A User's Guide to Calculus in $\mathbb{R}^n$

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This is a set of notes that I'm compiling on my way to understanding and learning Calculus in $\mathbb{R}^n$. Think of this as a user's guide to Calculus in $\mathbb{R}^n$.

0.1 Introduction

Calculus is hard. I'll be honest I understand more about General Topology (even Algebraic Topology) than I do basic Calculus at this point in time. Heck I think I even understand (truly) basic Category Theory better than basic Calculus at this point in time.

The reason for this, is I think the folllowing; Topology and Algebra are fields (no pun intended) of mathematics that are very structural in nature to me. You take a set, put a topology on it and voila you have a topological space.

Calculus is different, there's no real inherent structure (to me at this point in time).

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Chapter 1

Recap of Metric Spaces

Before we can get started with calculus proper, we need to recap some stuff from metric spaces specifically pertaining to continuity and limits.

1.1 Continuity

Definition 1.1.1 (Continuity)**.** Let (X, d_x) and (Y, d_Y) be metric spaces and let $f : (X, d_X) \to (Y, d_Y)$ be any function. Then f is continuous at $a \in X$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(a, x) < \delta \implies d_Y\left(f(a), f(x)\right) < \epsilon$$

for all $x \in B_{(X,d_X)}(a, \delta)$. We say that f is continuous (or continuous on X) if f is continuous for each $x \in X$.

Here are some really powerful theorems and corollaries that will make our lives way way way easier.

Theorem 1.1.1. Suppose that X, Y and Z are metric spaces and $f : X \to Y$, $g : f[X] \to Z$ are continuous mappings. Then the composition mapping $h = g \circ f : X \to Z$ defined by

$$h(x) = g(f(x))$$

for $x \in X$ is continuous.

Theorem 1.1.2. Let (X, d) be a metric space and let $f, g : X \to \mathbb{C}$ be continuous functions. Then $f + g$ and $f \cdot g$ are continuous on X. If in addition we have $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous on X.

Corollary 1.1.1. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous functions. Then $f + g$ and $f \cdot g$ are continuous on X. If in addition we have $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous on X.

Show how the above theorem implies the corollary. Use the fact that the restriction of f, g from the codomain of $\mathbb{C}$ to the set

$$\Gamma = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 0\}$$

is continuous, then use the fact that $\mathbb{R} \times \{0\}$ is homeomorphic to Γ and then the fact that $\mathbb{R} \times \{0\}$ is homeomorphic to $\mathbb{R}$ to conclude and arrive at the corollary.

1.2 Limits

Limits will be invaluable to our study of calculus in $\mathbb{R}^n$. For example to even define the derivative in $\mathbb{R}^1$ we need to know the definition(s) of a limit in metric spaces. The reason for this is that we need the notion of a limit to even define the derivative rigorously.

As you may remember there are two different definitions of a limit in metric spaces. They are

1. Limits of sequences

2. Limits of functions

Below we give definitions for each of them.

Definition 1.2.1 (Limit of a sequence)**.** We say that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq (X, d)$ converges to $x' \in X$ and write

$$\lim_{n\to\infty} x_n = x'$$

if for any $\epsilon > 0$ there exists a natural number $N > 0$ such that $d(x_n, x') < \epsilon$ for all $n > N$

Definition 1.2.2 (Limit of a function)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $E \subseteq X$. Let $f : E \to Y$ and suppose that p is a limit point of E. We say that

$$\lim_{x\to p} f(x) = q$$

if there exists a point $q \in Y$ with the property that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < d_X(x, p) < \delta \implies d_Y((f(x), q) < \epsilon$$

for any $x \in B_{(X,d_x)}(p, \delta)$

As it turns out we can express these two different notions of a limit in terms of another and vice versa in quite a nice way.

Theorem 1.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$ and let $f : E \to Y$ be any function and let p be a limit point of E. Then

$$\lim_{x \to p} f(x) = q$$

if and only if

$$\lim_{n \to \infty} f(p_n) = q$$

for every (non-constant) sequence $\{p_n\} \subseteq E$ such that

$$\lim_{n \to \infty} p_n = p$$

1. Show that in $\mathbb{R}^n$ with the usual topology it simplies to $p \in \overline{E}$

2. Theorems about limits in $\mathbb{R}^n$ from Hubbard and Hubbard [2]

3. Limits and continuity theorems

4. Techniques to show limits exists and calculate limits

The whole of chapter 4 in Rudin is a treasure trove of theorems that would be useful here.

This next theorem while fairly innocuous at first glance to a Topologist like myself but if calculus was a game, then this is the holy grail of cheat codes. (The reason why is that derivatives are defined as limits of functions and if we know that f is continuous, then we just evaluate f at p and we're done, we'd have just showed the existence and calculated the derivative like that, boom.)

Theorem 1.2.2. Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$ and let $f : E \to Y$. Assume that p is a limit point of E, then f is continuous at p if and only if

$$\lim_{x \to p} f(x) = f(p)$$

This has a extremely useful corollary to check for discontinuities of a function

Corollary 1.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$ and let $f : E \to Y$. Assume that p is a limit point of E, then f is not continuous at p if and only if

$$\lim_{x \to p} f(x) \neq f(p).$$

(Also if $\lim_{x\to p} f(x)$ does not exist then vacuously we have that $\lim_{x\to p} f(x) \neq f(p)$)

Theorem 1.2.3. Let X be a metric space and $E \subseteq X$. Suppose that p is a limit point of E and $f, g : X \to \mathbb{C}$ and that

$$\lim_{x\to p} f(x) = a \in \mathbb{C}, \quad \lim_{x\to p} g(x) = b \in \mathbb{C}$$

Then

$$\text{(i)} \quad \lim_{x\to p}(f+g)(x) = a + b;$$
$$\text{(ii)} \quad \lim_{x\to p}(f\cdot g)(x) = a \cdot b;$$
$$\text{(iii)} \quad \lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{a}{b} \quad \text{if} \;\; b \neq 0.$$

This yields the weaker result;

Theorem 1.2.4. Let X be a metric space and $E \subseteq X$. Suppose that p is a limit point of E and $f, g : X \to \mathbb{R}$ and that

$$\lim_{x\to p} f(x) = a \in \mathbb{R}, \quad \lim_{x\to p} g(x) = b \in \mathbb{R}.$$

Then

$$\text{(i)} \quad \lim_{x\to p}(f+g)(x) = a + b;$$
$$\text{(ii)} \quad \lim_{x\to p}(f\cdot g)(x) = a \cdot b;$$
$$\text{(iii)} \quad \lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{a}{b} \quad \text{if} \;\; b \neq 0.$$

1.2.1 A very useful theorem

We now provide the reader with an extremely useful theorem to calculate derivatives by their definition in the next chapter.

Theorem 1.2.5. Let $A \subseteq \mathbb{R}^n$ and let $f : A \setminus \{0\} \to \mathbb{R}^m$ and suppose that A contains a neighborhood of $0 \in \mathbb{R}^n$. Now let $B \subseteq \mathbb{R}^n$ such that $0 \in B$ and $A \subseteq B$ and let $g : B \to \mathbb{R}^m$ be a continuous function such that $g(x) = f(x)$ for all $x \in A \setminus \{0\}$, then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = g(0)$$

Proof. We claim that $\lim_{x\to 0} f(x) = g(0)$. By continuity of g and the fact that 0 is a limit point of B (because $A \subseteq B$ and A contains a neighborhood of 0), it follows that $\lim_{x\to 0} g(x) = 0$.

Thus for any $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < d_{\mathbb{R}^n}(x, 0) < \delta \implies d_{\mathbb{R}^m}(g(x), g(0)) < \epsilon$ for any $x \in B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, \delta)$. Now since A contains a neighborhood of $0 \in \mathbb{R}^n$, there exists an $r > 0$ such that $B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, r) \subseteq A$. Note also that $f(x) = g(x)$ for all $x \in B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, r) \setminus \{0\}$.

Now let $\epsilon > 0$ be given, then there exists a $\delta > 0$ (since $\lim_{x\to 0} g(x) = g(0)$) such that

$$0 < d_{\mathbb{R}^n}(x, 0) < \delta \implies d_{\mathbb{R}^m}(g(x), g(0)) = d_{\mathbb{R}^m}(f(x), g(0)) < \epsilon$$

for any $x \in B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, \delta) \setminus \{0\}$.

Now because of the fact that the domain of f might not contain all points of $B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, \delta) \setminus \{0\}$ (i.e. $B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, \delta) \setminus \{0\} \nsubseteq A \setminus \{0\}$) we need to ensure we can find a δ such that $f(x)$ is actually defined for every $x \in B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, \delta) \setminus \{0\}$.

We have two cases either $r \leq \delta$ or $r > \delta$.

If $r \leq \delta$ then it follows that $0 < d_{\mathbb{R}^n}(x, 0) < r \implies d_{\mathbb{R}^m}(f(x), g(0)) < \epsilon$ for any $x \in B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, r)$ and thus $\lim_{x\to 0} f(x) = g(0)$.

If $r > \delta$ then $B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, \delta) \subseteq B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, r)$ and so $f(x)$ is defined for any

$x \in B_{(\mathbb{R}^n, d_{\mathbb{R}^n})}(0, r)$ and there is nothing further to prove and it follows that $\lim_{x \to 0} f(x) = g(0)$.

Thus we've proven that

$$\lim_{x \to 0} f(x) = g(0).$$

□

We now give an example to show just how useful this theorem is.

Example 1.2.1. Consider the following function $f(x) : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{x^2}{x^4 + x^2}$$

prove that

$$\lim_{x \to 0} \ f(x) = 1.$$

Proof. Without the theorem we just proved above, we'd likely have to resort to proving this limit by the definition which would be a really painstaking task. With the theorem we proved above at hand though, we can simply do the following.

Note that for any $x \in \mathbb{R} \setminus \{0\}$ we have that

$$f(x) = \frac{x^2}{x^4 + x^2} = \frac{1}{x^2 + 1}$$

and note that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x^2+1}$ is continuous on $\mathbb{R}$ (and in particular it is continuous at $0 \in \mathbb{R}$) and thus

$$\lim_{x \to 0} g(x) = g(0) = \frac{1}{(0)^2 + 1} = 1.$$

Thus since $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus \{0\}$ and $\mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ it follows by the above theorem that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = g(0) = 1.$$

□

In fact as far as the author is aware, this theorem is used every single time someone calculates the derivative of a function by the definition of the derivative.

1.2.2 ”Limits from the Left” and ”Limits from the Right” in $\mathbb{R}$

Definition 1.2.3 (”Limit from the right”)**.** Let $f : (a, b) \to \mathbb{R}$. Consider any point $x \in [a, b)$. We write

$$\lim_{t \to x^+} f(t) = q$$

if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}_{n \in \mathbb{N}} \subseteq (x, b)$ such that $t_n \to x$.

Definition 1.2.4 (”Limit from the left”)**.** Let $f : (a, b) \to \mathbb{R}$. Consider any point $x \in (a, b]$. We write

$$\lim_{t \to x^-} f(t) = q$$

if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}_{n \in \mathbb{N}} \subseteq (a, x)$ such that $t_n \to x$.

Theorem 1.2.6. Let $f : (a, b) \to \mathbb{R}$ and consider any point $x \in (a, b)$. Then $\lim_{t \to x} f(t)$ exists if and only if

$$\lim_{t \to x^+} f(t) = \lim_{t \to x^-} f(t) = \lim_{t \to x} f(t)$$

1.2.3 Infinite Limits and Limits at Infinity

Definition 1.2.5 (Extended Real Number System)**.** The extended real numbers is obtained by adding two elements and $+\infty$ and $-\infty$ to $\mathbb{R}$. We'll denote the extended real numbers by the symbol $\overline{\mathbb{R}}$ (Not to be confused with the topological closure of $\mathbb{R}$)

Definition 1.2.6 (Neighborhoods of Infinity)**.** For any $c \in \mathbb{R}$, the set $\{x \in \mathbb{R} \mid x > c\}$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $\{x \in \mathbb{R} \mid x < c\}$ which will be denoted by $(-\infty, c)$ is called a neighborhood of $-\infty$.

Definition 1.2.7. Let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$ be any function. We say that

$$\lim_{t \to x} f(t) = \alpha$$

where α and x are in the extended real number system, if for every neighborhood U of α there exists a neighborhood V of x such that $V \cap E$ is non-empty and such that $f(t) \in U$ for all $t \in V \cap E$ for which $t \neq x$.

Remark. This definition above coincides with Definition 2.3 when α and x are real numbers.

Let's get our hands dirty with an example of this definition in action.

Example 1.2.2. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = e^x$. We claim that

$$\lim_{t\to\infty} f(t) = \infty$$

Proof. Let U be a neighbourhood of ∞. Then $U = (c, \infty)$ for some $c \in \mathbb{R}$. We need to find a neighbourhood V of ∞ such that $V \cap (\mathbb{R} \setminus \{0\}) \neq \emptyset$ and for which $f(t) \in (c, \infty)$ for all $t \in V \cap (\mathbb{R} \setminus \{0\})$ for which $t \neq \infty$.

Since f is only defined on $\mathbb{R} \setminus \{0\}$ and not on the extended reals, then the previous line simplifies to us needing to find a neighbourhood V of ∞ such that $V \cap (\mathbb{R} \setminus \{0\}) \neq \emptyset$. and $f(t) \in (c, \infty)$ for all $t \in V \cap (\mathbb{R} \setminus \{0\})$. We have two cases to analyse.

If $c < 1$ then let $V = (0, \infty)$ because then we have $V \cap \mathbb{R} \setminus \{0\} = V = (0, \infty)$ and for any $t \in V$ we have $f(t) = e^t > e^0 = 1 > c$ since f is monotonically increasing on V. Thus for any $t \in V = V \cap (\mathbb{R} \setminus \{0\})$ we have $f(t) \in (c, \infty) = U$ as desired.

If $c \geq 1$, then we need to find a neighborhood V such that $e^t \in (c, \infty)$ for all $t \in V \cap \mathbb{R} \setminus \{0\}$. Note then that $f(t) = e^t \in (c, \infty) = U \implies e^t > c \implies t > \ln(c)$. So if we let $V = (\ln(c), \infty)$ then for any $V \cap \mathbb{R} \setminus \{0\} = V$ we have $f(t) = e^t > e^{\ln(c)} = c$ since f is monotonically increasing on V as desired. $\square$

The analogue of Theorem 2.2.4 is still true

Theorem 1.2.7. Let X be a metric space and $E \subseteq X$. Suppose that $p \in \overline{\mathbb{R}}$ and $f, g : X \to \mathbb{R}$ and that

$$\lim_{x\to p} f(x) = a \in \overline{\mathbb{R}}, \quad \lim_{x\to p} g(x) = b \in \overline{\mathbb{R}}.$$

Then

$$\text{(i)} \lim_{x\to p} f(x) = a' \implies a' = a$$

$$\text{(ii)} \quad \lim_{x\to p}(f+g)(x) = a+b;$$

$$\text{(iii)} \quad \lim_{x\to p}(f\cdot g)(x) = a\cdot b;$$

$$\text{(iv)} \quad \lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{a}{b} \quad \text{if} \;\; b \neq 0.$$

Important Point. *Note that $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{a}{0}$ are not defined.*

We finish off this section with a user's guide to continuity in metric spaces.

1.3 User's guide to continuity in metric spaces

> Life is short, don't always prove continuity using ϵ's and δ's
>
> *Shimal Harichurn*
> *Your Author*

My quote above is basically saying that proving continuity of every function using the ϵ-δ definition of continuity (or even the topological definition of continuity) is a waste of time. We have some powerful theorems at hand that will make our lives a lot easier, so why not use them?

This section is intended to show the reader how we actually prove continuity of functions between metric spaces in practice. In practice nobody actually proves continuity by the ϵ-δ definition. What we actually do is the following.

1. We start off with a base pack of functions which we prove are continuous using the ϵ-δ definition of continuity.

2. We compose these base pack of functions to arrive at new functions which are continuous, or given some function which we want to prove is continuous we decompose it into a composition of functions which we know to be continuous

3. Then we appeal to the fact that composition of continuous functions are continuous (see Theorem 2.1.1)

4. We throw in the fact that addition of continuous functions are continuous (see Corollary 2.1.1 or Theorem 5.1.2)

5. Voila we've shown our new function is continuous

"But wait hang on there Shimal....what if we can't decompose a given function f into a composition of functions from our base pack of functions?" Well then you're going to have to prove continuity of that function f using the ϵ-δ definition of continuity, but the thing is that those functions are quite few and far between so in most cases we won' t have to resort to using the definition of continuity to prove continuity.

Base pack of continuous functions

Result 1. Let $c \in \mathbb{R}$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$. Then f is continuous.

Proof. Pick $a \in \mathbb{R}$ and let $\epsilon > 0$ be give. Put $\delta = 1$. Then $|a - x| < \delta \implies |f(a) - f(x)| = |c - c| = 0 < \epsilon$, thus f is continuous. □

Remark. Note that any value for δ in the above proof will actually work.

Result 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. Then f is continuous on $\mathbb{R}$.

Proof. Pick $a \in \mathbb{R}$ and let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$ for all $x \in B(a, \delta) = (a - \delta, a + \delta)$.

Since $|f(x) - f(a)| = |x - a|$ (because $f(x) = x$ for all $x \in \mathbb{R}$ by definition of f) this suggests we choose $\delta = \epsilon$.

Put $\delta = \epsilon$, then

$$\begin{aligned} |x - a| < \delta &\implies |x - a| < \epsilon \\ &\implies |f(x) - f(a)| < \epsilon \end{aligned}$$

for all $x \in B(a, \delta)$. Thus f is continuous at $a \in \mathbb{R}$ and since a was chosen arbitrarily we have that f is continuous on $\mathbb{R}$ □

Result 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Then f is continuous on $\mathbb{R}$.

To prove this next result we make use of the following theorem on metric spaces.

Theorem 1.3.1. Let (X, d) be a metric space and let $a \in X$ be any fixed point. Then the map $f : (X, d) \to \mathbb{R}$ defined by $f(x) = d(x, a)$ is continuous.

With this theorem at hand we can now prove the above result.

Proof. Note that $(\mathbb{R}, d)$ is a metric space when endowed with the usual Euclidean metric which is $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x, y|$ for all $x, y \in \mathbb{R}$. (Whenever we refer to $\mathbb{R}$ in this book we always refer to it as this metric space). Thus since $0 \in \mathbb{R}$, the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = d(x, 0) = |x|$ is continuous. □

Result 4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Then f is uniformly continuous (and thus continuous) on $\mathbb{R}$.

The following proof is taken from [1]

Proof. Let $\epsilon > 0$ be given. Put $\delta = \epsilon^2$. Then for $|x - y| < \delta$ we have

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y| < \epsilon^2 \implies |\sqrt{x} - \sqrt{y}| < \epsilon.$$

Thus f is uniformly continuous on $\mathbb{R}_+$ and thus continuous on $\mathbb{R}$ □

Trigonometric functions

Result 5 (Projections mappings are continuous)**.** Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\pi_i(x) = \pi_i(x_1, ..., x_n) = x_i$$

for $x = (x_1, .., x_n) \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$. Then π_i is continuous.

Proof. Let U be open in $\mathbb{R}$, then

$$\pi_i^{-1}[U] = \{x \in \mathbb{R}^n \mid \pi_i(x) \in U\} = \mathbb{R} \times \mathbb{R} \times \cdots \times \underbrace{U}_{i\text{th position}} \times \mathbb{R} \times \cdots \times \mathbb{R}$$

and since the topology on $\mathbb{R}^n$ induced by the euclidean metric and the product topology on $\mathbb{R}^n$ are equivalent, it follows that since $\mathbb{R}$ and U are open in $\mathbb{R}$ a finite product of them is open in $\mathbb{R}^n$. Thus $\pi_i^{-1}[U]$ is open in $\mathbb{R}^n$. Thus π_i is continuous. □

Chapter 2

Differentiation in $\mathbb{R}^1$

Let's dive head on into it.

2.1 Prerequisites

Now with the two notions of a limit defined we can define the derivative. This is quite an important point which warrants the need for this to have a bit of space so that the reader can see it and not forget it.

Important Point. *In the definition of the derivative the limit we will be using is the limit of a function!*

2.2 Defining the Derivative

Definition 2.2.1 (Derivative as defined in Munkres [4])**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be any function. Suppose that A contains a neighborhood of the point a. We define the derivative of f at a by the equation

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$$

provided that the limit exists.

But wait before we can go any further we **really need too** unpack this definition and see what's going on under the hood. The reader at this point is probably going *"Hey author, urmm how exactly are we using the definition* *of the limit of a function here? Heck what is the function we're even taking a limit of here"* and your author will get to that in a second, just bear with me.

Personally I would define the derivative in a much cleaner way (and I will do so later), but the above definition is basically the convention in mathematics as it's taught in high-school and first-year undergraduate courses and (hopefully) everybody is used to it.

Understanding this definition will not only bridge the gap between high-school level mathematics and more advanced pure mathematics, but it will also give the reader an idea into the amount of mathematical maturity (e.g asking the simplest questions, which people sometimes say are silly questions) needed to truly understand concepts. Okay I'll shut up for now and proceed to unpack this definition.

Observation 1. The neighbourhood of a that Munkres is referring to is an open set $U \subseteq A$ containing a that is open in $\mathbb{R}$ as opposed to U being open in the subspace A. But what would happen if we wanted the neighborhood of a to be an open set of the subspace A as opposed to $\mathbb{R}$? Well then we could have all kinds of wacky domains on which the function we're taking the limit of (not to be confused with f) would be defined and the limit would not exist. We'll give some examples of this a bit later on below

Observation 2. This is quite a subtle one. The fact that the definition above states that we need A to contain a neighborhood of a which, as we know by Observation 1., means that A needs to contain a subset $U \subseteq A$ which contains a and is also open in $\mathbb{R}$, but then this means that a needs to be an interior point of A, since U must be open in $\mathbb{R}$. So a must be an interior point of A for f to even meet the criteria needed for f to be differentiable at a. (So if $a \notin \text{Int}(A)$ then we don't even meet the criteria needed for f to be differentiable at a, so f can't be differentiable at a) Later on in my definition of the derivative I will just let A be an open set and be done.

This in essence shows that for a function to be differentiable on some domain, that domain must be particularly "nice", specifically we need the domain to be an open set in $\mathbb{R}$ and the only open sets in $\mathbb{R}$ are unions of open intervals and $\mathbb{R}$ and $\emptyset$.

Observation 3. In the definition of $f'(a)$ no explicit mention of the domain function we're taking the limit of is given, we know that the function we're taking the limit of is

$$\frac{f(a+t)-f(a)}{t}$$

but it is unclear over what domain this function is defined, and it's unclear if the criteria needed to take the limit of this function is satisfied.

Since A contains a neighborhood of a, which I'll call U and since U is open in $\mathbb{R}$, there exists an $r > 0$ such that $B(a,r) \subseteq U \subseteq A$. Then note that $B(a,r) = (a-r, a+r)$.

Now a **key** realization to make is that $\frac{f(a+t)-f(a)}{t}$ is defined for all $t \neq 0$ such that $a+t$ belongs to the neighborhood (which in this case becomes our open ball) of the point a. The set of all such $t \in \mathbb{R}$ satisfying the properties given above is given by

$$\{t \in \mathbb{R} \mid t \neq 0 \text{ and } a+t \in B(a,r) = (a-r, a+r)\} = B(0,r) \setminus \{0\}.$$

Thus we can define $\phi : B(0,r) \setminus \{0\} \to \mathbb{R}$ by

$$\phi(t) = \frac{f(a+t)-f(a)}{t}$$

and then we can rewrite the derivative of f as

$$f'(a) = \lim_{t\to 0} \phi(t).$$

Furthermore note that $0 \in \mathbb{R}$ is a limit point of the set $B(0,r) \setminus \{0\}$ since any neighborhood of 0 contains infinitely many members of $B(0,r) \setminus \{0\}$. Thus checking against our definition of a limit we see that all our criteria are satisfied so we can thus take the limit of ϕ as $t \to 0$

Example of a function that is defined on an open set but which is not differentiable

4. Give an example of a function with a domain that's not nice and show why it isn't differentiable in two ways. Show that firstly that no point $a \in A$ has a nbhd in $\mathbb{R}$ which is a subset of A then go ahead anyway and show that the limit doesn't exist. Showing that the limit doesn't exist shows that the function won't be differentiable and why we must have the nbhd property as a priori for differentiability. This explains what G&P meant when they said what they said in Differential Topology

Now below we'll get onto my definition of the derivative.

Definition 2.2.2 (My definition of the derivative of a function at a point)**.** Let $U \subseteq \mathbb{R}$ be an open set. Let $f : U \to \mathbb{R}$ be any function. Pick $a \in U$ and choose $r > 0$ such that $B(a, r) \subseteq U$. Let $\Omega_a = B(a, r) = (a - r, a + r)$ and let $\Theta_{r_a} = B(0, r) \setminus \{0\}$ Define $\phi : \Theta_{r_a} \to \mathbb{R}$ by

$$\phi(t) = \frac{f(a + t) - f(a)}{t}.$$

Then we define the *derivative of f at a* as

$$f'(a) = \lim_{t \to 0} \phi(t)$$

provided that the limit exists and we say that f is *differentiable at a.*

Remark (1)*.* If f is differentiable for every $a \in U$ we say that f is differentiable on U.

Remark (2)*.* We say that Θ_{r_a} **is a domain of definition for the quotient function** (since Θ_{r_a} is a domain which allows the quotient function ϕ to be well-defined), it depends on the r from the open ball Ω_a, so any domain of definition for a quotient function depends on finding an open ball in $\mathbb{R}$ containing a first. The subscripts on Θ_{r_a} shows this dependence, that the domain of definition of the quotient function depends on r, and that r depends on a.

Definition 2.2.3 (My definition of **the** derivative of a function on an open set)**.** Let $U \subseteq \mathbb{R}$ be an open set. Let $f : U \to \mathbb{R}$ be any function. If f is differentiable for every $a \in U$, then we define the **derivative of** f to be the function $f' : U \to \mathbb{R}$ defined by

$$f'(a) = \lim_{t \to 0} \phi_a(t)$$

where $\phi_a : \Theta_{r_a} \to \mathbb{R}$ is defined by

$$\phi_a(t) = \frac{f(a+t) - f(a)}{t}$$

and Θ_{r_a} is a domain of definition for the quotient function.

Remark. Note that in the above definition since f is differentiable on U, for every $a \in U$ it follows that $f'(a) = \lim_{t\to 0} \phi_a(t)$ exists, also it follows that since f is differentiable at for every $a \in U$, there exists a domain of definition Θ_{r_a} for the quotient function ϕ_a and so ϕ_a is well defined for every $a \in U$ so the above definition is correct and logical.

See how Shalins comments about the need for the domain of a linear map to be a vector space affect this definition .

One of the Professors I speak to online bugged me about even being able to talk about the differentiability of $f : [0,1] \to \mathbb{R}$ because $[0,1]$ is not an open interval. My method for dealing with that would've been to restrict f to it's interior and then analyse differentiability of the restricted function. (which I don't see anything wrong with) but some people might want a more general definition in which incorporate more general domains so they don't have to think about restrictions (but then again their functions on these general domains will only be differentiable on the interiors of those domains anyway *shrugs*)

Definition 2.2.4 (My more general definition of the derivative of a function at a point)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be any function. Pick $a \in \text{Int}(A)$ and choose $r > 0$ such that $B(a,r) \subseteq \text{Int}(A)$. Define $\phi : B_{(\mathbb{R},d)}(0,r)\backslash\{0\} \to \mathbb{R}$ by

$$\phi(t) = \frac{f(a+t) - f(a)}{t}.$$

Then we define the derivative of f at a as

$$f'(a) = \lim_{t\to 0} \phi(t)$$

provided that the limit exists.

Important Point. *An immediate consequence of my definition is that if* $\text{Int}(A) = \emptyset$ *then* f *is differentiable nowhere on* A*.*

2.3 Differentiation Theorems

Theorem 2.3.1. Suppose that $f, g : [a, b] \to \mathbb{R}$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, $f \cdot g$ and $\frac{f}{g}$ are differentiable at x and

(i)
$$(f + g)'(x) = f'(x) + g'(x)$$

(ii)
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

(iii)
$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - g'(x) \cdot f(x)}{g^2(x)}$$

2.4 The Chain Rule

Theorem 2.4.1 (The Chain Rule)**.** Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x)$ exists at some point $x \in [a, b]$ and $g : I \to \mathbb{R}$ where $I \supseteq f\left[[a, b]\right]$ and g is differentiable at the point $f(x)$. If $h : [a, b] \to \mathbb{R}$ is defined by $h(t) = g(f(t))$ for all $t \in [a, b]$ then h is differentiable at x and

$$h'(x) = g'\left(f(x)\right) \cdot f'(x)$$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(3x)$, prove that f is differentiable given that $\sin(x)$ is differentiable and compute the derivative

2.5 L'Hospital's Rule

2.6 A User's Guide to Differentiation in $\mathbb{R}^1$

Life is short, don't always prove
the derivative exists and
calculate it using the definition

Shimal Haricharn
Your Author

My quote above is basically saying that proving differentiability (and calculating the derivative) of every function using the definition of the derivative is a waste of time. We have some powerful theorems at hand that will make our lives a lot easier, so why not use them?

This section is intended to show the reader how we actually prove differentiability of functions between defined on $\mathbb{R}^1$ in practice.

1. We start off with a base pack of functions which we prove are differentiable using the definition of the derivative and this theorem on limits

2. We compose these base pack of functions to arrive at new functions which are differentiable using the chain rule, or given some function which we want to prove is continuous we decompose it into a composition of functions which we know to be differentiable

3. Then we appeal to the fact that composition of differentiable functions are differentiable

4. We throw in the fact that addition of differentiable functions are differentiable (see Corollary 2.1.1 or Theorem 5.1.2)

5. Voila we've shown our new function is differentiable and we can calculate it using the differentiation theorems

Chapter 3

Integration in $\mathbb{R}^1$

For an **excellent** treatment of Riemann Integration in $\mathbb{R}^1$ using Darboux sums (which make computing the upper and lower integral sums easier) see [6]. We list a bevy of results that are extremely useful in using the Fundamental Theorem of Calculus rigorously.

Theorem 3.0.1. Let $f : [a, b] \to \mathbb{R}$. If $f \in C([a, b])$, then $f \in R([a, b])$. (In words if f is continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$).

3.1 Fundamental Theorem of Calculus

> Life is short, don't compute the integrals of continuous functions by the definition...pls
>
> *Shimal Harichurn*

So now we come onto the famous Fundamental Theorem of Calculus. The result above shows that all continuous functions defined on a closed interval are Riemann integrable, but how do we compute the integral of such a continuous function. Do we use the definition? We could, but that would just waste a ton of time. Instead we have two extremely powerful theorems, the Fundamental Theorem of Calculus (part 1) and the Fundamental Theorem of Calculus (part 2) which allow us to compute the integral of any continuous function defined on a closed interval under some appropriate conditions. (Also see [7] where most of this was sourced from.)

Theorem 3.1.1 (Fundamental Theorem of Calculus Part 1.)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $F : [a, b] \to \mathbb{R}$ be defined by

$$F(x) = \int_a^x f(t)dt.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. That is

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

.

The above result basically shows that anti-derivatives of f always exist when f is continuous. The above result has a corollary

Corollary 3.1.1. If $f : [a, b] \to \mathbb{R}$ is continuous and F is an anti-derivative of f in $[a, b]$ then

$$\int_a^n f(t)dt = F(b) - F(a)$$

This corollary assumes continuity on the whole interval. The result is strengthened slightly in the second part of the Fundamntal throem of Calculus.

Theorem 3.1.2. Let $f : [a, b] \to \mathbb{R}$ and suppsoe that F is an anti-derivarive of f in $[a, b]$, that is

$$F'(x) = f(x)$$

for all $x \in [a, b]$. If f is Riemann integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second part is somewhat stronger than the corollary because it does not assume that f is continuous

3. Measure theory results

3.1.1 What is dx?

Patience you must have, young Padawan

Yoda

Chapter 4

Differentiation in $\mathbb{R}^n$

I only care about derivatives I can compose

Mozart...probably

4.1 Prerequisites

4.1.1 Some continuity theorems

First off this theorem is very useful in showing continuity of functions from any metric space into $\mathbb{R}^k$

Theorem 4.1.1. Let (X, d) be a metric space and let $f_1, ..., f_k$ be real valued functions on X (that is $f_i : X \to \mathbb{R}$ for $i \in \{1, \ldots k\}$) and let $\mathbf{f} : X \to \mathbb{R}^k$ be defined by

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \ldots, f_k$ is continuous.

Theorem 4.1.2. Let (X, d) be a metric space and suppose that $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^k$ are continuous functions, then $\mathbf{f} + \mathbf{g} : X \to \mathbb{R}^k$ defined by

$$(\mathbf{f} + \mathbf{g})(x) = (f_1(x) + g_1(x), \ldots, f_k(x) + g_k(x))$$

is continuous on X and $\mathbf{f} \cdot \mathbf{g} : X \to \mathbb{R}$ defined by

$$(\mathbf{f} \cdot \mathbf{g})(x) = (f_1(x), \ldots, f_k(x)) \odot (g_1(x), \ldots, g_k(x)) = f_1(x) \cdot g_1(x) + \ldots f_k(x) \cdot g_k(x)$$

is continuous on X.

Remark. Note that $\odot$ refers to the dot product on the vector space $\mathbb{R}^k$. The above theorem shows that addition of vector-valued functions whose domains are metric spaces (which could possibly be $\mathbb{R}^m$ for any $m \in \mathbb{N}$) are continuous, and so is taking the dot product of vector valued functions. Proofs of the above theorems can be found in [5]

4.2 The Directional Derivative

This is how the directional derivative is defined in [4].

To ensure that the quotient function in the derivative is well defined we first prove the following theorem.

Theorem 4.2.1. Let $f : A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Suppose that A contains a neighbourhood V of $a = (a_1, \ldots, a_m)$. Choose $r > 0$ such that

$$(a_1 - r, a_1 + r) \times \cdots \times (a_m - r, a_m + r) \subseteq V \subseteq A.$$

Pick $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ such that $u \neq 0 \in \mathbb{R}^m$ and put $R = \max\{|u_1|, \ldots, |u_m|\}$ then the open set

$$\Theta = \left(-\frac{r}{R}, \frac{r}{R}\right) \setminus \{0\}$$

is a set on which $\phi_u : \Theta \to \mathbb{R}^n$ defined by

$$\phi_u(t) = \frac{f(a + tu) - f(a)}{t}$$

is well defined.

Proof. To show that ϕ_u is well-defined it suffices to show that for each $t \in \Theta$ we have $a + tu \in VA$. So to that end pick $t \in \Theta$ and $i \in \{1, \ldots, m\}$. We claim that $a_i + tu_i \in (a_r - r, a_i + r)$. We have two cases to examine $u_i \geq 0$ or $u_i < 0$. Suppose that $u_i \geq 0$ then

$$a_i - r = a_i - \frac{r}{|u_i|} u_i \leq a_i - \frac{r}{R} u_i < a_i + tu_i < a_i + \frac{r}{R} u_i \leq a_i + \frac{r}{|u_i|} u_i = a_i + r.$$

if $u_i < 0$ then

$$\begin{aligned} a_i + tu_i &> a_i + \frac{r}{R}u_i \quad (\text{since } r > 0 \text{ and } R > 0 \text{ and } u_i < 0) \\ &\geq a_i + \frac{r}{|u_i|}u_i \\ &= a_i - r \end{aligned}$$

and

$$\begin{aligned} a_i + tu_i &< a_i - \frac{r}{R}u_i \quad (\text{since } r > 0 \text{ and } R > 0 \text{ and } u_i < 0 \text{ so } \frac{r}{R}u_i < 0) \\ &\leq a_i - \frac{r}{|u_i|}u_i \\ &= a_i - (-1)r \\ &= a_i + r \end{aligned}$$

In both cases we have

$$a_i - r < a_i + tu_i < a_i + r.$$

Thus $a_i + tu_i \in (a_i - r, a_i + r)$.‘Hence

$$a + tu \in (a_1 - r, a_1 + r) \times \cdots \times (a_m - r, a_m + r) \subseteq V$$

as desired. □

Remark. We call Θ a domain of definition for the quotient function ϕ_u.

With this theorem at hand we can now go ahead and define the directional derivative.

Definition 4.2.1 (Directional Derivative)**.** Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Suppose that A contains a neighbourhood of a. Given $u \in \mathbb{R}^m$ with $u \neq 0 \in \mathbb{R}^n$ define $\phi_u : \Theta \to \mathbb{R}^n$ by

$$\phi_u(t) = \frac{f(a + tu) - f(a)}{t}$$

where Θ is any domain of definition for the quotient function ϕ_u. Then we may define

$$f'_u(a) = \lim_{t \to 0} \phi_u(t)$$

(provided the limit exists). We call $f'_u(a)$ the directional derivative of f at a with respect to the vector u.

Remark. Note that the domain of definition Θ for the quotient function ϕ_u depends on both a and u!

4.3 The Total Derivative

This is THE true derivative in $\mathbb{R}^n$

Before we can get to defining the total derivative, we need to ensure that the definition we're going to give actually makes sense. To do that we need the following theorem. Technically an author could give the definition of the derivative and explain why it's well-defined but I feel this way gives the reader a better idea as to what's going on.

Theorem 4.3.1 (The quotient function in the derivative is well-defined on a certain neighborhood)**.** *Let $U \subseteq \mathbb{R}^m$ be an open set and let $f : U \to \mathbb{R}^n$ be any function. Pick $a = (a_1, \ldots, a_m) \in U$ and choose $r > 0$ so that*

$$\Omega = (a_1 - r, a_1 + r) \times (a_2 - r, a_2 + r) \times \cdots \times (a_m - r, a_m + r) \subseteq U.$$

Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be a linear map. Then define

$$\Theta = ((-r, r) \times (-r, r) \times \cdots \times (-r, r)) \setminus \{0\}$$

and then now define $\phi_L : \Theta \to \mathbb{R}^n$ by

$$\phi_L(h) = \frac{f(a+h) - f(a) - L(h)}{|h|.}$$

Then

$$\phi_L$$

is a well-defined function.

Proof. Pick $h \in \Theta$, then we have by definition $\phi_L(h) = \frac{f(a+h)-f(a)-L(h)}{|h|}$. Since L is defined on $\mathbb{R}^m$ we only need to check that $a + h \in U$ for every $h \in \Theta$, thus $f(a + h)$ will be well-defined (i.e. $a + h$ will be in the domain of f) if we can conclude this.

So to that end pick $h \in \Theta$, then $h = (h_1, \ldots, h_m)$ and we have $h_i \in (-r, r)$ for $i \in \{1, \ldots, m\}$. Then note that $a + h = (a_1 + h_1, \ldots, a_m + h_m)$. Now pick $i \in \{1, \ldots, m\}$ and consider $a_i + h_i$. Observe that $a_i + h_i \in (a_i - r, a_i + r)$ since $h_i \in (-r, r)$.

Since i was chosen arbitrarily from $\{1, ..., m\}$ we conclude that this holds for all $i \in \{1, .., m\}$ and thus it follows that $a + h \in \Gamma \subseteq U$. Thus we can conclude that ϕ_L is well defined on Θ as desired. $\square$

Remark. This above theorem shows that the quotient function ϕ_L is well-defined on the open set $\Theta \subseteq \mathbb{R}^m$. Note that this open set Θ depends on r.

Definition 4.3.1 (Rigorous definition of the Derivative in $\mathbb{R}^n$)**.** Let $U \subseteq \mathbb{R}^m$ be an open set. Let $f : U \to \mathbb{R}^m$ be any function. Pick $a = (a_1, \ldots, a_m) \in U$ and choose $r > 0$ so that

$$\Omega = (a_1 - r, a_1 + r) \times (a_2 - r, a_2 + r) \times \cdots \times (a_m - r, a_m + r) \subseteq U.$$

Let $\Gamma(\mathbb{R}^m, \mathbb{R}^n)$ denote the set of linear maps from $\mathbb{R}^m \to \mathbb{R}^n$. Define

$$\Theta = ((-r, r) \times (-r, r) \times \cdots \times (-r, r)) \setminus \{0\}$$

and then now define $\phi_L : \Theta \to \mathbb{R}^n$ by

$$\phi_L(h) = \frac{f(a + h) - f(a) - L(h)}{|h|}$$

for $L \in \Gamma(\mathbb{R}^m, \mathbb{R}^n)$. By the above theorem ϕ_L is well-defined. Then if there exists a $L \in \Gamma(\mathbb{R}^m, \mathbb{R}^n)$ such that

$$\lim_{h \to 0} \phi_L(h) = 0$$

we say that f is differentiable at a and we call the linear map L, the derivative of f at a.

Remark. Now this is a rigorous but fairly cumbersome way to define the derivative. We'll show now how we'll work our way from this definition to the usual definition you'd see in a book like Munkres.

Firstly note that $0 \in \mathbb{R}^n$ is a limit point for the domain of ϕ_L so it makes sense to take the limit of ϕ_L as $h \to 0 \in \mathbb{R}^n$ as defined in the definition of the limit of a function

Secondly note that since U is open, there will **always** exist an $r > 0$ such that $\Omega \subseteq U$. And since the well-definedness of the function ϕ_L depends on this $r > 0$ and such an r always exists, we may safely remove it from our definition (as most textbook authors do) and thus remove the explicit need to introduce the new function ϕ_L.

Finally since we don't need to explicitly introduce the new function ϕ_L which depends on a specific linear map, we can rewrite our definition as follows below.

Important Point. *As long as we understand that*

1. *the quotient function ϕ_L is defined on the open set Θ*
2. *and that Θ depends on the $r > 0$ such that $\Omega \subseteq U$*
3. *and that existence of such an r is always guaranteed since U is open*

and that everything that we're doing (e.g taking limits of the quotient function ϕ_L) takes place on these open sets, Ω and Θ, then ϕ_L is always well-defined. The textbook authors understand this, but don't explicitly state it, and state the definition below, which is perfectly okay as long as one understands what is going on behind the scenes.

Important Point. *Everything above put together basically says that taking the limit is defined locally (and thus the derivative?), (see how Ω and Θ affect the domain of the quotient function (and thus the derivative?))*

Important Point. *Set-theoretically just a domain of Θ works for the linear map that we call the derivative, why do book authors assert the need for domain to be all of $\mathbb{R}^m$? The reason for this is a linear map/linear transformation must have as its domain a vector space and Θ may not be a vector space (for example it might not contain $0 \in \mathbb{R}^m$) and thus it's natural to just consider the domain to be $\mathbb{R}^m$.*

All of this put together allows us to arrive at the following book definition of the derivative.

Definition 4.3.2 (Conventional Book definition of the Derivative in $\mathbb{R}^n$)**.** Let $U \subseteq \mathbb{R}^m$ be an open set. Let $f : U \to \mathbb{R}^n$ be any function. Pick $a \in U$.

We say that f is differentiable at a if there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{|h|} = 0.$$

The linear map L is called the derivative of f at a and is denoted $Df(a)$.

So from this point on now it's perfectly okay to use the above conventional definition.

Important Point. *Since every linear map $L \in \Gamma(\mathbb{R}^m, \mathbb{R}^n)$ can be expressed (for every $x \in \mathbb{R}^m$) as $L(x) = A \cdot x$ for some fixed $n \times m$ matrix A (and where $\cdot$ represents matrix multiplication) authors will write $Df(a) \cdot v$ where in this case they identify the derivative of f at a which is the linear map $Df(a)$ with its corresponding matrix instead of writing $(Df(a))(v)$. This is the case in the theorem that follows.*

Theorem 4.3.2. Let $U \subseteq \mathbb{R}^m$ be an open set and let $f : U \to \mathbb{R}^n$. If f is differentiable at $a \in U$, then all the directional derivatives of f at a exist and

$$f'_a(u) = Df(a) \cdot u$$

Chapter 5

C^k and C^∞ functions

$C^\infty(U)$ is a commutative ring from [3]

5.1 A User's guide to C^∞ functions

Chapter 6

Differential Forms

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