We jump to the proposition.

Proposition 1 (Multiple integration)**.** *If $f(x)$ is smooth enough, we have*

$$I_n f(x) := I_0^{x_0} \prod_{i=1}^{n} (I_0^x) f(x) = \int_0^{x_0} \frac{(x_0 - x)^n}{n!} f(x)\,\mathrm{d}x, \tag{1}$$

where $n \in \mathbb{N}$.

Proof. We extend the definition of I as

$$I_{n,m} f(x) := I_0^{x_0} g_m(x) \prod_{i=1}^{n} (I_0^x) f(x), \tag{2}$$

where $g_m(x)$ is defined as

$$g_m(x) := \frac{(x_0 - x)^m}{m!}, \tag{3}$$

and we have

$$\begin{aligned}
I_{n,m} f(x) &= \int_0^{x_0} g_m(x) \underbrace{\int_0^x \cdots \int_0^x}_{n} f(x) \underbrace{\mathrm{d}x \cdots \mathrm{d}x}_{n+1} \\
&= g_{m+1}(x) \underbrace{\int_0^x \cdots \int_0^x}_{n} f(x) \underbrace{\mathrm{d}x \cdots \mathrm{d}x}_{n} \Big|_{x_0}^{0} + \int_0^{x_0} g_{m+1}(x) \underbrace{\int_0^x \cdots \int_0^x}_{n-1} f(x) \underbrace{\mathrm{d}x \cdots \mathrm{d}x}_{n} \\
&= \int_0^{x_0} g_{m+1}(x) \underbrace{\int_0^x \cdots \int_0^x}_{n-1} f(x) \underbrace{\mathrm{d}x \cdots \mathrm{d}x}_{n} \\
&= I_{n-1,m+1} f(x) \\
&= \cdots \\
&= I_{0,m+n} f(x) \\
&= \int_0^{x_0} g_{m+n}(x) f(x)\,\mathrm{d}x.
\end{aligned}$$

Also, note that $I_n f(x) = I_{n,0} f(x)$, which implies $I_n f(x) = I_{0,n} f(x)$, completing the proof. □