We start with a lemma.

Lemma 1 (Exchanging the order of integration)**.** *If $f(x)$ is smooth enough, we have*

$$\int_0^a\int_0^x f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^a\int_y^a f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

where $a \geq 0$.

The 'proof' can be shown in Figure 1.

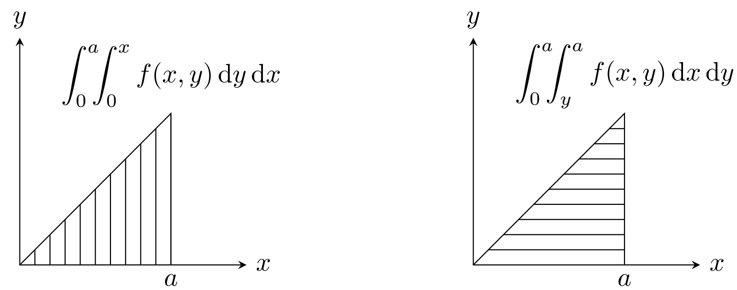

Figure 1: 'Proof' of Lemma 1.

And a definition.

Definition 2 (The integrator I)**.** *$I_a^b f(x)$ is defined as*

$$I_a^b f(x) := \int_a^b f(x)\,\mathrm{d}x. \tag{2}$$

Then comes the proposition.

Proposition 3 (Multiple integration)**.** *If $f(x)$ is smooth enough, we have*

$$I_n f(x) := I_0^{x_0}\prod_{i=1}^{n}(I_0^x)f(x) = \int_0^{x_0}\frac{(x_0-x)^n}{n!}f(x)\,\mathrm{d}x, \tag{3}$$

where $n \in \mathbb{N}$.

Proof. First, the proposition holds for $n = 0$. Then we assume it holds for $n = k$, and we have

$$\begin{aligned}
I_{k+1}f(x) = I_k\big(I_0^x f(x)\big) &= \int_0^{x_0}\frac{(x_0-x)^k}{k!}\left(\int_0^x f(t)\,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_0^{x_0}\int_0^x \frac{(x_0-x)^k}{k!}f(t)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^{x_0}\int_t^{x_0} \frac{(x_0-x)^k}{k!}f(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^{x_0}\frac{(x_0-t)^{k+1}}{(k+1)!}f(t)\,\mathrm{d}t \\
&= \int_0^{x_0}\frac{(x_0-x)^{k+1}}{(k+1)!}f(x)\,\mathrm{d}x. \quad \square
\end{aligned}$$