1. Exercise

Claim: $\nabla \times (\nabla \cdot \phi) = 0 = \text{rot}(\text{grad}(\phi))$

Proof: $$\nabla \times (\nabla \cdot \phi) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix} = \begin{pmatrix} \partial_y \partial_z \phi - \partial_z \partial_y \phi \\ \partial_z \partial_x \phi - \partial_x \partial_z \phi \\ \partial_x \partial_y \phi - \partial_y \partial_x \phi \end{pmatrix} = \begin{pmatrix} \partial_y \partial_z \phi - \partial_y \partial_z \phi \\ \partial_z \partial_x \phi - \partial_z \partial_x \phi \\ \partial_x \partial_y \phi - \partial_x \partial_y \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$ □

Claim: $\nabla \cdot (\nabla \times A) = 0 = \text{div}(\text{rot}(A))$

Proof: $$\nabla \cdot (\nabla \times A) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \left[\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \right] = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} =$$

$$= \partial_x \partial_y A_z - \partial_x \partial_z A_y + \partial_y \partial_z A_x - \partial_y \partial_x A_z + \partial_z \partial_x A_y - \partial_z \partial_y A_x =$$

$$= \underbrace{\partial_y \partial_z A_x - \partial_z \partial_y A_x}_{=0} + \underbrace{\partial_z \partial_x A_y - \partial_x \partial_z A_y}_{=0} + \underbrace{\partial_x \partial_y A_z - \partial_y \partial_x A_z}_{=0} = 0$$ □

Claim: $\nabla \times (\nabla \times A) = \nabla \cdot (\nabla \cdot A) - \Delta A$

Proof:

$$\nabla \times (\nabla \times A) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \left[\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \right] = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} =$$

$$= \begin{pmatrix} (\partial_x \partial_y A_y - \partial_y^2 A_x) - (\partial_z^2 A_x - \partial_x \partial_z A_z) \\ (\partial_y \partial_z A_z - \partial_z^2 A_y) - (\partial_x^2 A_y - \partial_x \partial_y A_x) \\ (\partial_x \partial_z A_x - \partial_x^2 A_z) - (\partial_y^2 A_z - \partial_y \partial_z A_y) \end{pmatrix} =$$

$$= \begin{pmatrix} \partial_x (\partial_y A_y + \partial_z A_z) - (\partial_y^2 + \partial_z^2) A_x \\ \partial_y (\partial_x A_x + \partial_z A_z) - (\partial_x^2 + \partial_z^2) A_y \\ \partial_z (\partial_x A_x + \partial_y A_y) - (\partial_x^2 + \partial_y^2) A_z \end{pmatrix} \pm \begin{pmatrix} \partial_x^2 A_x \\ \partial_y^2 A_y \\ \partial_z^2 A_z \end{pmatrix} =$$

$$= \begin{pmatrix} \partial_x \, \nabla \cdot A - \Delta A_x \\ \partial_y \, \nabla \cdot A - \Delta A_y \\ \partial_z \, \nabla \cdot A - \Delta A_z \end{pmatrix} = \nabla \cdot (\nabla \cdot A) - \Delta A$$ □