

# Dynamic Programming: Preliminaries

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# Preliminary topics

- Reminders, notation
- The curse of dimensionality
- (General state) Markov chains
- Nonlinear functional equations

## References

- Stokey and Lucas (1989)
- Stachurski (2009)

## Reminder 1: Distributions

Let  $S$  be a nonempty set

A **distribution**  $\phi$  on  $S$  is a function that assigns probabilities to subsets of  $S$ :

$$\phi(B) = \text{probability mass assigned to } B \subset S$$

I'll often use notation such as

$$\int g(x)\phi(\mathrm{d}x)$$

Think of this as  $\mathbb{E} g(X)$  when  $X \sim \phi$

**Example.** If  $S = \mathbb{R}$  and  $\phi$  has a density  $f$ , then

$$\int g(x)\phi(\mathrm{d}x) = \int_{-\infty}^{\infty} g(x)f(x) \mathrm{d}x$$

**Example.** If  $S = \mathbb{R}^2$  and  $\phi$  has a density  $f$ , then

$$\int g(x)\phi(\mathrm{d}x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2)f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$$

**Example.** If  $S = \{1, 2, \dots, n\}$ , then

$$\int g(x)\phi(\mathrm{d}x) = \sum_{i=1}^n g(x_i)\phi(x_i)$$

## Reminder 2: Metric Spaces

Let  $\mathcal{G}$  be a nonempty set and let  $\rho$  map  $\mathcal{G} \times \mathcal{G}$  to  $\mathbb{R}$

The pair  $(\mathcal{G}, \rho)$  is called a **metric space** if, for any  $x, y, z$  in  $\mathcal{G}$ ,

- $\rho(x, y) = 0$  if and only if  $x = y$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

**Example.**  $\mathcal{G} = \mathbb{R}^n$  and  $\rho(x, y) = \|x - y\|$

**Example.**  $S \subset \mathbb{R}^n$  and  $\mathcal{C}$  is all continuous bounded functions from  $S$  to  $\mathbb{R}$ ,

$$\rho(f, g) := \sup_{x \in S} |f(x) - g(x)|$$

The three axioms hold for  $(\mathcal{C}, \rho)$

For example, if  $f, g$  and  $h$  are in  $\mathcal{C}$  and  $x \in S$ , then

$$\begin{aligned} |f(x) - h(x)| &= |f(x) - g(x) - (g(x) - h(x))| \\ &\leq |f(x) - g(x)| + |g(x) - h(x)| \\ &\leq \rho(f, g) + \rho(g, h) \end{aligned}$$

$$\therefore \rho(f, h) \leq \rho(f, g) + \rho(g, h)$$

# Optimization and Computers

Some optimization problems are pretty easy

- All functions are differentiable
- Few choice variables (low dimensional)
- Concave (for max) or convex (for min)
- First order / tangency conditions relatively simple

Textbook examples often chosen to have this structure

In reality many problems don't have this structure

- Can't take derivatives
- No analytical solution for FOCs
- Many choice variables (high dimensional)
- Neither concave nor convex — local maxima and minima



# Can Computers Save Us?

For any function we can always try brute force optimization

Here's an example for the following function

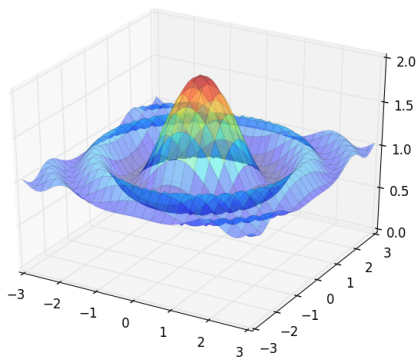
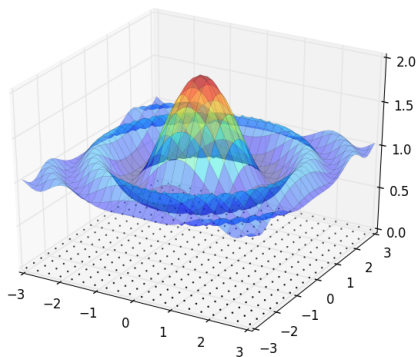


Figure: The function to maximize



**Figure:** Grid of points to evaluate the function at

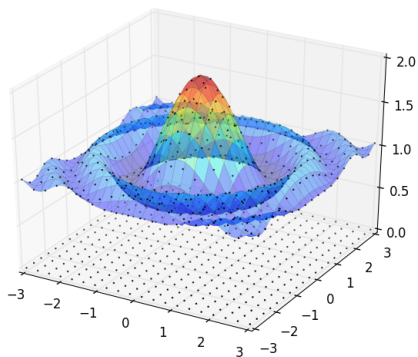


Figure: Evaluations

Grid size =  $20 \times 20 = 400$

## Outcomes

- Number of function evaluations = 400
- Time taken = almost zero
- Maximal value recorded = 1.951
- True maximum = 2

Not bad and we can easily do better

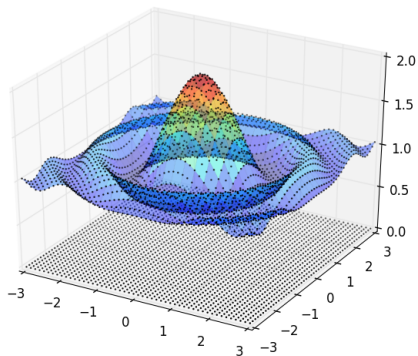


Figure:  $50^2 = 2500$  evaluations

- Number of function evaluations =  $50^2$
- Time taken =  $400 \mu s$
- Maximal value recorded = 1.992
- True maximum = 2

So why even study optimization?

The problem is mainly with larger numbers of choice variables

- 3 vars:  $\max f(x_1, x_2, x_3)$
- 4 vars:  $\max f(x_1, x_2, x_3, x_4)$
- ...

If we have 50 grid points per variable and

- 2 variables then evaluations  $= 50^2 = 2500$
- 3 variables then evaluations  $= 50^3 = 125,000$
- 4 variables then evaluations  $= 50^4 = 6,250,000$
- 5 variables then evaluations  $= 50^5 = 312,500,000$
- ...



**Example.** Recent study: Optimal placement of drinks across vending machines in Tokyo

Approximate dimensions of problem:

- Number of choices for each variable = 2
- Number of choice variables = 1000

Hence number of possibilities =  $2^{1000}$

How big is that?

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```
In [10]: 2**1000
```

```
Out[10]:
```

```
107150860718626732094842504906000181056140481170  
553360744375038837035105112493612249319837881569  
585812759467291755314682518714528569231404359845  
775746985748039345677748242309854210746050623711  
418779541821530464749835819412673987675591655439  
460770629145711964776865421676604298316526243868  
37205668069376
```

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Let's say my machine can evaluate about 1 billion possibilities per second

How long would that take?

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```
In [16]: (2**1000 / 10**9) / 31556926  # In years
```

```
Out[16]:
```

```
339547840365144349278007955863635707280678989995  
899349462539661933596146571733926965255861364854  
060286985707326991591901311029244639453805988092  
045933072657455119924381235072941549332310199388  
301571394569707026437986448403352049168514244509  
939816790601568621661265174170019913588941596
```

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## What about high performance computing?

- more powerful hardware
- faster CPUs
- GPUs
- vector processors
- cloud computing
- massively parallel supercomputers
- ...

Let's say speed up is  $10^{12}$  (wildly optimistic)

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```
In [19]: (2**1000 / 10**(9 + 12)) / 31556926
```

```
Out[19]:
```

```
3395478403651443492780079558636357072806789899958  
9934946253966193359614657173392696525586136485406  
0286985707326991591901311029244639453805988092045  
9330726574551199243812350729415493323101993883015  
7139456970702643798644840335204916851424450993981  
6790601568621661265174170019
```

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For comparison:

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```
In [20]: 5 * 10**9 # Expected lifespan of sun
```

```
Out[20]: 5000000000
```

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Message: There are serious limits to computation

What's required is clever analysis

Exploit what information we have

- without information (oracle) we're stuck
- with information / structure we can do clever things

Examples later on...

# Markov Chains on General Spaces

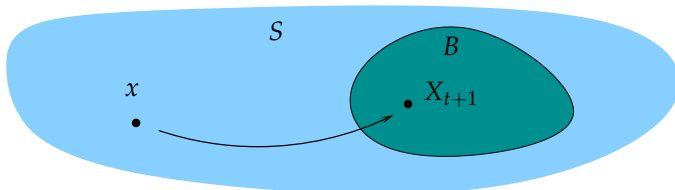
Let

- $S$  be any set (called the **state space**)
- $P(x, dy)$  be a **stochastic kernel** on  $S$  — a distribution over  $S$  for each  $x \in S$

If  $\{X_t\}$  is a stochastic process satisfying

$$P(x, B) = \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$$

then called a **Markov process** with stochastic kernel  $P$





**Example.** Let  $\{W_t\}$  be an IID sequence with distribution  $\phi$

Consider the stochastic difference equation

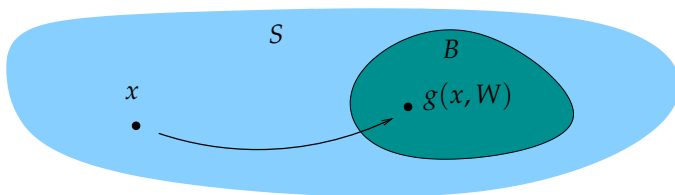
$$X_{t+1} = g(X_t, W_{t+1}) \quad \text{with} \quad X_0 = x_0$$

Each  $X_t$  takes values in  $S$ , a set of

- vectors in  $\mathbb{R}^n$ , or
- scalars in  $\mathbb{R}$ , or
- something else...

This is a Markov process with stochastic kernel

$$P(x, B) = \phi\{w \in \mathbb{W} : g(x, w) \in B\}$$



**Example.** Let  $S = \mathbb{R}$ , let  $\{W_t\} \stackrel{\text{iid}}{\sim} N(0, 1)$  and let

$$X_{t+1} = aX_t + b + \sigma W_{t+1}$$

This is a **linear Gaussian** Markov process with kernel

$$P(x, dy) := N(ax + b, \sigma^2)$$

That is,  $P(x, B) = \int_B p(x, y) dy$  where

$$p(x, y) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(y - ax - b)^2}{\sigma^2} \right\}$$

**Example.** Consider the Solow–Swan model

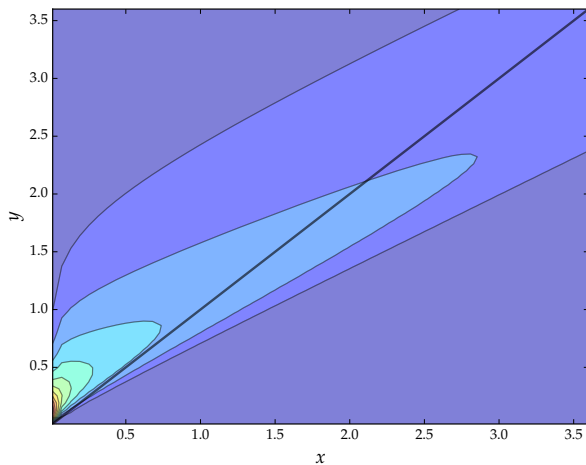
$$k_{t+1} = sf(k_t)W_{t+1} + (1 - \delta)k_t \quad \{W_t\}_{t \geq 1} \stackrel{\text{iid}}{\sim} \phi$$

Here

- $k_t$  takes values in  $S = (0, \infty)$
- $s, \delta \in (0, 1)$  and  $f(k) > 0$  when  $k > 0$

The stochastic kernel is

$$P(k, B) = \phi \{w \in \mathbb{W} \mid sf(k)w + (1 - \delta)k \in B\}$$



## Higher Order Kernels

Fix a Markov process  $\{X_t\}$  with stochastic kernel  $P$

Let  $P^n$  be the  $n$ -step kernel:

$$P^n(x, B) = \mathbb{P}\{X_{t+n} \in B \mid X_t = x\}$$

**Fact.**  $\{P^k\}$  satisfies the **Chapman–Kolmogorov relation**

$$P^{n+k}(x, B) = \int P^k(x, dz) P^n(z, B)$$

**Example.** Recall the process

$$X_{t+1} = g(X_t, W_{t+1}) \quad \text{with} \quad X_0 = x$$

The  $n = 2$  kernel is

$$P^2(x, B) =$$

$$(\phi \times \phi) \{ (w_1, w_2) \in \mathbb{W} \times \mathbb{W} \mid g(g(x, w_1), w_2) \in B \}$$

Higher order kernels

- $P^3(x, dy) \stackrel{\mathcal{D}}{=} g(g(g(x, W_1), W_2), W_3)$
- etc.

# Markov Operators

Given stochastic kernel  $P$  on  $S$  and  $h: S \rightarrow \mathbb{R}$ , let

$$(Ph)(x) = \int h(y)P(x, \mathrm{d}y) \quad (x \in S)$$

Called the **Markov operator** corresponding to  $P$

**Example.** If  $P$  corresponds to  $X_{t+1} = g(X_t, W_{t+1})$ , then

$$(Ph)(x) = \int h(g(x, w))\phi(\mathrm{d}w)$$



Interpretations:

$$(Ph)(x) = \mathbb{E} [h(X_{t+1}) \mid X_t = x]$$

$$(P^n h)(x) = \mathbb{E} [h(X_{t+n}) \mid X_t = x]$$

# Solving Equations

**Discussion:** When does this **vector equation** in  $\mathbb{R}^n$  have a unique solution?

$$x = Ax + b$$

When does the **method of successive approximations** converge?

1. pick any  $x_0 \in \mathbb{R}^n$
2.  $x_{n+1} = Ax_n + b$

How else could we find a solution?

**Discussion:** Is there a unique  $k \in (0, 1)$  that solves

$$k = sk^\alpha + (1 - \delta)k$$

Does  $k_{n+1} = sk_n^\alpha + (1 - \delta)k_n$  converge to the solution? When?

**Discussion:** Consider the **asset price equation**

$$q_t = \beta \mathbb{E}_t[q_{t+1} + \delta(X_{t+1})]$$

Here  $\{X_t\}$  is Markov  $\sim P$  and  $\delta(X_t)$  is current dividend

Guess  $q_t = q(X_t)$  and rewrite as

$$q(X_t) = \beta \mathbb{E}_t[q(X_{t+1}) + \delta(X_{t+1})]$$

or as the **functional equation**

$$q(x) = \beta \int q(y)P(x, dy) + \beta \int \delta(y)P(x, dy) \quad (x \in S)$$

- Unique solution? How to solve?

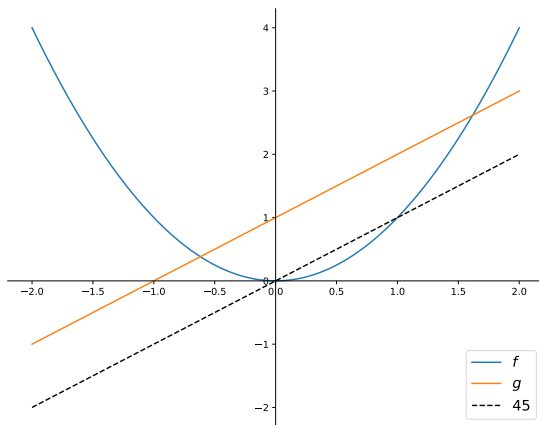
# Fixed Points

Let  $(\mathcal{G}, \rho)$  be a metric space and let  $T: \mathcal{G} \rightarrow \mathcal{G}$

A **fixed point** of  $T$  is a point  $x^* \in \mathcal{G}$  such that  $Tx^* = x^*$

## Examples.

- If  $f(x) = x^2$  on  $\mathbb{R}$ , then 0 and 1 are fixed points
- If  $g(x) = x + 1$  on  $\mathbb{R}$ , then  $g$  has no fixed points on  $\mathbb{R}$



$T$  is called a **contraction map** on  $(\mathcal{G}, \rho)$  if

$$\exists \alpha < 1 \quad \text{such that} \quad \rho(Tx, Ty) \leq \alpha \rho(x, y), \quad \forall x, y \in \mathcal{G}$$

**Example.**  $f(x) = \alpha x + b$  on metric space  $(\mathbb{R}, |\cdot|)$  with  $|\alpha| < 1$ , since

$$|f(x) - f(y)| = |\alpha x - \alpha y| = |\alpha| |x - y|$$

**Fact.** Every contraction  $T$  is continuous on  $\mathcal{G}$

Proof: If  $x_n \rightarrow x$  in  $(\mathcal{G}, \rho)$ , then

$$\rho(Tx_n, Tx) \leq \alpha \rho(x_n, x) \rightarrow 0$$

**Fact.** If  $T$  is a contraction map on  $(\mathcal{G}, \rho)$  and  $x \in \mathcal{G}$ , then  $\{T^k x\}$  is Cauchy

Sketch of proof: Along the trajectory  $\{T^k x\}$  from  $x$ , we have

$$\begin{aligned}\rho(T^{k+1}x, T^k x) &\leq \alpha \rho(T^k x, T^{k-1}x) \\ &\leq \alpha^2 \rho(T^{k-1}x, T^{k-2}x) \\ &\vdots \\ &\leq \alpha^k \rho(Tx, x)\end{aligned}$$



# Banach's Fixed Point Theorem

**Theorem.** If  $(\mathcal{G}, \rho)$  is complete and  $T$  is a contraction, then  $T$  has a unique fixed point  $x^*$  in  $\mathcal{G}$  and, for all  $x \in \mathcal{G}$ ,

$$\lim_{k \rightarrow \infty} \rho(T^k x, x^*) = 0$$

Proof: Pick any  $x \in \mathcal{G}$

The sequence  $\{T^k x\}$  is Cauchy and hence converges to some  $x^*$

The point  $x^*$  is a fixed point, since

$$Tx^* = T(\lim_k T^k x) = \lim_k T(T^k x) = \lim_k T^{k+1} x = x^*$$

Regarding uniqueness, if  $x^*$  and  $x^{**}$  are fixed points of  $T$ , then

$$\rho(x^*, x^{**}) = \rho(Tx^*, Tx^{**}) \leq \alpha \rho(x^*, x^{**})$$

$$\therefore \rho(x^*, x^{**}) = 0$$

$$\therefore x^* = x^{**}$$

## Application: Asset Pricing

Recall the asset pricing equation

$$q(x) = \beta Pq(x) + \beta P\delta(x)$$

Let  $\mathcal{C}$  be all continuous bounded functions on  $S$  with metric

$$\rho(f, g) := \sup_{x \in S} |f(x) - g(x)|$$

Let  $P$  have the **Feller property**, which is to say that

$$h \in \mathcal{C} \implies Ph \in \mathcal{C}$$

Let  $\delta \in \mathcal{C}$  and let  $\beta \in (0, 1)$

**Claim:** The asset pricing equation

$$q(x) = \beta Pq(x) + \beta P\delta(x) \quad (x \in S)$$

has a unique solution  $q^* \in \mathcal{C}$

Remarks:

- We often write this as  $q = \beta Pq + \beta P\delta$
- Equivalent: the operator  $T: \mathcal{C} \rightarrow \mathcal{C}$  defined by

$$Tq = \beta Pq + \beta P\delta$$

has a unique fixed point in  $\mathcal{C}$

To prove this we need to show that

1.  $Tq = \beta Pq + \beta P\delta$  is in  $\mathcal{C}$  when  $q \in \mathcal{C}$
2. the pair  $(\mathcal{C}, \rho)$  forms a complete metric space
3.  $T$  is a contraction map on  $(\mathcal{C}, \rho)$

Here (1) follows from assumption and the proof of (2) is omitted

Regarding (3), fix any  $q, q'$  in  $\mathcal{C}$  and any  $x \in S$

We have,

$$\begin{aligned} |Tq(x) - Tq'(x)| &= |\beta Pq(x) + \beta P\delta(x) - \beta Pq'(x) - \beta P\delta(x)| \\ &= \beta \left| \int q(y)P(x, dy) - \int q'(y)P(x, dy) \right| \\ &= \beta \left| \int [q(y) - q'(y)]P(x, dy) \right| \\ &\leq \beta \int |q(y) - q'(y)|P(x, dy) \\ &\leq \beta \sup_y |q(y) - q'(y)| = \beta \rho(q, q') \end{aligned}$$

Taking the supremum with respect to  $x$  completes the proof