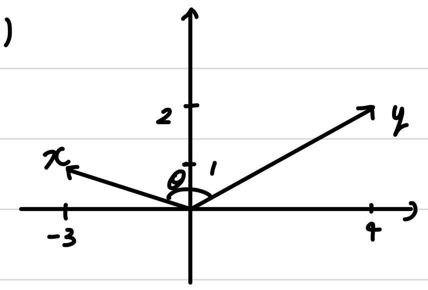
2019150445 신백흑

Q1.

a)



b) $L_x = \sqrt{9+1} = \sqrt{10}$

c) $\cos\theta = \dfrac{x'y}{L_x L_y} = \dfrac{-10}{10\sqrt{2}} = -\dfrac{1}{\sqrt{2}}$

$\therefore \theta = \frac{2}{4}\pi$

d) Projection of x on $y = \dfrac{x'y}{L_y} \cdot \dfrac{1}{L_y} y$

$= -\frac{1}{2}(4,2)$

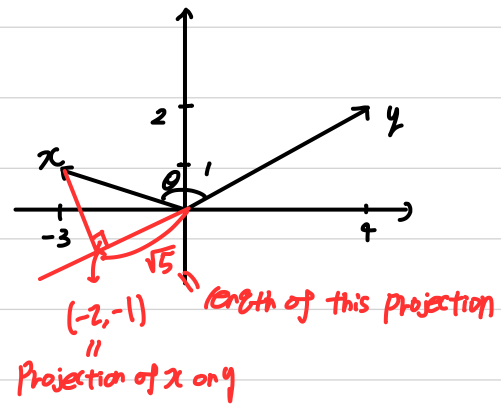
$= (-2,-1)$

Length of Projection $= L(x)|\cos\theta|$

$= \sqrt{10} \cdot \frac{1}{\sqrt{2}}$

$= \sqrt{5}$

e)



Q2.

$u_1 = x_1 = (-1,1,0) \xrightarrow{\text{unit length}} \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$u_2 = x_2 - \dfrac{(x_2'u_1)}{u_1'u_1} u_1$

$= (3,0,-1) - \dfrac{-3}{2}(-1,1,0)$

$= \left(\frac{3}{2}, \frac{3}{2}, -1\right) \xrightarrow{\text{u.l}} \left(\frac{3}{\sqrt{22}}, \frac{3}{\sqrt{22}}, -\frac{2}{\sqrt{22}}\right)$

$u_3 = (1,1,-2) - \frac{0}{2}(-1,1,0) - \dfrac{5}{11/2}\left(\frac{3}{2}, \frac{3}{2}, -1\right)$

$= \left(-\frac{4}{11}, -\frac{4}{11}, -\frac{12}{11}\right) \xrightarrow{\text{unit length}} \left(-\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}\right)$

Q3.

a) $\begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\rightarrow -x_1 + x_2 - 2x_3 = 0$

$x_1 + x_2 + 2x_3 = 0$

$-x_1 - x_2 + 2x_3 = 0$

$\rightarrow x_1 = x_2 = x_3 = 0$

b) $\begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\rightarrow$ rank=3

c)

#rows = #rank $\Rightarrow$ nonsingular.

Q4.

$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

a) $|A| = 1 \times \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$= (-1-1) - 2 \times (1+1)$

$\neq 0$

$\Rightarrow$ Columns of the matrix A are mutually perpendicular.

b) $C = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$

c) It is mutually perpendicular & have unit lengths.

$\Rightarrow$ A is orthogonal

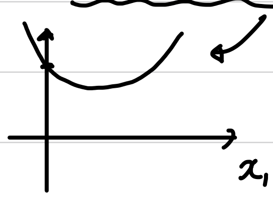
Q5.

a) $x'Ax = (x_1\ x_2)\begin{pmatrix}9 & -2\\ -2 & 6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

$= (9x_1-2x_2\ ,\ -2x_1+6x_2)\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

$= 9x_1^2-2x_1x_2-2x_1x_2+6x_2^2$

$= \underbrace{9x_1^2-4x_2x_1+6x_2^2}$



Always positive $\forall x \neq \begin{pmatrix}0\\ 0\end{pmatrix}$

b) $\begin{vmatrix}9-\lambda & -2\\ -2 & 6-\lambda\end{vmatrix} = 0$

$\rightarrow \lambda^2-15\lambda+50=0$

$\rightarrow \lambda_1=10, \lambda_2=5$ (eigenvalues)

$\begin{pmatrix}9 & -2\\ -2 & 6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 10\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ $\quad$ $\begin{pmatrix}9 & -2\\ -2 & 6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 5\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

$\rightarrow 9x_1-2x_2=10x_1$ & $-2x_1+6x_2=10x_2$ $\quad$ $\rightarrow 9x_1-2x_2=5x_1$ & $-2x_1+6x_2=5x_2$

$\rightarrow x_1=-2x_2$ $\quad$ $\rightarrow 2x_1=x_2$

$\therefore x=\begin{pmatrix}-2/\sqrt{5}\\ 1/\sqrt{5}\end{pmatrix}$ $\quad$ $\therefore x=\begin{pmatrix}1/\sqrt{5}\\ 2/\sqrt{5}\end{pmatrix}$ (eigenvectors)

c)

$A=\begin{pmatrix}9 & -2\\ -2 & 6\end{pmatrix}=\sum_{i=1}^{2}\lambda_i x_i x_i'$

$=10\begin{pmatrix}-2/\sqrt{5}\\ 1/\sqrt{5}\end{pmatrix}\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}+5\begin{pmatrix}1/\sqrt{5}\\ 2/\sqrt{5}\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}$

d) $A^{-1/2}=\sum_{i=1}^{2}\frac{1}{\lambda_i^2}x_i x_i'$

$=\frac{1}{100}\begin{pmatrix}4/5 & -2/5\\ -2/5 & 1/5\end{pmatrix}+\frac{1}{25}\begin{pmatrix}1/5 & 2/5\\ 2/5 & 4/5\end{pmatrix}$

$=\begin{pmatrix}\frac{8}{500} & \frac{6}{500}\\ \frac{6}{500} & \frac{17}{500}\end{pmatrix}$

e)

Eigenvalues = 10^5, 5^5

eigenvectors = $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

Q6.

a) $\underbrace{(x_1, x_2)}_{x'} \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$

$D<0 \Rightarrow$ Positive definite

b) $\underbrace{(x_1, x_2)}_{x'} \underbrace{\begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$

$D<0 \Rightarrow$ Positive definite

c) $\underbrace{(x_1, x_2)}_{x'} \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$

$D>0 \Rightarrow$ positive definite ×

Q7

$$\max_{x \neq 0} \frac{(x'd)^2}{x'Ax} = d'A^{-1}d$$

$$= (-1\ \ 2)\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= (-1\ \ 3)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= 7$$

Q8.

$$4x_1^2+4x_2^2-6x_1x_2 = \frac{(x_1, x_2)\overset{\text{let } B}{\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{(x_1, x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \frac{x'Bx}{x'x}$$

$$|B-\lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ -3 & 4-\lambda \end{vmatrix} \overset{\text{set}}{=} 0$$

$\rightarrow 16-8\lambda+\lambda^2-9=0$

$\rightarrow \lambda^2-8\lambda+7=0. \rightarrow \lambda_1=7, \lambda_2=1$

$$\max_{x\neq 0} \frac{x'Bx}{x'x} = \lambda_1 = 7$$

$$\min_{x\neq 0} \frac{x'Bx}{x'x} = \lambda_2 = 1$$

Q9.

$$|A-\lambda I| = \begin{vmatrix} 10-\lambda & 2 & -2 \\ 2 & 13-\lambda & -4 \\ -2 & -4 & 10-\lambda \end{vmatrix} \overset{\text{set}}{=} 0$$

$$= \begin{vmatrix} 10-\lambda & 2 & -2 \\ 2 & 13-\lambda & -4 \\ 0 & 9-\lambda & 6-\lambda \end{vmatrix}$$

$$= -(9-\lambda)\begin{vmatrix} 10-\lambda & -2 \\ 2 & -4 \end{vmatrix} + (6-\lambda)\begin{vmatrix} 10-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix}$$

$$= (9-\lambda)(36-4\lambda) + (6-\lambda)(126-23\lambda+\lambda^2) = 0$$

$$4(\lambda-9)^2 \qquad (\lambda-9)(\lambda-14)$$

$$= (\lambda-9)(4\lambda-36-\lambda^2+20\lambda-84) = 0$$

$$= (\lambda-9)(\lambda^2-24\lambda+120) = 0$$

$$\lambda = 9,\ 12 \pm 2\sqrt{6}$$

$$\rightarrow \lambda_1 = 12+2\sqrt{6},\ \lambda_2 = 9,\ \lambda_3 = 12-2\sqrt{6}.$$

$$\rightarrow \max_{x\neq 0} \frac{x'Ax}{x'x} = 12+2\sqrt{6},$$

$$\min_{x\neq 0} \frac{x'Ax}{x'x} = 12-2\sqrt{6}$$

Q10.

$$\frac{z'Bz}{z'z} = \frac{z'P\Lambda P'z}{z'P\underbrace{P'z}_{=y}}$$

$$= \frac{y'\Lambda y}{y'y}$$

$$= \frac{\sum_{i=1}^{k} \lambda_i y_i^2}{\sum_{i=1}^{k} y_i^2} \geq \frac{\sum_{i=1}^{k} \lambda_k y_i^2}{\sum_{i=1}^{k} y_i^2} \quad (\lambda_k \leq \lambda_{k-1} \cdots \leq \lambda_1)$$

$$= \lambda_k.$$

$$\therefore \min_{z \neq 0} \frac{z'Bz}{z'z} = \lambda_k.$$

With $z = x_k$,

$$y = P'x_k = (x_1 \cdots x_k)'x_k = \begin{pmatrix} x_1'x_k \\ \vdots \\ \vdots \\ x_k'x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (\because x_k'x_k = 1,\ x_i'x_k = 0,\ i \neq k)$$

$$\Rightarrow y'\Lambda y = (0 \cdots 1)\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_k \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \lambda_k.$$

$\therefore$ when $z = x_k$, $\frac{z'Bz}{z'z} = \lambda_k$