(x)$$

for any $U \in \mathrm{SO}(n)$. It is also self-adjoint by

$$\langle A_t f, g\rangle = \mathop{\mathbb{E}}_{x\in\mathbb{S}^{n-1}}[g(x)A_t f(x)] = \mathop{\mathbb{E}}_{x\in\mathbb{S}^{n-1}}\left[\mathop{\mathbb{E}}_{y:x\cdot y=e^{-t}}[g(x)f(y)]\right] = \mathop{\mathbb{E}}_{y\in\mathbb{S}^{n-1}}\left[\mathop{\mathbb{E}}_{x:x\cdot y=e^{-t}}[g(x)f(y)]\right] = \langle f, A_t g\rangle .$$

for all $f, g \in L^2(\mathbb{S}^{n-1})$.

Using standard techniques from representation theory, we establish that the spaces $\mathcal{S}_k$ are eigenspaces of A_t:

Lemma 2.1. *For any $k \geq 0$ and $Y_k \in \mathcal{S}_k$, we have*

$$A_t(Y_k) = \mu_{k,t} Y_k,$$

where $\mu_{k,t} = G_k(e^{-t})$, with $G_k : [-1,1] \to \mathbb{R}$ denoting the degree-k Gegenbauer polynomial (see,e.g., [Mül06]), given by

$$G_k(r) = \mathbb{E}\left[\left(r + iX_1\sqrt{1-r^2}\right)^k\right],$$

where $X = (X_1, \ldots, X_{n-1})$ is uniformly distributed on $\mathbb{S}^{n-2}$ and X_1 denotes any fixed coordinate.

Proof. By Schur's Lemma (see e.g. [S+77]), any linear operator commuting with the group action must act as a scalar on each irreducible representation. Since A_t commutes with rotations and $\mathcal{S}_k$ are inequivalent irreducible representations for different k, the restriction of A_t to $\mathcal{S}_k$ must be a scalar multiple of the identity.

To identify the eigenvalue, consider the function $f_k(x) := G_k(x \cdot v)$ for some fixed $v \in \mathbb{S}^{n-1}$. It is known that $f_k \in \mathcal{S}_k$ [Mül06]. Then

$$(A_t f_k)(v) = G_k(e^{-t}), \quad \text{and} \quad f_k(v) = G_k(1) = 1,$$

showing that $\mu_{k,t} = G_k(e^{-t})$. ∎

2.2 Eigenvalue Estimates

We now estimate the eigenvalues $\mu_{k,t}$.

Lemma 2.2. *The eigenvalues $\mu_{k,t}$ satisfy*

$$|\mu_{k,t} - e^{-kt}| = O_t\left(\frac{1}{n}\right),$$

as $n \to \infty$, where the implicit constant depends only on t but is independent of k.

Proof. Let us first analyze the k-th moments of X_1. Notice that the density of X_1 is supported in $[-1,1]$ and proportional to $(1-x^2)^{(n-4)/2}$. Since the density X_1 is an odd function, the odd moments vanishes, $\mathbb{E}[X_1^k] = 0$ for odd k. For even k, there is an estimate

$$\mathbb{E}[X_1^k] \leq \left(\frac{k}{n-4}\right)^{k/2}$$

given by [RK11] Lemma 5.5. We will estimate the eigenvalues $\mu_{k,t}$ for different regimes of k.

When $k \leq \frac{n}{10}$, a direct computation gives

$$\begin{aligned}
\mu_{k,t} &= G_k(e^{-t}) \\
&= \mathbb{E}\left[\left(e^{-t} + iX_1\sqrt{1-e^{-2t}}\right)^k\right] \\
&= \sum_{a=0}^{k}\left(i\sqrt{1-e^{-2t}}\right)^a e^{-t(k-a)}E[X_1^a] \\
&= \sum_{a=0}^{\lfloor k/2\rfloor}(-1)^a(1-e^{-2t})^a e^{-t(k-2a)}E[X_1^{2a}] \\
&= e^{-kt} + \sum_{a=1}^{\lfloor k/2\rfloor}(-1)^a(1-e^{-2t})^a e^{-t(k-2a)}E[X_1^{2a}]\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
|\mu_{k,t} - e^{-kt}| &\leq \sum_{a=1}^{\lfloor k/2\rfloor}(1-e^{-2t})^a e^{-t(k-2a)}\left(\frac{2a}{n-4}\right)^a \\
&\leq \frac{2}{n-4} + \sum_{a=2}^{\lfloor k/2\rfloor}(1-e^{-2t})^a e^{-t(k-2a)}\left(\frac{2a}{n-4}\right)^a \\
&\leq \frac{2}{n-4} + \sum_{a=2}^{\lfloor k/2\rfloor}\left(\frac{2a}{n-4}\right)^a
\end{aligned}$$

Since $\left(\frac{2a}{n-4}\right)^a$ is decreasing with a when $2 \leq a \leq \lfloor k/2\rfloor \leq \frac{n}{20}$, we can bound

$$|\mu_{k,t} - e^{-kt}| \leq \frac{2}{n-4} + (\lfloor k/2\rfloor - 2)\left(\frac{4}{n-4}\right)^2 \leq \frac{2}{n-4} + \frac{n}{20}\left(\frac{4}{n-4}\right)^2 \leq O\left(\frac{1}{n}\right).$$

For the case where $k \geq \frac{n}{10}$, by Markov's inequality

$$\mathbf{Pr}(|X_1| \geq \frac{1}{2}) \leq \frac{\mathbb{E}\left[|X_1|^{2\lceil n/20\rceil}\right]}{(1/2)^{2\lceil n/20\rceil}} \leq \left(\frac{4\lceil n/20\rceil}{n-4}\right)^{n/20} \leq O\left(\frac{1}{n}\right)$$

which leads us to

$$\begin{aligned}
&|\mu_{k,t} - e^{-kt}| \\
&\leq |G_k(e^{-t})| + |e^{-kt}| \\
&\leq \mathbb{E}\left[\left|e^{-t} + iX_1\sqrt{1-e^{-2t}}\right|^k\right] + |e^{-kt}| \\
&\leq \mathbb{E}\left[\left(e^{-2t} + (1-e^{-2t})|X_1|\right)^k\right] + |e^{-kt}| \\
&\leq \mathbf{Pr}(|X_1| < \frac{1}{2})\left(e^{-2t} + \frac{1}{2}(1-e^{-2t})\right)^k + \mathbf{Pr}(|X_1| \geq \frac{1}{2}) + |e^{-\frac{n}{10}t}| \\
&\leq O\left(\left(\frac{1+e^{-2t}}{2}\right)^{n/10}\right) + O(\frac{1}{n}) + O_t(\frac{1}{n}) \\
&\leq O_t(\frac{1}{n})
\end{aligned}$$

as $n \to \infty$. $\square$

3 Reverse Hypercontractivity of A_t

Hypercontractive inequalities play a central role in modern analysis, probability, and theoretical computer science, quantifying how semigroups "smooth out" irregularities over time by contracting L^q norms to L^p norms. A semigroup $(T_t)_{t\geq 0}$ is hypercontractive if for all real-valued functions f,

$$\|T_t f\|_p \leq \|f\|_q,$$

for $1 < q < p < \infty$, whenever $t \geq \tau(p, q)$. These inequalities are deeply connected to logarithmic Sobolev inequalities [Gro75] and underlie results on quantum information theory [Mon12], diffusion processes [BÉ06], and mixing times [BT03].

On the Boolean hypercube $\{0,1\}^n$, a canonical example is the *Bonami–Beckner semigroup* T_ρ, where $0 \leq \rho \leq 1$, defined as

$$T_\rho f(x) = \mathbb{E}[f(y)], \quad \text{where } y \sim_\rho x,$$

and y is obtained by flipping each bit of x independently with probability $\frac{1-\rho}{2}$. The operator satisfies the sharp hypercontractive inequality [Bon70, Nel73, Bec75, Gro75]:

$$\|T_\rho f\|_p \leq \|f\|_q \quad \text{if } \rho \leq \sqrt{\frac{q-1}{p-1}}$$

for $1 < q < p < \infty$. This inequality underpins many foundational results in the analysis of Boolean functions, such as the KKL theorem [MOS13], the invariance principle [MOS13].

In contrast, reverse hypercontractivity, introduced by Mossel, Oleszkiewicz, and Sen [MOS13], captures a complementary "anti-smoothing" phenomenon. A semigroup $(T_t)_{t\geq 0}$ is reverse hypercontractive if for all non-negative functions f,

$$\|T_t f\|_q \geq \|f\|_p,$$

for $0 < q < p < 1$, whenever $t \geq \tau(p, q)$. This inequality lower-bounds the dispersion of mass under the semigroup and connects to reverse log-Sobolev inequalities [MOS13], Gaussian isoperimetry [MN15], and tail bounds in correlated settings.

On the Boolean hypercube, the Bonami–Beckner semigroup T_ρ also satisfies reverse hypercontractive inequalities [MOS13]: for all non-negative $f : \{0,1\}^n \to \mathbb{R}_{\geq 0}$,

$$\|T_\rho f\|_q \geq \|f\|_p \quad \text{if } \rho < \sqrt{\frac{1-p}{1-q}}$$

for all $0 < q < p < 1$.

Reverse hypercontractivity has enabled several significant applications on the hypercube. It was used to prove the "It Ain't Over Till It's Over" conjecture [Kho02] from social choice theory [MOO05], and dimension-free Gaussian isoperimetric inequalities [MN15].

The operator A_t serves as the analogue of the Bonami–Beckner operator on the sphere. In this section, we demonstrate that A_t is close to the Poisson Markov semigroup P_t in the L^2 norm. Using the logarithmic Sobolev inequality for P_t, we then establish that P_t satisfies reverse hypercontractivity inequalities, which in turn imply reverse hypercontractivity inequalities for A_t.

3.1 Poisson Markov Semigroup

Definition 3.1. *A family* $(P_t)_{t\geq 0}$ *of operators on real-valued measurable functions on* $\mathbb{S}^{n-1}$ *is called a* Markov semigroup *if it satisfies the following properties:*

(i) For each $t \geq 0$*,* P_t *is a linear operator mapping bounded measurable functions to bounded measurable functions.*

(ii) $P_t(1) = 1$*, where* 1 *denotes the constant function on* $\mathbb{S}^{n-1}$ *(mass conservation).*

(iii) If $f \geq 0$*, then* $P_t f \geq 0$ *(positivity preservation).*

(iv) $P_0 = \mathrm{Id}$*, the identity operator.*

(v) For any $s, t \geq 0$*,* $P_{s+t} = P_s \circ P_t$ *(semigroup property).*

(vi) For each f*, the map* $t \mapsto P_t f$ *is continuous.*

Operators satisfying properties (i)–(iii) are called *Markov operators.* For any function $f \in L^2(\mathbb{S}^{n-1})$, we can express it in terms of the spherical harmonic basis as

$$f(x) = \sum_{k,l} \widehat{f_{k,l}} Y_{k,l}(x),$$

where $\widehat{f_{k,l}} = \langle f, Y_{k,l} \rangle$. Define the *Poisson semigroup*[1] $(P_t)_{t\geq 0} : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ by

$$P_t f(x) = \sum_{k,l} e^{-kt} \widehat{f_{k,l}} Y_{k,l}(x).$$

As shown in [Bec92], this can equivalently be written in terms of the Poisson kernel

$$K_r(x, y) = \frac{1 - r^2}{|x - ry|^n}, \quad (-1 \leq r \leq 1),$$

so that

$$P_t f(x) = \int_{\mathbb{S}^{n-1}} K_{e^{-t}}(x, y) f(y)\, d\sigma(y).$$

It is straightforward to verify that P_t satisfies the Markov semigroup properties.

We now show that A_t is close to P_t in the L^2-norm.

Lemma 3.2. *For any* $f \in L^2(\mathbb{S}^{n-1})$,

$$\|A_t f - P_t f\|_2 = O_t(n^{-2}) \|f\|_2$$

as $n \to \infty$.

[1] In the literature, e.g., [Bec92], the Poisson semigroup is typically defined by $P_r f(x) = \sum_{k,l} r^k \widehat{f_{k,l}} Y_{k,l}(x)$ with multiplicative semigroup property $P_{rs} = P_r \circ P_s$. We adopt a slightly modified definition here to match the additive semigroup convention of Markov semigroups.

Proof. Expressing f in the spherical harmonic basis as $f(x) = \sum_{k,l} \widehat{f_{k,l}} Y_{k,l}(x)$, we have

$$\|A_t f - P_t f\|_2 = \left\| \sum_{k,l} (\mu_{k,t} - e^{-kt}) \widehat{f_{k,l}} Y_{k,l} \right\|_2 = \sum_{k,l} (\mu_{k,t} - e^{-kt})^2 \left(\widehat{f_{k,l}}\right)^2 .$$

By Lemma 2.2,

$$(\mu_{k,t} - e^{-kt})^2 \leq O_t(n^{-2})$$

where we can conclude

$$\|A_t f - P_t f\|_2^2 \leq O_t(n^{-2}) \sum_{k,l} \left(\widehat{f_{k,l}}\right)^2 = O_t(n^{-2}) \|f\|_2^2 .$$ □

3.2 Reverse Hypercontractivity

Mossel, Oleszkiewicz, and Sen [MOS13] showed that for Markov semigroups, reverse hypercontractivity inequalities follow from log-Sobolev inequalities. We will use this fact to establish reverse hypercontractivity for the Poisson semigroup P_t. For any positive function $f > 0$, define its entropy by

$$\mathbf{Ent}(f) = \mathbb{E}[f \log f] - \mathbb{E}[f] \cdot \log \mathbb{E}[f].$$

For a Markov semigroup $(P_t)_{t \geq 0}$ with generator L, the Dirichlet form is defined as

$$\mathcal{E}(f, g) = \mathbb{E}[f L g] = \mathbb{E}[g L f] = \mathcal{E}(g, f) = - \frac{d}{dt} \mathbb{E}[f P_t g] \Big|_{t=0} .$$

Lemma 3.3 ([Bec92], Theorem 1)**.** *The Poisson semigroup $(P_t)_{t \geq 0}$ satisfies the log-Sobolev inequality:*

$$\mathbf{Ent}(f^2) \leq C \mathcal{E}(f, f),$$

for some constant $C \leq \frac{1}{2}$.

Lemma 3.4 ([MOS13], Theorem 1.10)**.** *If a Markov semigroup $(P_t)_{t \geq 0}$ satisfies the log-Sobolev inequality with constant C, then for all $q < p < 1$ and every positive function $f : \mathbb{S}^{n-1} \to \mathbb{R}$, the following inequality holds for all $t \geq \frac{C}{4} \log \frac{1-q}{1-p}$:*

$$\|P_t f\|_q \geq \|f\|_p.$$

We now deduce a reverse hypercontractivity inequality for the operator A_t, leveraging its proximity to P_t in the L^2 norm.

Theorem 3.5. *For any positive functions $f, g \in L^2(\mathbb{S}^{n-1})$, the following reverse hypercontractivity inequality holds:*

$$\mathop{\mathbb{E}}_{x \cdot y = e^{-t}} [f(x) g(y)] = \langle f, A_t g \rangle \geq \|f\|_p \|g\|_p - O_t(n^{-2}) \|f\|_2 \|g\|_2,$$

as $n \to \infty$, for all $0 < p \leq 1 - e^{-4t}$.

Proof. By Lemma 3.2, we have

$$\mathop{\mathbb{E}}_{x\cdot y=e^{-t}}[f(x)g(x)] = \langle f, A_t g\rangle \geq \langle f, P_t g\rangle - O_t(n^{-2})\|f\|_2\|g\|_2.$$

Let $p' = \frac{p}{p-1}$ be the Hölder conjugate of p, so that $1/p + 1/p' = 1$. Applying the reverse Hölder inequality (see [HLP52], Theorem 13), we obtain

$$\langle f, P_t g\rangle = \|fP_t g\|_1 \geq \|f\|_p \|P_t g\|_{p'}.$$

From Lemma 3.3, the semigroup $(P_t)_{t\geq 0}$ satisfies a log-Sobolev inequality with constant $C \leq \frac{1}{2}$. Thus, by Lemma 3.4,

$$\|P_t g\|_{p'} \geq \|g\|_p$$

provided that $t \geq \frac{1}{8}\log\frac{1-p'}{1-p} = -\frac{1}{4}\log(1-p)$. Therefore, the inequality holds for all $p \leq 1 - e^{-4t}$, completing the proof. □

4 Density Sphere Avoidance

In this section, we present our main results: the density version of the Frankl–Rödl theorem on the sphere (Theorem 4.3) and the density sphere avoidance result for inductive configurations (Theorem 4.6).

4.1 Inductive Configurations

Let $(v_1, \cdots, v_k)$ be a configuration on $\mathbb{S}^{n-1}$, where $v_i \in \mathbb{S}^{n-1}$ are distinct vectors satisfying $\langle v_i, v_j\rangle = r_i$ for $i < j$. Specifically, writing v_i as column vectors, we consider configurations whose covariance matrix has the form

$$R(r_1, \cdots, r_{k-1}) := (v_1, \cdots, v_k)^T (v_1, \cdots, v_k) = \begin{pmatrix} 1 & r_1 & r_1 & \cdots & r_1 & r_1 \\ r_1 & 1 & r_2 & \cdots & r_2 & r_2 \\ r_1 & r_2 & 1 & \cdots & r_3 & r_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1 & r_2 & r_3 & \cdots & 1 & r_{k-1} \\ r_1 & r_2 & r_3 & \cdots & r_{k-1} & 1 \end{pmatrix}.$$

Let $\Delta(n, R) = \{(x_1, \cdots, x_k) : (x_1, \cdots, x_k)^T (x_1, \cdots, x_k) = R\} \subseteq (\mathbb{S}^{n-1})^k$ denote the set of tuples that are congruent to the configuration $(v_1, \cdots, v_k)$ defined by R. We refer to such configurations as *inductive configurations*, as they can be constructed recursively as follows:

Start with any vector v_1 and a choice of $-1 < r_1 < 1$. Choose v_2 on the $(n-2)$-dimensional subsphere $S_{v_1,r_1} = \{x \in \mathbb{S}^{n-1} : \langle x, v_1\rangle = r_1\}$. Next, select r_2 such that the intersection $S_{v_1,r_1} \cap S_{v_2,r_2}$ is nonempty, and choose v_3 on this $(n-3)$-dimensional subsphere. Proceed inductively: for each v_i, choose it on the $(n-i)$-dimensional subsphere $\bigcap_{j<i} S_{v_j,r_j}$.

Throughout the paper, we exclude a special case of inductive configuration–configurations where the length of $v_k - v_{k-1}$ is the diameter of $\bigcap_{j<k-1} S_{v_j,r_j}$—for reasons that will become clear in the later proof. Notably, both the length of $v_k - v_{k-1}$ and the diameter of $\bigcap_{j<k-1} S_{v_j,r_j}$ are independent

of the ambient dimension n, so this is a constraint on the configuration R itself.

We will show that, fixing any inductive configuration R (excluding the special case), any set $A \subseteq \mathbb{S}^{n-1}$ of constant spherical measure will, as $n \to \infty$, contain many congruent copies of R. Moreover, we provide an explicit lower bound on the probability that a randomly chosen congruent copy of R lies entirely within A.

4.2 Configuration with Pairwise Orthogonal Vectors

Let $\Delta_k(n,0) := \Delta(n, I_k) = \{(x_1, \cdots, x_k) \in (\mathbb{S}^{n-1})^k : \langle v_i, v_j \rangle = 0, \ \forall i \neq j\}$ denote the set of all k-tuples of pairwise orthogonal vectors in $\mathbb{S}^{n-1}$. Recall that $S_{x,r}$ denotes the $(n-2)$-subsphere $\{y \in \mathbb{S}^{n-1} : x \cdot y = r\}$ and $\sigma_{x,r}$ is the uniform probability measure on $S_{x,r}$.

Lemma 4.1 ([RK11], Theorem 2.2)**.** *For any measurable set $A \subseteq \mathbb{S}^{n-1}$, the condition*

$$\left| \frac{\sigma_{x,0}(A \cap S_{x,0})}{\sigma(A)} \right| \leq 0.1$$

holds for all $x \in \mathbb{S}^{n-1}$, except on a subset of measure at most $C \exp(-cn^{1/3})$, where $C, c > 0$ are universal constants.

Theorem 4.2. *Fix an integer $k \geq 2$. For any measurable set $A \subseteq \mathbb{S}^{n-1}$, the probability that a tuple $(v_1, \cdots, v_k)$ drawn uniformly at random from $\Delta_k(n,0)$ lies entirely in A is at least*

$$\mathop{\mathbf{Pr}}_{(x_1,\cdots,x_k)\in\Delta_k(n,0)}(x_1 \in A, \cdots, x_k \in A) \geq \Omega_k \left(\sigma(A)^k - C_k \exp(-cn^{1/3}) \sigma(A)^{k-1} \right),$$

where $c > 0$ is a universal constant and C_k is a constant depending only on k.

Proof. We proceed by induction on k. Let $B \subseteq \mathbb{S}^{n-1}$ denote the set of all x_1 for which

$$\left| \frac{\sigma_{x_1,0}(A \cap S_{x_1,0})}{\sigma(A)} \right| \leq 0.1.$$

By Lemma 4.1, we have $\sigma(B) \geq 1 - C \exp(-cn^{1/3})$, where $C, c > 0$ are universal constants.

When $k = 2$,

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{(x_1,x_2)\in\Delta_2(n,0)}(x_1 \in A, x_2 \in A) \\
\geq &\mathop{\mathbf{Pr}}_{(x_1,x_2)\in\Delta_2(n,0)}(x_1 \in A \cap B, x_2 \in A) \\
= &\ \mathbf{Pr}(x_1 \in A \cap B) \cdot \mathbf{Pr}(x_2 \in A \mid x_1 \in A \cap B) \\
= &\ \mathbf{Pr}(x_1 \in A \cap B) \cdot \mathbb{E}[\sigma_{x_1,0}(A \cap S_{x_1,0}) \mid x_1 \in A \cap B] \\
\geq &\left(\sigma(A) - C \exp(-cn^{1/3}) \right) \cdot 0.9\sigma(A) \\
\geq &\ \Omega \left(\sigma(A)^2 - C \exp(-cn^{1/3}) \sigma(A) \right)
\end{aligned}$$

When $k > 2$, assume the claim holds for $k-1$. Then:

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{(x_1,\cdots,x_k)\in\Delta_k(n,0)}(x_1 \in A, \cdots, x_k \in A) \\
\geq &\ \mathbf{Pr}(x_1 \in A \cap B) \cdot \mathbf{Pr}(x_2, \cdots, x_k \in A \mid x_1 \in A \cap B) \\
= &\ \mathbf{Pr}(x_1 \in A \cap B) \cdot \mathbf{Pr}(x_2, \cdots, x_k \in A \cap S_{x_1,0} \mid x_1 \in A \cap B).
\end{aligned}$$

By the inductive hypothesis applied within the subsphere $S_{x_1,0}$, we have:

$$\begin{aligned}
&\mathbf{Pr}(x_2, \cdots, x_k \in A \cap S_{x_1,0} \mid x_1 \in A \cap B) \\
&\geq \Omega_k \left((0.9\sigma(A))^{k-1} - C_k \exp(-cn^{1/3})(0.9\sigma(A))^{k-2} \right) \\
&\geq \Omega_k \left(\sigma(A)^{k-1} - C_k \exp(-cn^{1/3})\sigma(A)^{k-2} \right).
\end{aligned}$$

Multiplying with $\mathbf{Pr}(x_1 \in A \cap B) \geq \sigma(A) - C \exp(-cn^{1/3})$, we obtain:

$$\begin{aligned}
&\mathbf{Pr}(x_1, \cdots, x_k \in A) \\
&\geq \Omega_k \left(\sigma(A)^{k-1} - C_k \exp(-cn^{1/3})\sigma(A)^{k-2} \right) \cdot \left(\sigma(A) - C \exp(-cn^{1/3}) \right) \\
&\geq \Omega_k \left(\sigma(A)^{k} - C'_k \exp(-cn^{1/3})\sigma(A)^{k-1} \right),
\end{aligned}$$

completing the induction. □

4.3 Main Results

We first prove a two-set version of the density sphere Frankl-Rödl theorem which will be later used for induction on inductive configurations.

Theorem 4.3 (Density Sphere Frankl-Rödl)**.** *For any measurable sets $A, B \subseteq \mathbb{S}^{n-1}$ and any $r \in (-1,1)$, we have*

$$\mathop{\mathbf{Pr}}_{x \cdot y = r}(x \in A,\ y \in B) \geq (\sigma(A)\sigma(B))^{1/(1-r^4)} - O_r(n^{-2})\sqrt{\sigma(A)\sigma(B)}.$$

Proof. The case $r = 0$ follows from a stronger version of Lemma 4.1, given by [RK11, Theorem 5.1]. Now consider $r \neq 0$. Let

$$f = \mathbb{1}_A, \quad \text{and} \quad g(x) = \begin{cases} \mathbb{1}_B(x) & \text{if } r > 0, \\ \mathbb{1}_B(-x) & \text{if } r < 0. \end{cases}$$

Applying Theorem 3.5 with $p = 1 - r^4$, we obtain

$$\begin{aligned}
\mathop{\mathbf{Pr}}_{x \cdot y = r}(x \in A,\ y \in B) &= \mathop{\mathbb{E}}_{x \cdot y = |r|}[f(x)g(y)] \\
&\geq \|f\|_p \|g\|_p - O_r(n^{-2})\|f\|_2 \|g\|_2 \\
&= (\sigma(A)\sigma(B))^{1/(1-r^4)} - O_r(n^{-2})\sqrt{\sigma(A)\sigma(B)}\ .
\end{aligned}$$

□

Fix an inductive configuration R and let $A \subseteq \mathbb{S}^{n-1}$ be measurable. For any $x \in \mathbb{S}^{n-1}$, recall that $\sigma_{x,r}$ denotes the uniform measure on $S_{x,r}$. We say that a vector $x \in A$ is *good* if

$$\sigma_{x,r}(A \cap S_{x,r}) \geq \frac{1}{2}\sigma(A)^{(1+r^4)/(1-r^4)}.$$

Define the set of good vectors:

$$A_{\text{good}} = \left\{ x \in \mathbb{S}^{n-1} : \sigma_{x,r}(A \cap S_{x,r}) \geq \frac{1}{2}\sigma(A)^{(1+r^4)/(1-r^4)} \right\}.$$

Using Theorem 4.3, we can show that many such good vectors must exist.

Proposition 4.4. *The set of good vectors satisfies*

$$\sigma(A_{\text{good}}) \geq \frac{1}{2}\sigma(A)^{2/(1-r^4)} - O_r(n^{-2})\sigma(A).$$

Proof. By Theorem 4.3, we have

$$\mathop{\mathbb{E}}_{x}[\sigma_{x,r}(A \cap S_{x,r}) \mid x \in A] = \mathop{\mathbf{Pr}}_{x \cdot y = r}(y \in A \mid x \in A) \geq \sigma(A)^{(1+r^4)/(1-r^4)} - O_r(n^{-2}).$$

Split the expectation by conditioning on A_{good} and its complement:

$$\begin{aligned}\mathop{\mathbb{E}}_{x}[\sigma_{x,r}(A \cap S_{x,r}) \mid x \in A] &= \mathbb{E}[\sigma_{x,r}(A \cap S_{x,r}) \mid x \in A_{\text{good}}]\, \mathbf{Pr}(x \in A_{\text{good}} \mid x \in A) \\ &\quad + \mathbb{E}[\sigma_{x,r}(A \cap S_{x,r}) \mid x \notin A_{\text{good}}]\, \mathbf{Pr}(x \notin A_{\text{good}} \mid x \in A) \\ &\leq 1 \cdot \mathbf{Pr}(x \in A_{\text{good}} \mid x \in A) + \frac{1}{2}\sigma(A)^{(1+r^4)/(1-r^4)}.\end{aligned}$$

Rearranging yields

$$\mathbf{Pr}(x \in A_{\text{good}} \mid x \in A) \geq \mathop{\mathbb{E}}_{x}[\sigma_{x,r}(A \cap S_{x,r}) \mid x \in A] - \frac{1}{2}\sigma(A)^{(1+r^4)/(1-r^4)},$$

and thus

$$\sigma(A_{\text{good}}) = \mathbf{Pr}(x \in A_{\text{good}} \mid x \in A) \cdot \sigma(A) \geq \frac{1}{2}\sigma(A)^{2/(1-r^4)} - O_r(n^{-2})\sigma(A) \ . \qquad \square$$

The following proposition shows the relation between inner product on $\mathbb{S}^{n-1}$ and the normalized inner product on $S_{x,c}$:

Proposition 4.5. *Let $x_1, x_2, x_3 \in \mathbb{S}^{n-1}$ be such that $\langle x_1, x_2 \rangle = \langle x_1, x_3 \rangle = c$ and $\langle x_2, x_3 \rangle = r$. For $i = 2, 3$, we can write*

$$x_i = cx_1 + \sqrt{1-c^2}\, y_i,$$

where $y_i = \frac{x_i - cx_1}{\|x_i - cx_1\|_2}$ satisfies $\langle y_i, x_1 \rangle = 0$ and

$$\langle y_2, y_3 \rangle = f_c(r),$$

with $f_c(r) = \frac{r-c^2}{1-c^2}$.

Proof. A direct computation shows that y_i is orthogonal to x_1 and that $\|x_i - cx_1\|_2 = \sqrt{1-c^2}$. Thus,

$$\begin{aligned}\langle y_2, y_3 \rangle &= \frac{\langle x_2 - cx_1, x_3 - cx_1 \rangle}{1-c^2} \\ &= \frac{\langle x_2, x_3 \rangle - c\langle x_1, x_3 \rangle - c\langle x_1, x_2 \rangle + c^2\|x_1\|_2^2}{1-c^2} \\ &= \frac{r-c^2}{1-c^2} \ . \end{aligned} \qquad \square$$

We now apply the above propositions to prove our main result using induction.

Theorem 4.6. *Let $(v_1, \ldots, v_k)$ be an inductive configuration with covariance matrix R, and assume that*

$$\|v_k - v_{k-1}\|_2 \neq \operatorname{diam}\left(\bigcap_{j<k-1} S_{v_j, r_j}\right).$$

Then for any measurable set $A \subseteq \mathbb{S}^{n-1}$ with $\sigma(A) \geq \omega_R(n^{-\epsilon_R})$, the probability that a uniformly random tuple $(x_1, \ldots, x_k) \in \Delta(n, R)$ lies entirely in A satisfies

$$\mathop{\mathbf{Pr}}_{(x_1,\ldots,x_k)\in\Delta(n,R)}(x_1 \in A, \ldots, x_k \in A) \geq \Omega_R(\sigma(A)^{C_R}),$$

as $n \to \infty$. The constant C_R is given by

$$C_R = \sum_{i=1}^{k-1} \frac{2}{1-c_i^4} \prod_{j=1}^{i-1} \frac{1+c_j^4}{1-c_j^4},$$

with

$$c_1 = r_1, \quad c_i = (f_{c_{i-1}} \circ \cdots \circ f_{c_1})(r_i), \quad i = 2, \ldots, k-1,$$

and $f_c(r) = \frac{r-c^2}{1-c^2}$. The constant ϵ_R is given by

$$\epsilon_R = 2\prod_{i=1}^{k-1} \frac{1-c_i^4}{1+c_i^4}.$$

Proof. We proceed by induction on k. The base case $k = 2$ follows directly from Theorem 4.3. The condition $v_2 - v_1 \neq \operatorname{diam}(\mathbb{S}^{n-1})$ ensures $r = \langle v_1, v_2 \rangle > -1$ and the condition $\sigma(A) \geq \omega_R(n^{-\epsilon_R})$ ensures that the error term goes to 0.

Assume the result holds for all configurations of size $< k$, and let $c_1 = r_1$. Define

$$A_{\text{good}} = \left\{ x \in \mathbb{S}^{n-1} : \sigma_{x,c_1}(A \cap S_{x,c_1}) \geq \frac{1}{2}\sigma(A)^{(1+c_1^4)/(1-c_1^4)} \right\}.$$

Then,

$$\begin{aligned}\mathop{\mathbf{Pr}}_{(x_1,\ldots,x_k)\in\Delta(n,R)}(x_1, \ldots, x_k \in A) &= \mathbf{Pr}(x_1 \in A) \cdot \mathbf{Pr}(x_2, \ldots, x_k \in A \mid x_1 \in A) \\ &\geq \mathbf{Pr}(x_1 \in A_{\text{good}}) \cdot \mathbf{Pr}(x_2, \ldots, x_k \in A \mid x_1 \in A_{\text{good}}).\end{aligned}$$

Given x_1, we write $x_i = c_1 x_1 + \sqrt{1-c_1^2}\, y_i$ for $i \geq 2$, where $y_i = \frac{x_i - r_1 x_1}{\|x_i - r_1 x_1\|_2}$. Conditioned on x_1, according to Proposition 4.5, the induced configuration $(y_2, \ldots, y_k)$ follows from a uniform distribution over $\Delta(n-1, R')$, where

$$R' = R(f_{c_1}(r_2), \ldots, f_{c_1}(r_k)).$$

Given x_1, the event $x_2 \in A, \cdots, x_k \in A$ is equal to $y_2 \in A \cap S_{x_1,c_1}, \cdots, y_k \in A \cap S_{x_1,c_1}$. Conditioned on $x_1 \in A_{\text{good}}$, we have $\sigma_{x_1,c_1}(A \cap S_{x_1,c_1}) \geq \frac{1}{2}\sigma(A)^{(1+c_1^4)/(1-c_1^4)} \geq \omega_{R'}(n^{-\epsilon_{R'}})$, so by the inductive hypothesis:

$$\mathbf{Pr}(y_2, \ldots, y_k \in A \cap S_{x_1,c_1}) \geq \Omega_R\left(\sigma(A)^{C_{R'}(1+c_1^4)/(1-c_1^4)}\right).$$

with

$$C_{R'} = \sum_{i=2}^{k-1} \frac{2}{1-c_i^4} \prod_{j=2}^{i-1} \frac{1+c_j^4}{1-c_j^4}$$

where $c_2 = f_{c_1}(r_2)$ and

$$c_i = (f_{c_{i-1}} \circ \cdots \circ f_{c_2})(f_{c_1}(r_i)) = (f_{c_{i-1}} \circ \cdots \circ f_{c_1})(r_i),\ i = 3, \cdots, k-1$$

By Proposition 4.4, we also have

$$\mathbf{Pr}(x_1 \in A_{\text{good}}) \geq \Omega_R(\sigma(A)^{2/(1-c_1^4)}).$$

Multiplying the two bounds give

$$\mathbf{Pr}(x_1, \dots, x_k \in A) \geq \Omega_R\left(\sigma(A)^{C_R}\right),$$

where

$$C_R = \frac{2}{1-c_1^4} + \frac{1+c_1^4}{1-c_1^4} C_{R'} = \sum_{i=1}^{k-1} \frac{2}{1-c_i^4} \prod_{j=1}^{i-1} \frac{1+c_j^4}{1-c_j^4}\ . \qquad \square$$

Remark 4.7. *The condition*

$$\|v_k - v_{k-1}\|_2 \neq \operatorname{diam}\left(\bigcap_{j<k-1} S_{v_j,r_j}\right)$$

is equivalent to $c_{k-1} \neq -1$, which is necessary for applying Theorem 4.3. For a given configuration $(v_1, \dots, v_k)$, the quantity c_i corresponds to the inner product between v_i and later vectors v_j, after projecting to the $(n-i)$-subsphere $\bigcap_{l<i} S_{v_l,r_l}$ and normalizing. Since $v_1, \dots, v_k$ are distinct, we have $-1 < c_i < 1$ for all $i \leq k-2$ and $c_{k-1} < 1$. The only forbidden case is $c_{k-1} = -1$, which occurs precisely when $v_k - v_{k-1}$ equals the diameter of the intersection $\bigcap_{j<k-1} S_{v_j,r_j}$.

A configuration is *sphere Ramsey* if for any $c > 0$, any c-coloring of the sphere contains a monochromatic congruent copy of the configuration,. Since any c-coloring must contain a monochromatic set A with $\sigma(A) \geq 1/c$, an immediate corollary to Theorem 4.6 is that all inductive configurations (excluding the special case) are sphere Ramsey.

Corollary 4.8. *Let $(v_1, \dots, v_k)$ be an inductive configuration with covariance matrix R, and assume that*

$$v_k - v_{k-1} \neq \operatorname{diam}\left(\bigcap_{j<k-1} S_{v_j,r_j}\right).$$

Then R is sphere Ramsey.

A particularly important class of inductive configurations are k-simplices:

$$\Delta_k(n,r) := \Delta(n, R(r, \dots, r)) = \{(x_1, \dots, x_k) \in (\mathbb{S}^{n-1})^k : \langle x_i, x_j \rangle = r,\ \forall i \neq j\}.$$

In this case, the coefficients c_i admit the closed-form expression $c_i = \frac{r}{1+(i-1)r}$. The condition $c_{k-1} > -1$ is equivalent to $r > -\frac{1}{k-1}$. We thus obtain the following corollary:

Corollary 4.9. *Fix any $r \in \left(-\frac{1}{k-1}, 1\right)$. For any measurable set $A \subseteq \mathbb{S}^{n-1}$ with $\sigma(A) \geq \omega_{k,r}(n^{-\epsilon_{k,r}})$, the probability that a uniformly random tuple $(x_1, \ldots, x_k) \in \Delta_k(n, r)$ lies entirely in A satisfies*

$$\Pr_{(x_1,\ldots,x_k)\in\Delta_k(n,r)}(x_1 \in A, \ldots, x_k \in A) \geq \Omega_{k,r}(\sigma(A)^{C_{k,r}}),$$

as $n \to \infty$. The constants $C_{k,r}$, $\epsilon_{k,r}$ are given by

$$C_{k,r} = \sum_{i=1}^{k-1} \frac{2}{1-c_i^4} \prod_{j=1}^{i-1} \frac{1+c_j^4}{1-c_j^4} \qquad \epsilon_{k,r} = 2\prod_{i=1}^{k-1} \frac{1-c_i^4}{1+c_i^4},$$

where

$$c_i = \frac{r}{1+(i-1)r}, \quad \text{for } i = 1, \ldots, k-1.$$

5 Open Problems

Our density version of the Frankl–Rödl theorem on the sphere (Theorem 4.3) requires the set $A \subseteq \mathbb{S}^{n-1}$ to have spherical measure $\sigma(A) > \omega(n^{-2})$ in order to guarantee a non-trivial lower bound. In contrast, the classical Frankl–Rödl theorem asserts that for any $-1 < r < 1$, there exists a constant $\epsilon = \epsilon(r) > 0$ such that

$$\Pr_{x\cdot y=r}(x \in A, y \in A) > 0$$

holds for all sets A with $\sigma(A) > \Omega(\epsilon^n)$. Notably, when $r = 0$, our result in Theorem 4.2 recovers the classical theorem for exponentially small sets. A natural open problem is to extend our techniques and obtain explicit lower bounds on

$$\Pr_{x\cdot y=r}(x \in A, y \in A)$$

for all $-1 < r < 1$, even when $\sigma(A)$ is exponentially small in n.

Another direction concerns the class of configurations we consider. Our density result applies to *inductive configurations*, but Matoušek and Rödl [MR95] showed that fix constant c and for any configuration P with circumradius less than 1, any c coloring of $\mathbb{S}^{n-1}$ contains a monochromatic copy of P. It remains open whether one can establish a *density* version of this theorem—that is, to show

$$\Pr_{(x_1,\ldots,x_k)\in P}(x_i \in A\ \forall i) > \epsilon(\sigma(A))$$

for any set $A \subseteq \mathbb{S}^{n-1}$ with constant density $\sigma(A) > \epsilon(P)$ where $\epsilon(\sigma(A))$ is a lower bound on the probability in terms of $\sigma(A)$.

Finally, our current results can be almost extended to 3-point configurations. For example, using Theorem 4.3, we can obtain bounds on

$$\Pr_{\substack{x\cdot y=r_1\\ x\cdot z=r_2}}(x \in A, y \in A, z \in A),$$

but this does not yet yield control over 3-point configurations, which require bounding

$$\Pr_{\substack{x\cdot y=r_1\\ x\cdot z=r_2\\ y\cdot z=r_3}}(x \in A, y \in A, z \in A).$$

It is an open question whether a density theorem can be proved for general 3-point configurations on the sphere.

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