

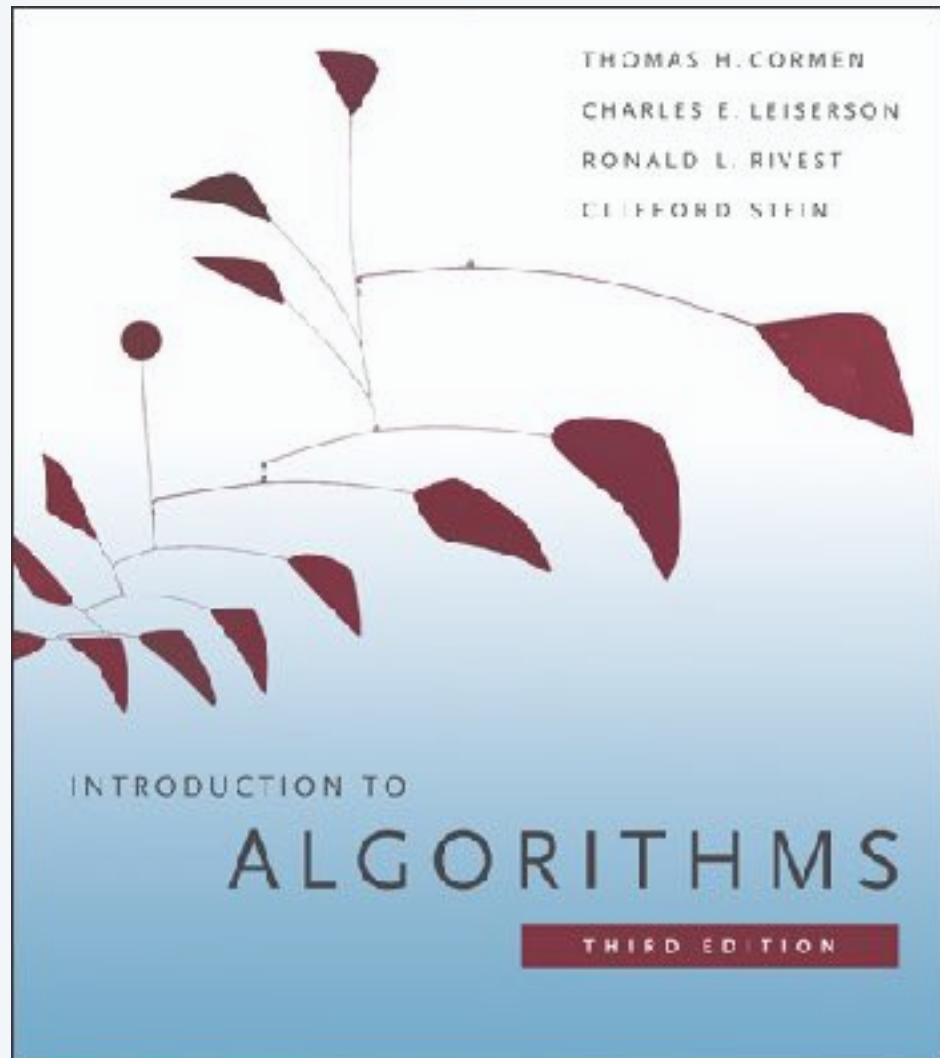
DIVIDE AND CONQUER II

- ▶ *master theorem*
- ▶ *integer multiplication*
- ▶ *matrix multiplication*
- ▶ *convolution and FFT*

Lecture slides by Kevin Wayne

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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>



SECTIONS 4.3–4.6

DIVIDE AND CONQUER II

- ▶ *master theorem*
- ▶ *integer multiplication*
- ▶ *matrix multiplication*
- ▶ *convolution and FFT*

Master method

Goal. Recipe for solving common divide-and-conquer recurrences:

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

Terms.

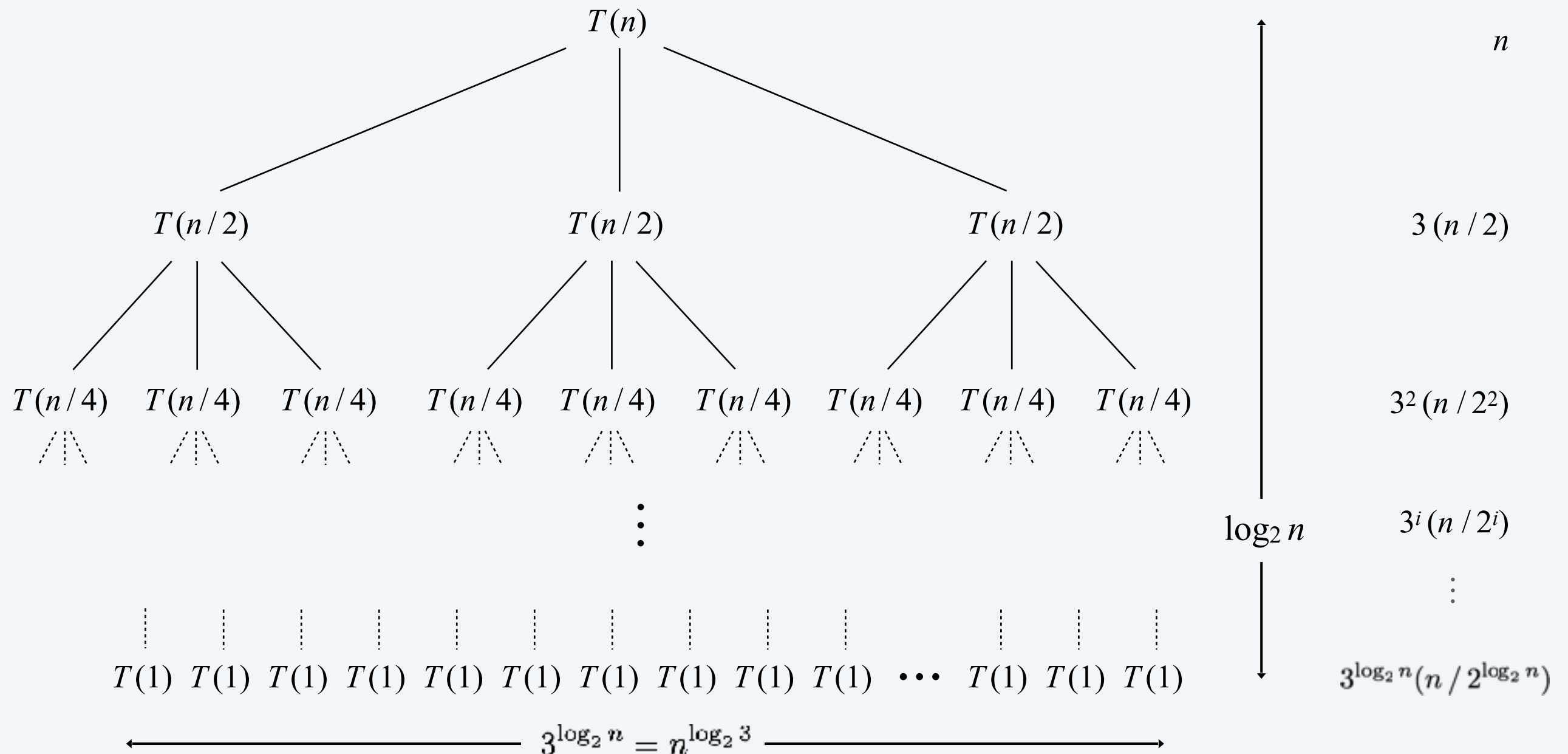
- $a \geq 1$ is the number of subproblems.
- $b > 0$ is the factor by which the subproblem size decreases.
- $f(n)$ = work to divide/merge subproblems.

Recursion tree.

- $k = \log_b n$ levels.
- a^i = number of subproblems at level i .
- n / b^i = size of subproblem at level i .

Case 1: total cost dominated by cost of leaves

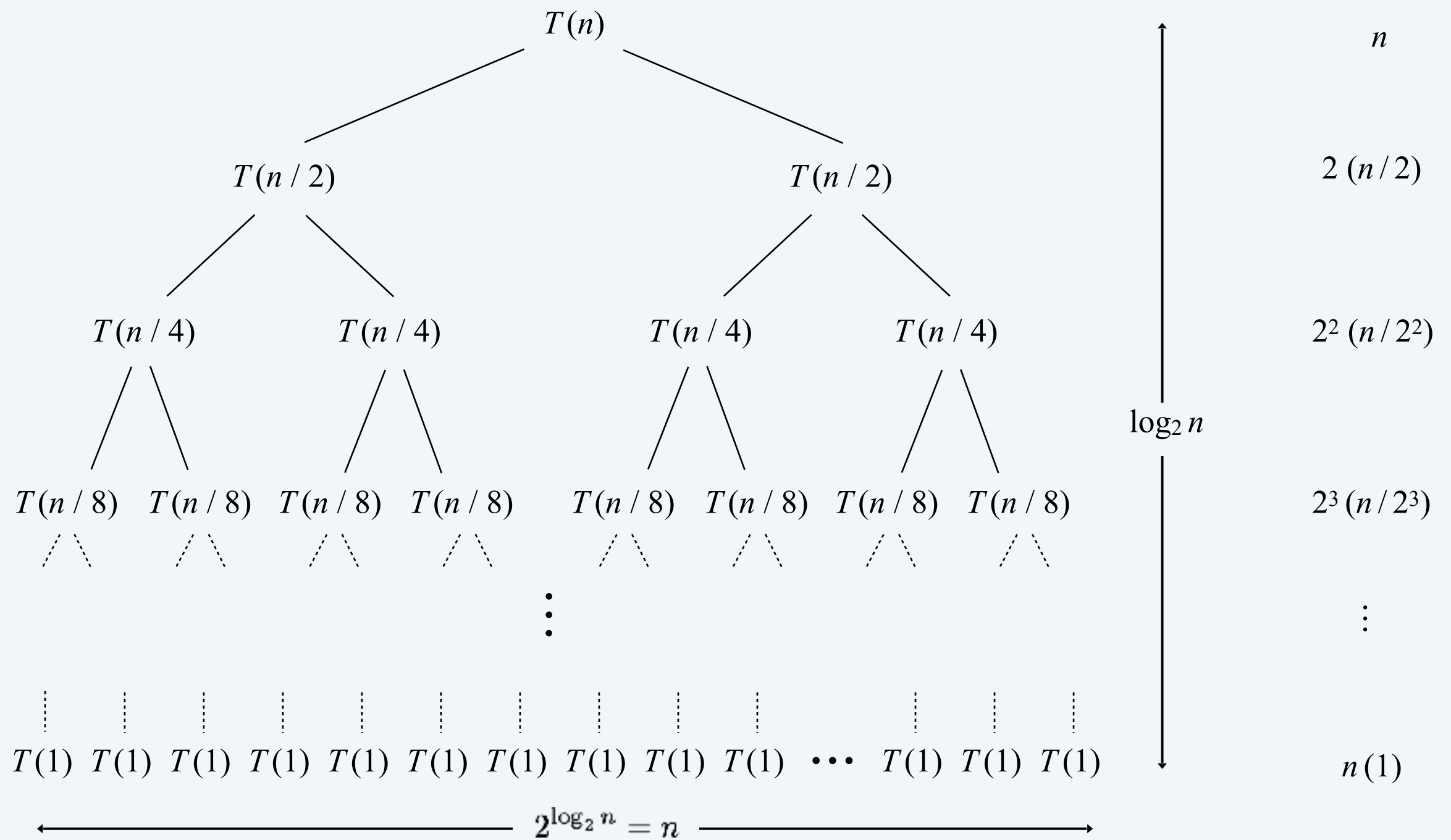
Ex 1. If $T(n)$ satisfies $T(n) = 3 T(n / 2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n^{\lg 3})$.



$$r = 3 / 2 > 1 \quad T(n) = (1 + r + r^2 + r^3 + \dots + r^{\log_2 n}) n = \frac{r^{1+\log_2 n} - 1}{r - 1} n = 3n^{\log_2 3} - 2n$$

Case 2: total cost evenly distributed among levels

Ex 2. If $T(n)$ satisfies $T(n) = 2 T(n / 2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n \log n)$.

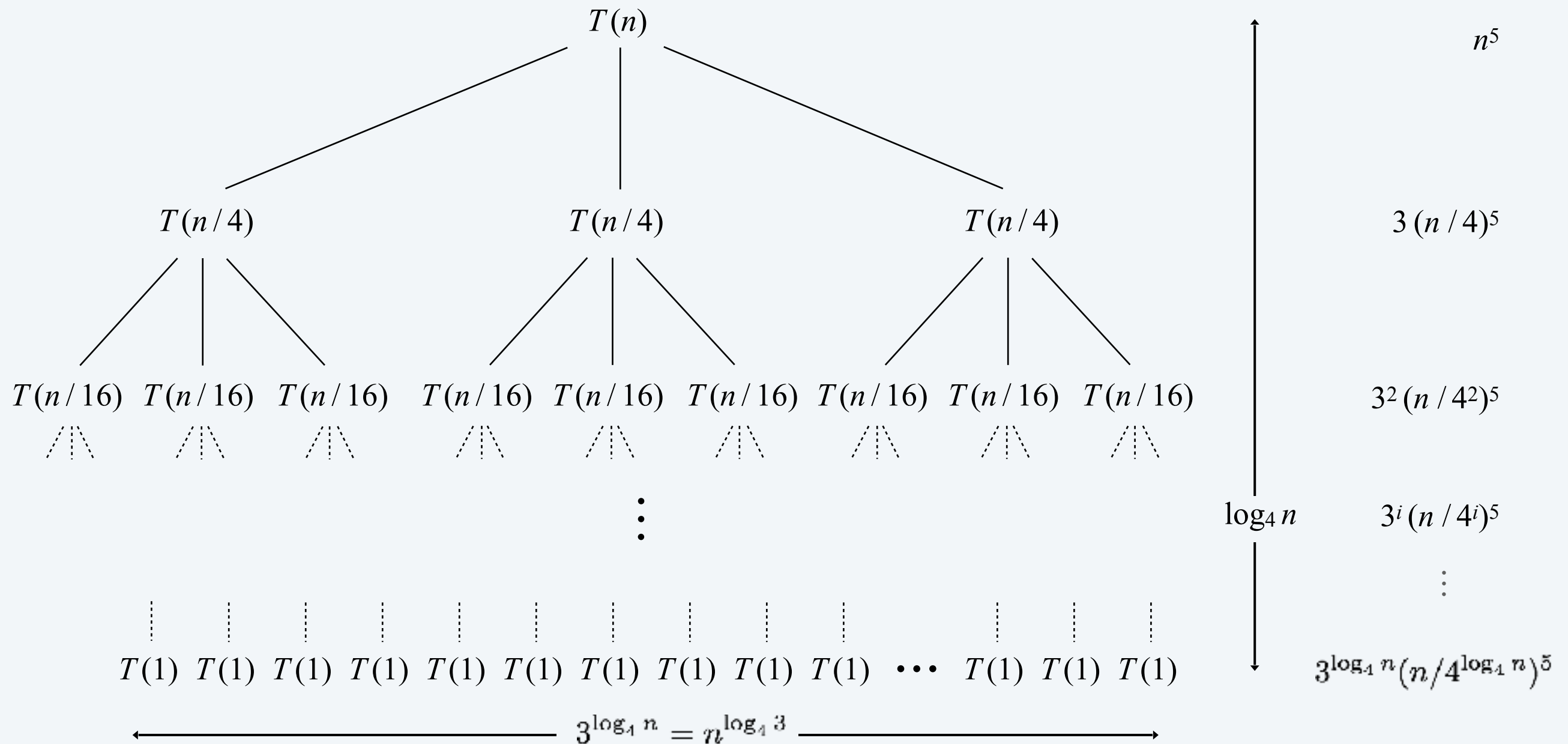


$$r = 1$$

$$T(n) = (1 + r + r^2 + r^3 + \dots + r^{\log_2 n}) n = n (\log_2 n + 1)$$

Case 3: total cost dominated by cost of root

Ex 3. If $T(n)$ satisfies $T(n) = 3 T(n / 4) + n^5$, with $T(1) = 1$, then $T(n) = \Theta(n^5)$.



$$r = 3 / 4^5 < 1 \quad n^5 \leq T(n) \leq (1 + r + r^2 + r^3 + \dots) n^5 \leq \frac{1}{1 - r} n^5$$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Ex. $T(n) = 3 T(n/2) + n$.

- $a = 3, b = 2, f(n) = n, k = \log_2 3$.
- $T(n) = \Theta(n^{\lg 3})$.

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Ex. $T(n) = 2 T(n/2) + \Theta(n \log n)$.

- $a = 2, b = 2, f(n) = 17n, k = \log_2 2 = 1, p = 1$.
- $T(n) = \Theta(n \log^2 n)$.


Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

regularity condition holds
if $f(n) = \Theta(n^{k+\varepsilon})$



Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a f(n/b) \leq c f(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$.

Ex. $T(n) = 3 T(n/4) + n^5$.

- $a = 3, b = 4, f(n) = n^5, k = \log_4 3$.
- $T(n) = \Theta(n^5)$.

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

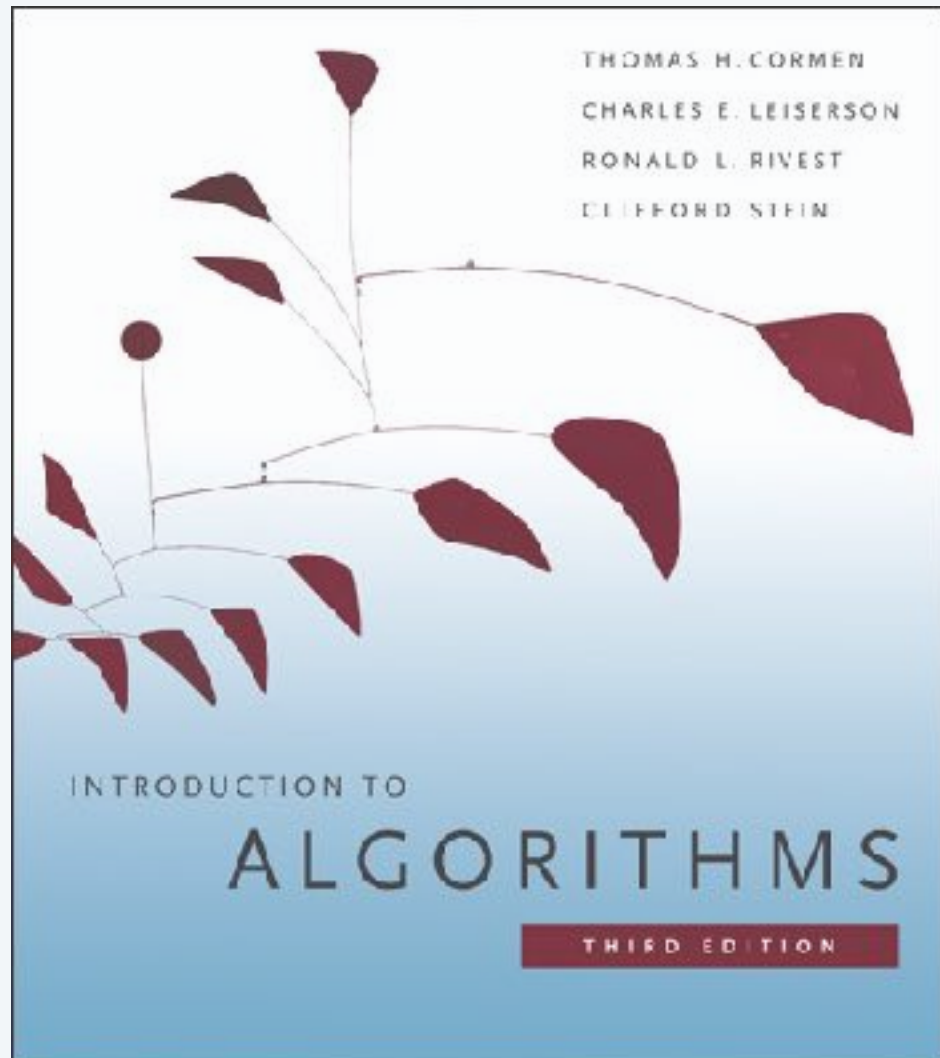
Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a f(n/b) \leq c f(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$.

Pf sketch.

- Use recursion tree to sum up terms (assuming n is an exact power of b).
- Three cases for geometric series.
- Deal with floors and ceilings.



SECTION 4.2


DIVIDE AND CONQUER II

- ▶ *master theorem*
- ▶ *integer multiplication*
- ▶ ***matrix multiplication***
- ▶ *convolution and FFT*

Dot product

Dot product. Given two length n vectors a and b , compute $c = a \cdot b$.

Grade-school. $\Theta(n)$ arithmetic operations.


$$a \cdot b = \sum_{i=1}^n a_i b_i$$

$$a = [.70 \quad .20 \quad .10]$$

$$b = [.30 \quad .40 \quad .30]$$

$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

Remark. Grade-school dot product algorithm is asymptotically optimal.

Matrix multiplication

Matrix multiplication. Given two n -by- n matrices A and B , compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Is grade-school matrix multiplication algorithm asymptotically optimal?

Block matrix multiplication

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix multiplication: warmup

To multiply two n -by- n matrices A and B :

- Divide: partition A and B into $\frac{1}{2}n$ -by- $\frac{1}{2}n$ blocks.
- Conquer: multiply 8 pairs of $\frac{1}{2}n$ -by- $\frac{1}{2}n$ matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$
$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

Running time. Apply case 1 of Master Theorem.

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Strassen's trick

Key idea. multiply 2-by-2 blocks with only **7** multiplications.

(plus 11 additions and 7 subtractions)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

$$C_{12} = P_1 + P_2$$

$$C_{21} = P_3 + P_4$$

$$C_{22} = P_1 + P_5 - P_3 - P_7$$

$$P_1 \leftarrow A_{11} \times (B_{12} - B_{22})$$

$$P_2 \leftarrow (A_{11} + A_{12}) \times B_{22}$$

$$P_3 \leftarrow (A_{21} + A_{22}) \times B_{11}$$

$$P_4 \leftarrow A_{22} \times (B_{21} - B_{11})$$

$$P_5 \leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22})$$

$$P_6 \leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22})$$

$$P_7 \leftarrow (A_{11} - A_{21}) \times (B_{11} + B_{12})$$

Pf. $C_{12} = P_1 + P_2$
 $= A_{11} \times (B_{12} - B_{22}) + (A_{11} + A_{12}) \times B_{22}$
 $= A_{11} \times B_{12} + A_{12} \times B_{22}. \checkmark$

Strassen's algorithm

STRASSEN(n, A, B)

IF ($n = 1$) RETURN $A \times B$.

Partition A and B into 2-by-2 block matrices.

$P_1 \leftarrow \text{STRASSEN}(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow \text{STRASSEN}(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow \text{STRASSEN}(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow \text{STRASSEN}(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow \text{STRASSEN}(n / 2, (A_{11} + A_{22}) \times (B_{11} + B_{22}))$.

$P_6 \leftarrow \text{STRASSEN}(n / 2, (A_{12} - A_{22}) \times (B_{21} + B_{22}))$.

$P_7 \leftarrow \text{STRASSEN}(n / 2, (A_{11} - A_{21}) \times (B_{11} + B_{12}))$.

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C .

assume n is
a power of 2

keep track of indices of submatrices
(don't copy matrix entries)

Analysis of Strassen's algorithm

Theorem. Strassen's algorithm requires $O(n^{2.81})$ arithmetic operations to multiply two n -by- n matrices.

Pf. Apply case 1 of the master theorem to the recurrence:

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Q. What if n is not a power of 2 ?

A. Could pad matrices with zeros.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 10 & 11 & 12 & 0 \\ 13 & 14 & 15 & 0 \\ 16 & 17 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 84 & 90 & 96 & 0 \\ 201 & 216 & 231 & 0 \\ 318 & 342 & 366 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Strassen's algorithm: practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm when n is "small" .

Common misperception. “*Strassen is only a theoretical curiosity.*”

- Apple reports 8x speedup on G4 Velocity Engine when $n \approx 2,048$.
- Range of instances where it's useful is a subject of controversy.

Linear algebra reductions

Matrix multiplication. Given two n -by- n matrices, compute their product.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	$\Theta(MM(n))$
matrix inversion	A^{-1}	$\Theta(MM(n))$
determinant	$ A $	$\Theta(MM(n))$
system of linear equations	$Ax = b$	$\Theta(MM(n))$
LU decomposition	$A = LU$	$\Theta(MM(n))$
least squares	$\min \ Ax - b\ _2$	$\Theta(MM(n))$

numerical linear algebra problems with the same complexity as matrix multiplication

Fast matrix multiplication: theory

Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?

A. Yes! [Strassen 1969]

$$\Theta(n^{\log_2 7}) = O(n^{2.807})$$

Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?

A. Impossible. [Hopcroft and Kerr 1971]

$$\Theta(n^{\log_2 6}) = O(n^{2.59})$$

Q. Multiply two 3-by-3 matrices with 21 scalar multiplications?

A. Unknown.

$$\Theta(n^{\log_3 21}) = O(n^{2.77})$$

Begun, the decimal wars have. [Pan, Bini et al, Schönhage, ...]

- Two 20-by-20 matrices with 4,460 scalar multiplications.

$$O(n^{2.805})$$

- Two 48-by-48 matrices with 47,217 scalar multiplications.

$$O(n^{2.7801})$$

- A year later.

$$O(n^{2.7799})$$

- December 1979.

$$O(n^{2.521813})$$

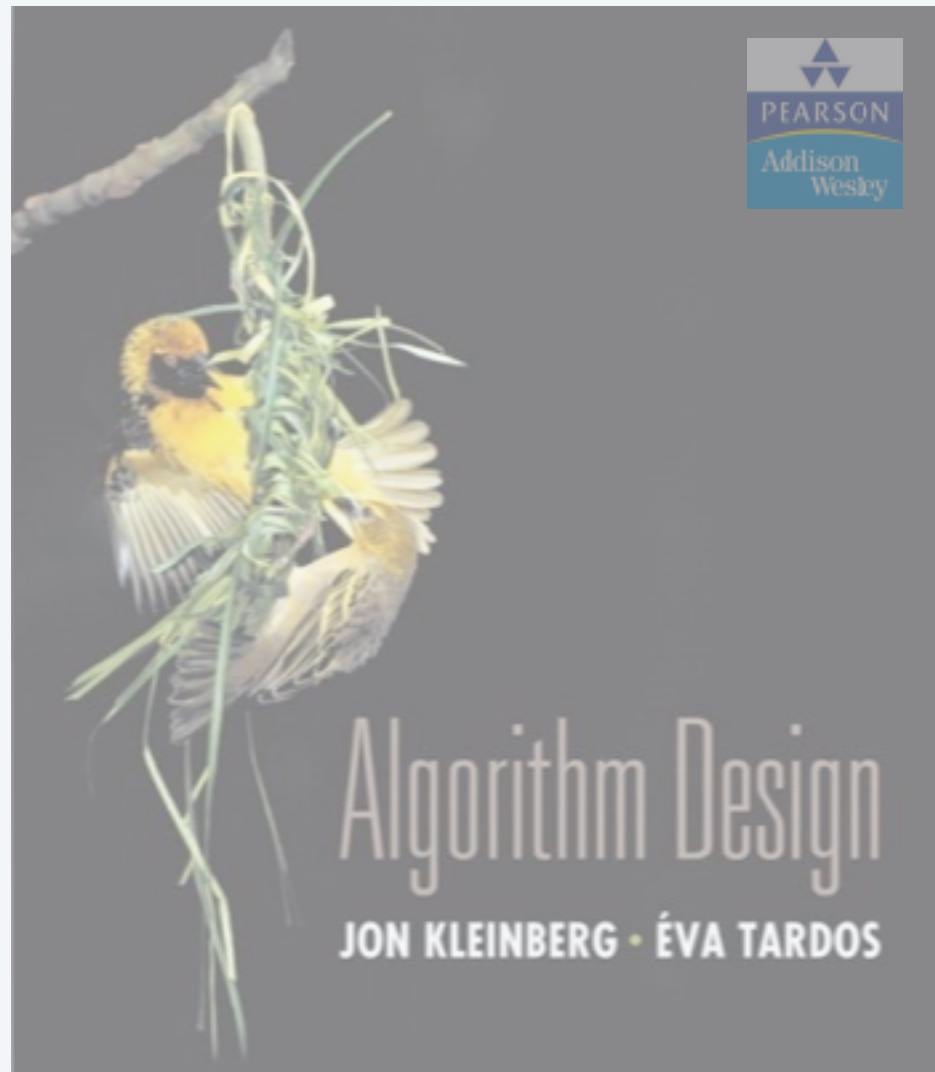
- January 1980.

$$O(n^{2.521801})$$

History of asymptotic complexity of matrix multiplication

year	algorithm	order of growth
?	brute force	$O(n^3)$
1969	Strassen	$O(n^{2.808})$
1978	Pan	$O(n^{2.796})$
1979	Bini	$O(n^{2.780})$
1981	Schönhage	$O(n^{2.522})$
1982	Romani	$O(n^{2.517})$
1982	Coppersmith-Winograd	$O(n^{2.496})$
1986	Strassen	$O(n^{2.479})$
1989	Coppersmith-Winograd	$O(n^{2.376})$
2010	Strother	$O(n^{2.3737})$
2011	Williams	$O(n^{2.3727})$
?	?	$O(n^{2+\epsilon})$

number of floating-point operations to multiply two n -by- n matrices

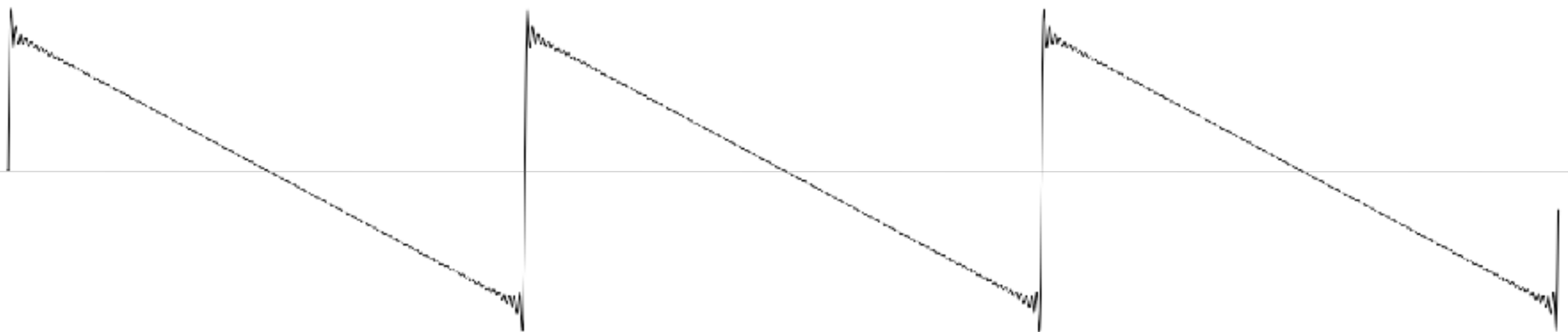


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Fourier analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any (sufficiently smooth) periodic function can be expressed as the sum of a series of sinusoids.



$$y(t) = \frac{2}{\pi} \sum_{k=1}^N \frac{\sin kt}{k} \quad N = 100$$

Euler's identity

Euler's identity. $e^{ix} = \cos x + i \sin x$.

Sinusoids. Sum of sine and cosines = sum of complex exponentials.

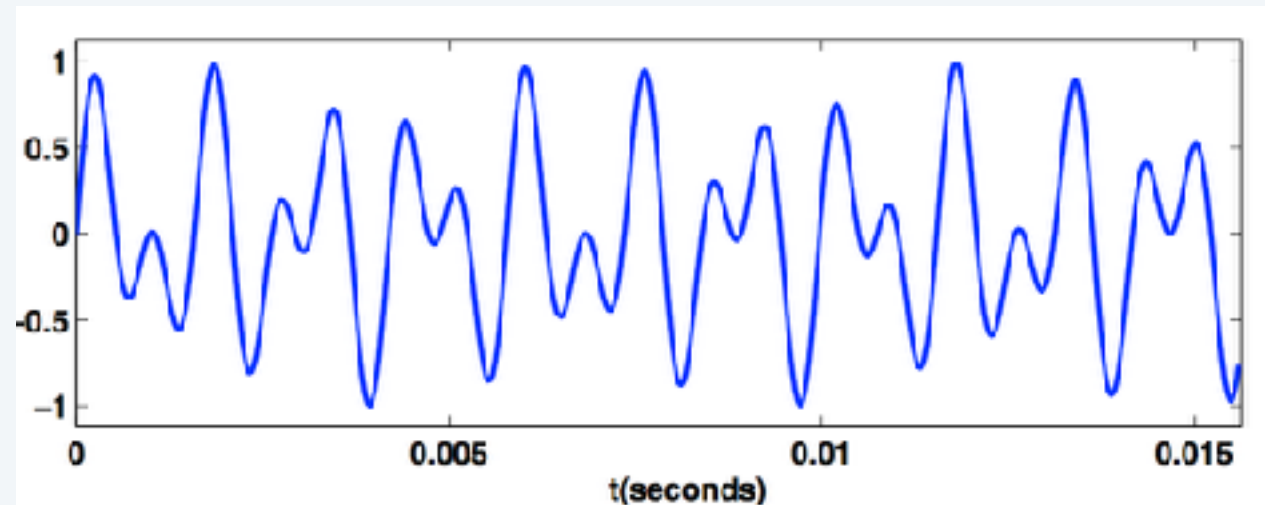
Time domain vs. frequency domain

Signal. [touch tone button 1]

$$y(t) = \frac{1}{2} \sin(2\pi \cdot 697 t) + \frac{1}{2} \sin(2\pi \cdot 1209 t)$$

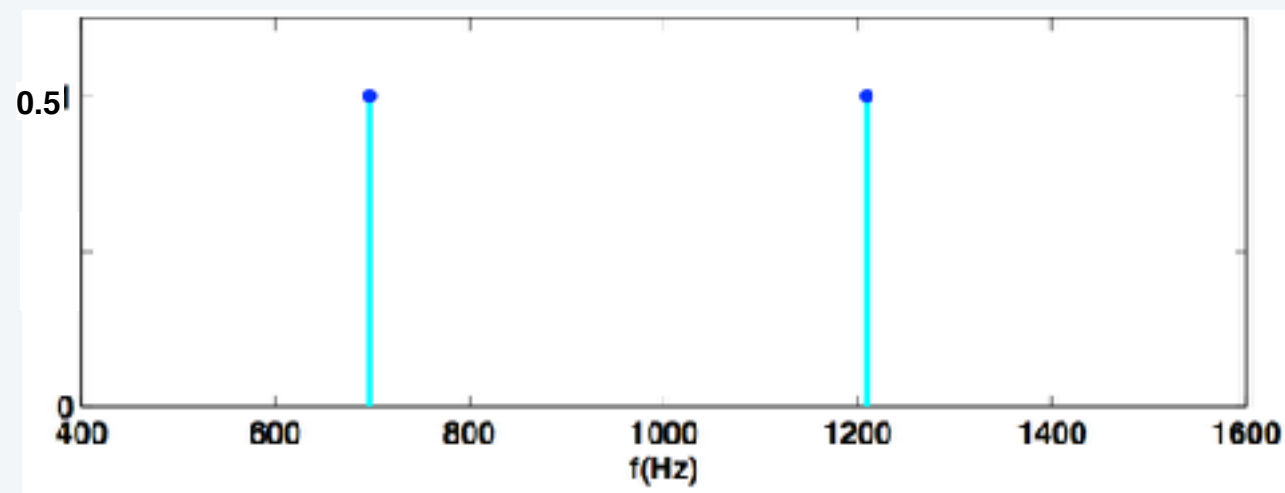
Time domain.

sound
pressure



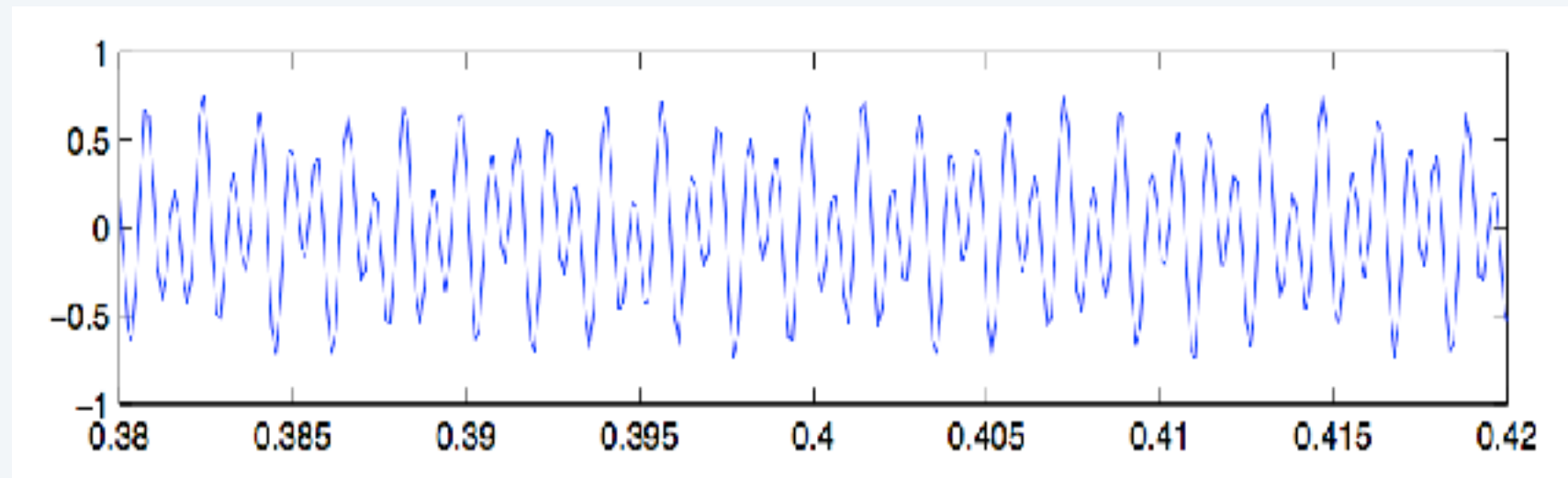
Frequency domain.

amplitude

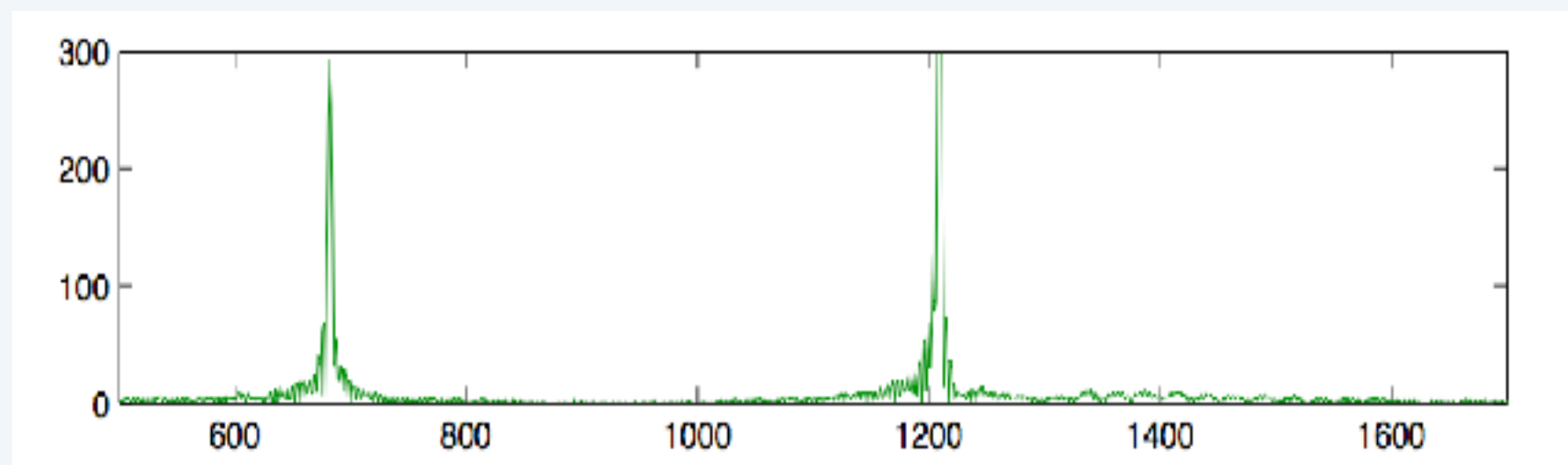


Time domain vs. frequency domain

Signal. [recording, 8192 samples per second]



Magnitude of discrete Fourier transform.



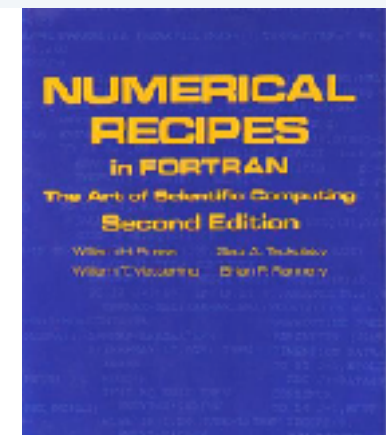
Fast Fourier transform

FFT. Fast way to convert between time-domain and frequency-domain.

Alternate viewpoint. Fast way to multiply and evaluate **polynomials**.

we take this approach

“ If you speed up any nontrivial algorithm by a factor of a million or so the world will beat a path towards finding useful applications for it. ” — Numerical Recipes



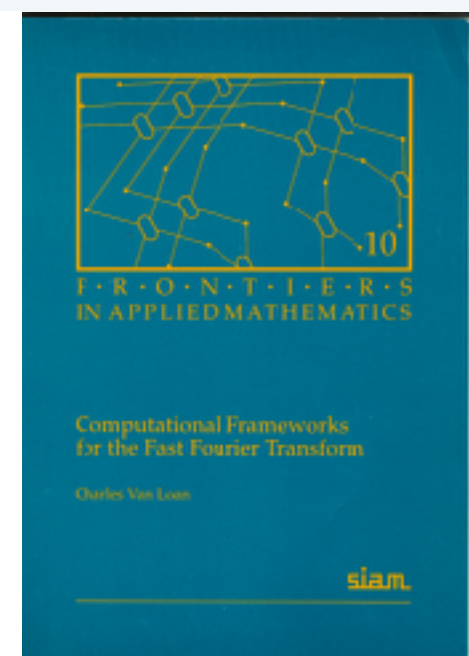
Fast Fourier transform: applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry, ...
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Integer and polynomial multiplication.
- Shor's quantum factoring algorithm.
- ...

“ The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. ”

— Charles van Loan



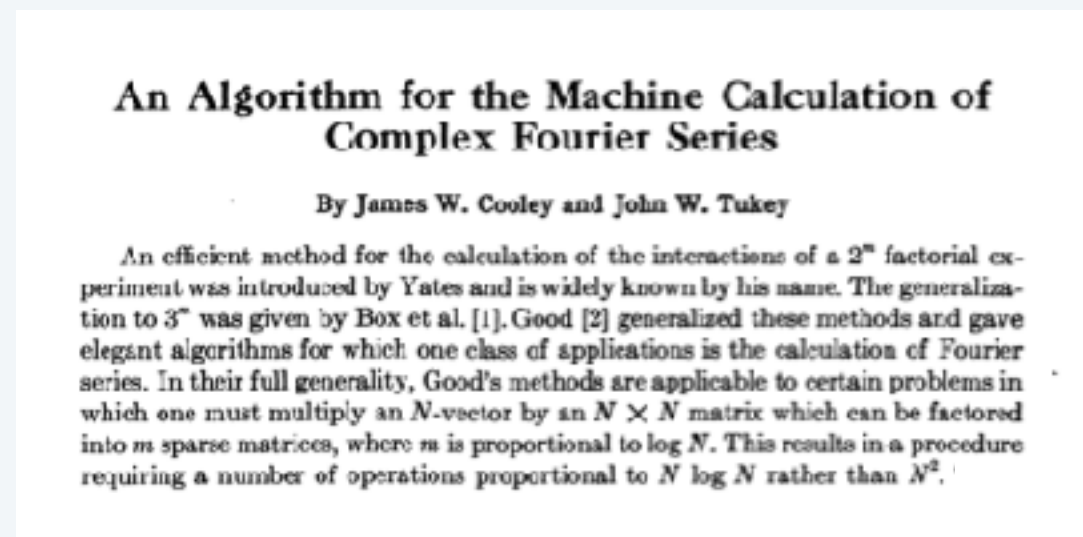
Fast Fourier transform: brief history

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm, x-ray crystallography.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.



paper published only after IBM lawyers decided not to
set precedent of patenting numerical algorithms
(conventional wisdom: nobody could make money selling software!)

Importance not fully realized until advent of digital computers.

Polynomials: coefficient representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

$$B(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1}$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate. $O(n)$ using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))) \cdots))$$

Multiply (convolve). $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^i a_j b_{i-j}$$

A modest Ph.D. dissertation title

DEMONSTRATIO NOVA
THEOREMATIS
OMNEM FUNCTIONEM ALGEBRAICAM
RATIONALEM INTEGRAM
UNIVS VARIABLE
IN FACTORES REALES PRIMI VEL SECUNDI GRADVS
RESOLVI POSSE

AVCTORE
CAROLO FRIDERICO GAUSS
HELMSTADII
APVD C. G. FLECKEISEN. 1799

1.

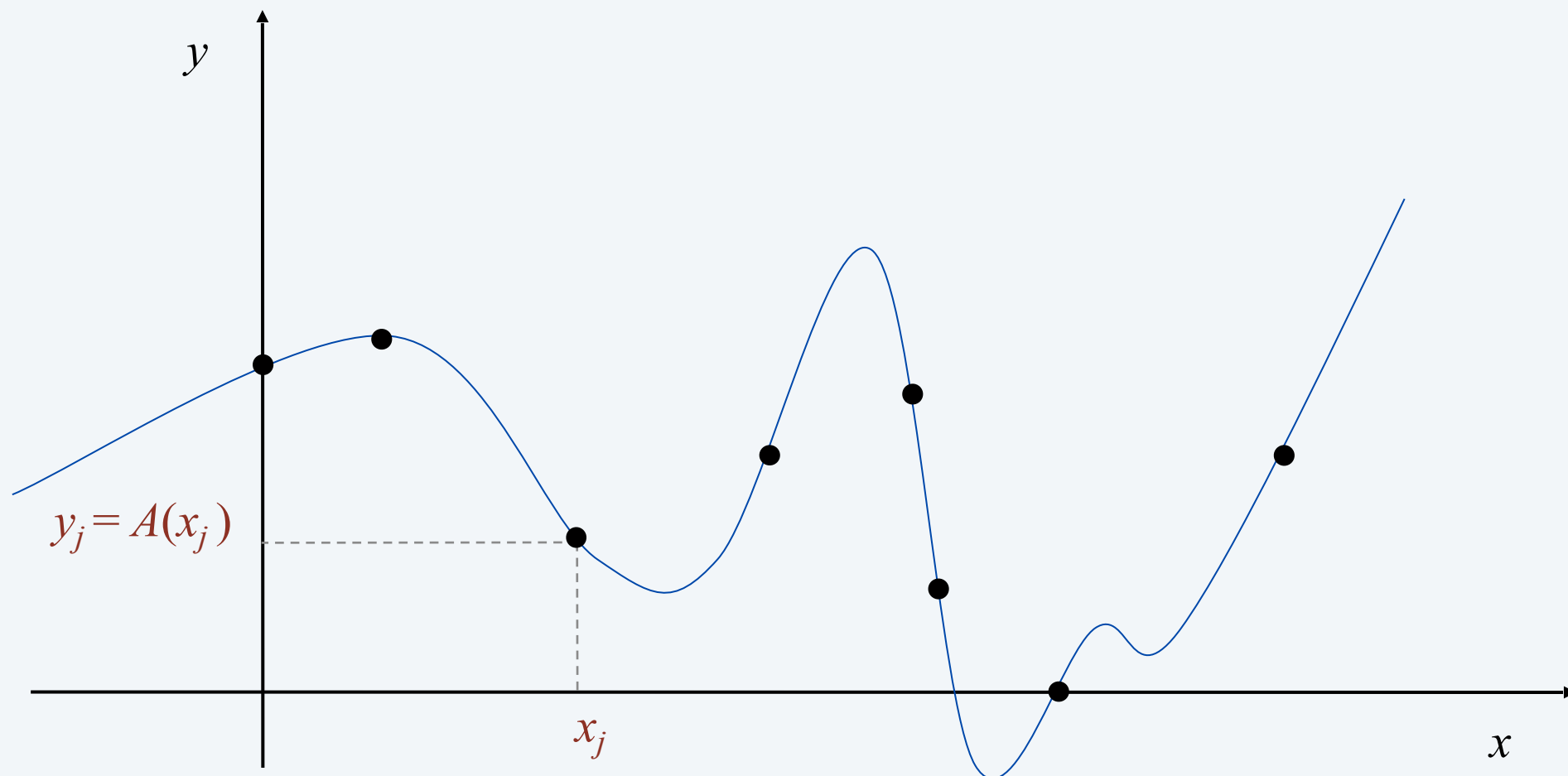
Quaelibet aequatio algebraica determinata reduci potest ad formam $x^m + Ax^{m-1} + Bx^{m-2} + \text{etc.} + M = 0$, ita ut m sit numerus integer positivus. Si partem primam huius aequationis per X denotamus, aequationique $X=0$ per plures valores inaequales ipsius x satisfieri supponimus, puta ponendo $x=a$, $x=\beta$, $x=\gamma$ etc. functio X per productum e factoribus $x-a$, $x-\beta$, $x-\gamma$ etc. divisibilis erit. Vice versa, si productum e pluribus factoribus simplicibus $x-a$, $x-\beta$, $x-\gamma$ etc. functionem X metitur: aequationi $X=0$ satisfiet, aequando ipsam x cuicunque quantitatibus a , β , γ etc. Denique si X producto ex m factoribus talibus simplicibus aequalis est (sive omnes diversi sint, sive quidam ex ipsis identici): alii factores simplices praeter hos functionem X metiri non poterunt. Quamobrem aequatio m^{ta} gradus plures quam m radices habere nequit; simul vero patet, aequationem m^{ta} gradus pauciores radices habere posse, etsi X in m factores simplices resolubilis sit:

"New proof of the theorem that every algebraic rational integral function in one variable can be resolved into real factors of the first or the second degree."

Polynomials: point-value representation

Fundamental theorem of algebra. A degree n polynomial with complex coefficients has exactly n complex roots.

Corollary. A degree $n - 1$ polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x



Polynomials: point-value representation

Polynomial. [point-value representation]

$$A(x): (x_0, y_0), \dots, (x_{n-1}, y_{n-1})$$

$$B(x): (x_0, z_0), \dots, (x_{n-1}, z_{n-1})$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x): (x_0, y_0 + z_0), \dots, (x_{n-1}, y_{n-1} + z_{n-1})$$

Multiply (convolve). $O(n)$, but need $2n - 1$ points.

$$A(x) \times B(x): (x_0, y_0 \times z_0), \dots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

Evaluate. $O(n^2)$ using Lagrange's formula.

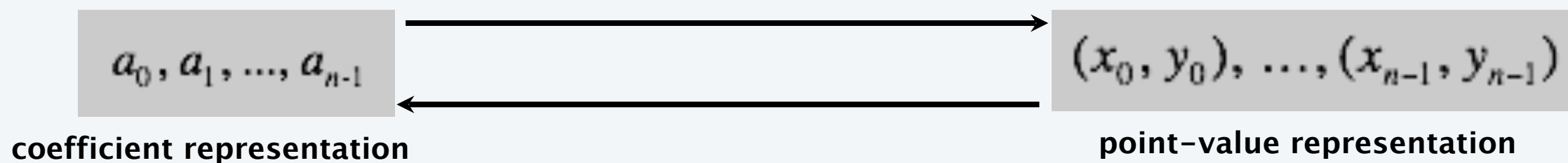
$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting between two representations

Tradeoff. Fast evaluation **or** fast multiplication. We want both!

representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

Goal. Efficient conversion between two representations \Rightarrow all ops fast.



Converting between two representations: brute force

Coefficient \Rightarrow point-value. Given a polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$, evaluate it at n distinct points x_0, \dots, x_{n-1} .

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Running time. $O(n^2)$ for matrix-vector multiply (or n Horner's).

Converting between two representations: brute force

Point-value \Rightarrow coefficient. Given n distinct points x_0, \dots, x_{n-1} and values y_0, \dots, y_{n-1} , find unique polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff x_i distinct

Running time. $O(n^3)$ for Gaussian elimination.

or $O(n^{2.3727})$ via fast matrix multiplication

Divide-and-conquer

Polynomial interpolation: any polynomial of degree d can be reconstructed from its values on a set of $d+1$ or more points.


Decimation in frequency. Break up polynomial into low and high powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{low}(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$
- $A_{high}(x) = a_4 + a_5x + a_6x^2 + a_7x^3.$
- $A(x) = A_{low}(x) + x^4 A_{high}(x).$

Decimation in time. Break up polynomial into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3.$
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3.$
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2).$

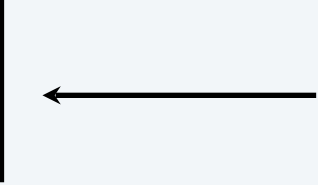
Coefficient to point-value representation: intuition

Coefficient \Rightarrow point-value. Given a polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$, evaluate it at n distinct points x_0, \dots, x_{n-1} .  we get to choose which ones!


Divide. Break up polynomial into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{\text{even}}(x) = a_0 + a_2x + a_4x^2 + a_6x^3.$
- $A_{\text{odd}}(x) = a_1 + a_3x + a_5x^2 + a_7x^3.$
- $A(x) = A_{\text{even}}(x^2) + x A_{\text{odd}}(x^2).$
- $A(-x) = A_{\text{even}}(x^2) - x A_{\text{odd}}(x^2).$

Intuition. Choose two points to be ± 1 .

- $A(1) = A_{\text{even}}(1) + 1 A_{\text{odd}}(1).$
 - $A(-1) = A_{\text{even}}(1) - 1 A_{\text{odd}}(1).$
- 
- Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.**

Coefficient to point-value representation: intuition

Coefficient \Rightarrow point-value. Given a polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$, evaluate it at n distinct points x_0, \dots, x_{n-1} .  we get to choose which ones!


Divide. Break up polynomial into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{\text{even}}(x) = a_0 + a_2x + a_4x^2 + a_6x^3.$
- $A_{\text{odd}}(x) = a_1 + a_3x + a_5x^2 + a_7x^3.$
- $A(x) = A_{\text{even}}(x^2) + x A_{\text{odd}}(x^2).$
- $A(-x) = A_{\text{even}}(x^2) - x A_{\text{odd}}(x^2).$

Intuition. Choose four **complex** points to be $\pm 1, \pm i$.

Key idea: find a set that are intimately related such that evaluating the divided parts of $A(x)$ on them can be shared!

- $A(1) = A_{\text{even}}(1) + 1 A_{\text{odd}}(1).$
- $A(-1) = A_{\text{even}}(1) - 1 A_{\text{odd}}(1).$
- $A(i) = A_{\text{even}}(-1) + i A_{\text{odd}}(-1).$
- $A(-i) = A_{\text{even}}(-1) - i A_{\text{odd}}(-1).$

 Can evaluate polynomial of degree $\leq n$ at 4 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 2 point.

Discrete Fourier transform

Coefficient \Rightarrow point-value. Given a polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$, evaluate it at n distinct points x_0, \dots, x_{n-1} .

Key idea. Choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

\uparrow
DFT

\uparrow
Fourier matrix F_n

Roots of unity

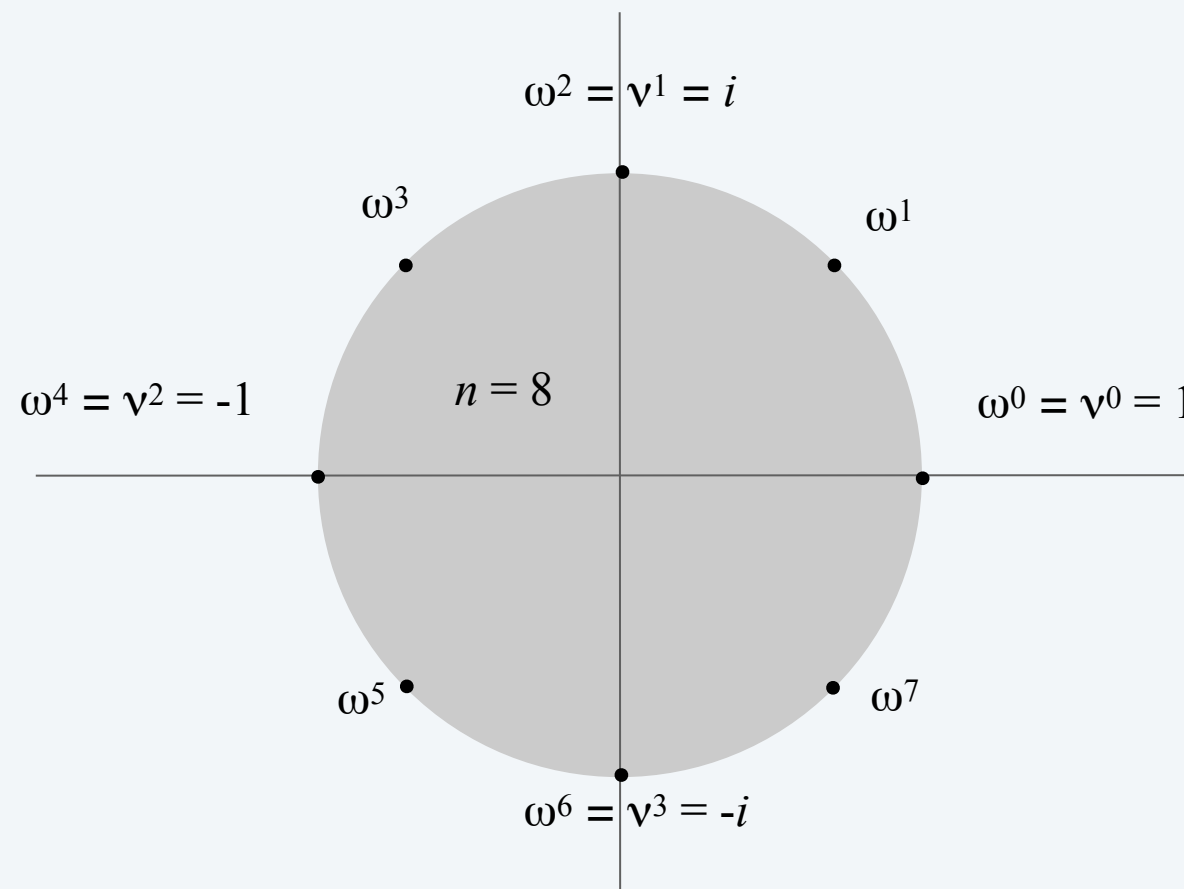
Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \dots, \omega^{n-1}$ where $\omega = e^{2\pi i/n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k/n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $1/2 n^{\text{th}}$ roots of unity are: $\nu^0, \nu^1, \dots, \nu^{n/2-1}$ where $\nu = \omega^2 = e^{4\pi i/n}$.

Complex number can be viewed as lying in the complex plane using polar; Axes represents their real and imaginary parts, using polar coordinates



Fast Fourier transform

Goal. Evaluate a degree $n - 1$ polynomial $A(x) = a_0 + \dots + a_{n-1} x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \dots, \omega^{n-1}$.

Divide. Break up polynomial into even and odd powers.

- $A_{\text{even}}(x) = a_0 + a_2x + a_4x^2 + \dots + a_{n-2}x^{n/2-1}$.
- $A_{\text{odd}}(x) = a_1 + a_3x + a_5x^2 + \dots + a_{n-1}x^{n/2-1}$.
- $A(x) = A_{\text{even}}(x^2) + x A_{\text{odd}}(x^2)$.

Conquer. Evaluate $A_{\text{even}}(x)$ and $A_{\text{odd}}(x)$ at the $1/2n^{\text{th}}$ roots of unity: $\nu^0, \nu^1, \dots, \nu^{n/2-1}$.

Combine.

- $A(\omega^k) = A_{\text{even}}(\nu^k) + \omega^k A_{\text{odd}}(\nu^k), \quad 0 \leq k < n/2$
- $A(\omega^{k+1/2n}) = A_{\text{even}}(\nu^k) - \omega^k A_{\text{odd}}(\nu^k), \quad 0 \leq k < n/2$

Diagram illustrating the relationship between the roots of unity used in the FFT algorithm:

- $\nu^k = (\omega^k)^2$ (indicated by a red arrow pointing from ν^k to $(\omega^k)^2$)
- $\nu^k = (\omega^{k+1/2n})^2$ (indicated by a red arrow pointing from ν^k to $(\omega^{k+1/2n})^2$)
- $\omega^{k+1/2n} = -\omega^k$ (indicated by a red arrow pointing from $\omega^{k+1/2n}$ to $-\omega^k$)

FFT: implementation

FFT ($n, a_0, a_1, a_2, \dots, a_{n-1}$)

IF ($n = 1$) **RETURN** a_0 .

$(e_0, e_1, \dots, e_{n/2-1}) \leftarrow \text{FFT}(n/2, a_0, a_2, a_4, \dots, a_{n-2})$.

$(d_0, d_1, \dots, d_{n/2-1}) \leftarrow \text{FFT}(n/2, a_1, a_3, a_5, \dots, a_{n-1})$.

FOR $k = 0$ **TO** $n/2 - 1$.

$\omega^k \leftarrow e^{2\pi i k/n}$.

$y_k \leftarrow e_k + \omega^k d_k$.

$y_{k+n/2} \leftarrow e_k - \omega^k d_k$.

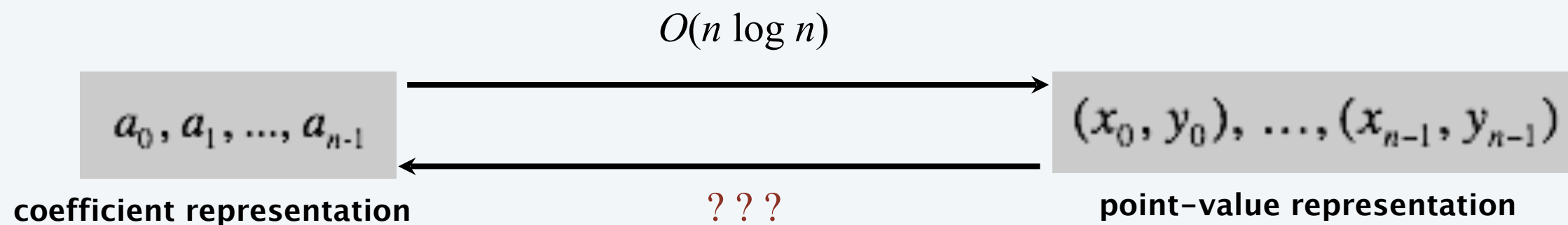
RETURN $(y_0, y_1, y_2, \dots, y_{n-1})$.

FFT: summary

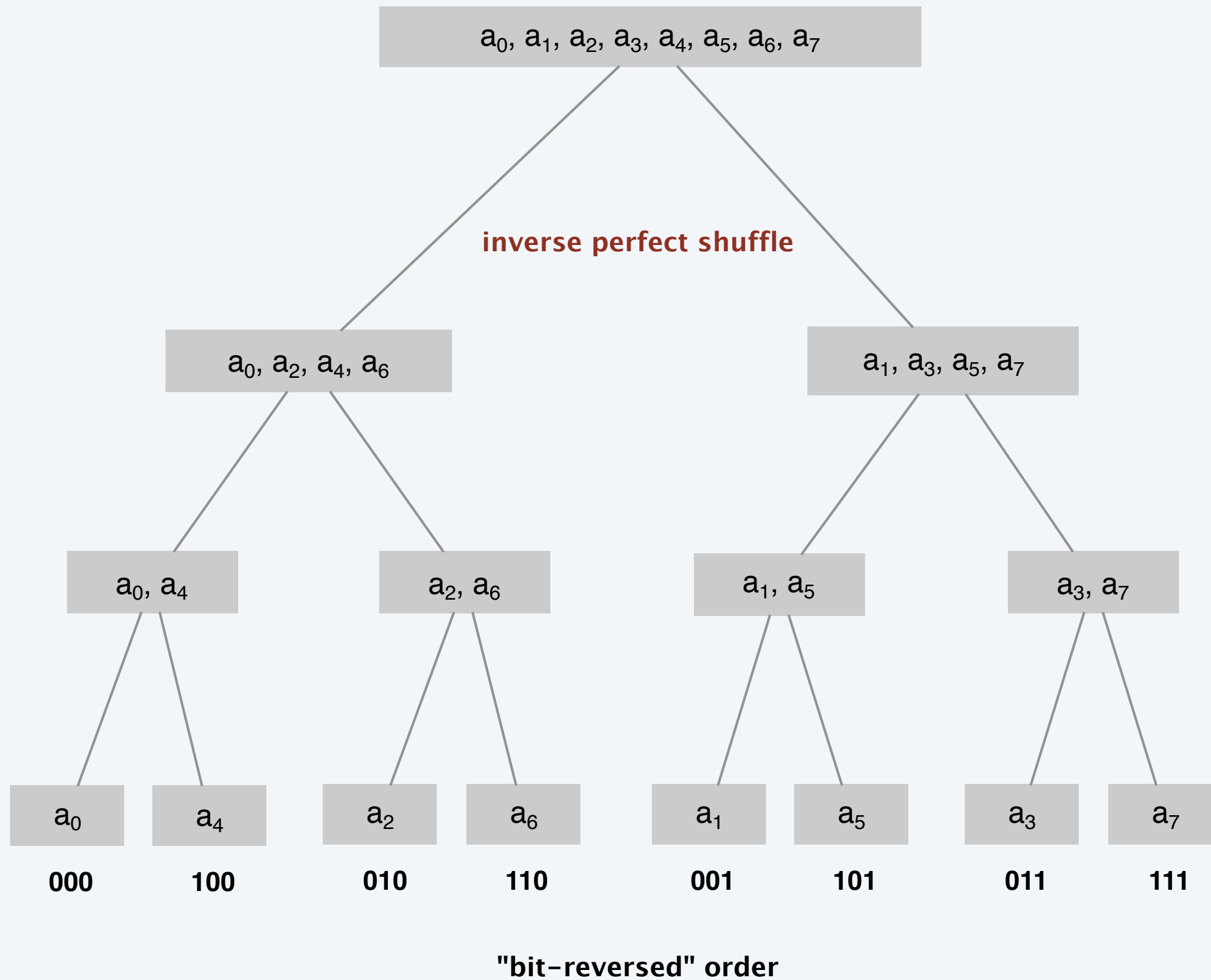
Theorem. The FFT algorithm evaluates a degree $n - 1$ polynomial at each of the n^{th} roots of unity in $O(n \log n)$ steps and $O(n)$ extra space.

Pf. $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$

assumes n is a power of 2



FFT: recursion tree



Inverse discrete Fourier transform

Point-value \Rightarrow coefficient. Given n distinct points x_0, \dots, x_{n-1} and values y_0, \dots, y_{n-1} , find unique polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$



Inverse DFT



Fourier matrix inverse $(F_n)^{-1}$

Inverse discrete Fourier transform

Claim. Inverse of Fourier matrix F_n is given by following formula:

$$G_n = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \dots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \dots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \dots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \dots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

F_n / \sqrt{n} is a unitary matrix

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide the result by n).

Inverse FFT: proof of correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n G_n\right)_{kk'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{kj} \omega^{-jk'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma (below)

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \Rightarrow$ series sums to n .
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)(1 + x + x^2 + \dots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \dots + \omega^{k(n-1)} = 0 \Rightarrow$ series sums to 0. ■

Inverse FFT: implementation

Note. Need to divide result by n .

INVERSE-FFT ($n, y_0, y_1, y_2, \dots, y_{n-1}$)

IF ($n = 1$) **RETURN** y_0 .

$(e_0, e_1, \dots, e_{n/2-1}) \leftarrow \text{INVERSE-FFT}(n/2, y_0, y_2, y_4, \dots, y_{n-2}).$

$(d_0, d_1, \dots, d_{n/2-1}) \leftarrow \text{INVERSE-FFT}(n/2, y_1, y_3, y_5, \dots, y_{n-1}).$

FOR $k = 0$ **TO** $n/2 - 1$.

$\omega^k \leftarrow e^{-2\pi i k/n}.$

$a_k \leftarrow (e_k + \omega^k d_k).$

$a_{k+n/2} \leftarrow (e_k - \omega^k d_k).$

RETURN $(a_0, a_1, a_2, \dots, a_{n-1}).$

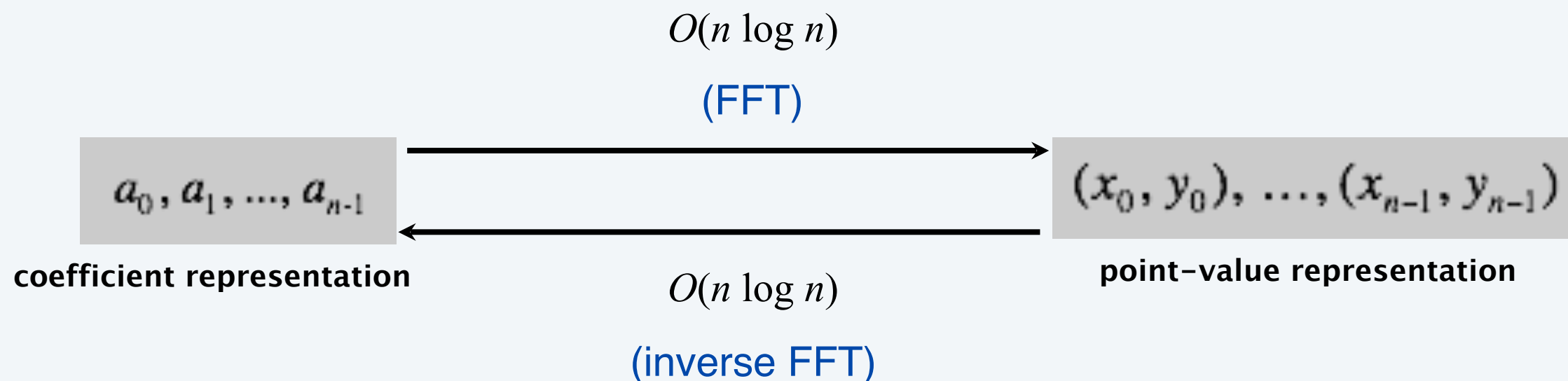
switch roles of a_i and y_i



Inverse FFT: summary

Theorem. The inverse FFT algorithm interpolates a degree $n - 1$ polynomial given values at each of the n^{th} roots of unity in $O(n \log n)$ steps.

↑
assumes n is a power of 2

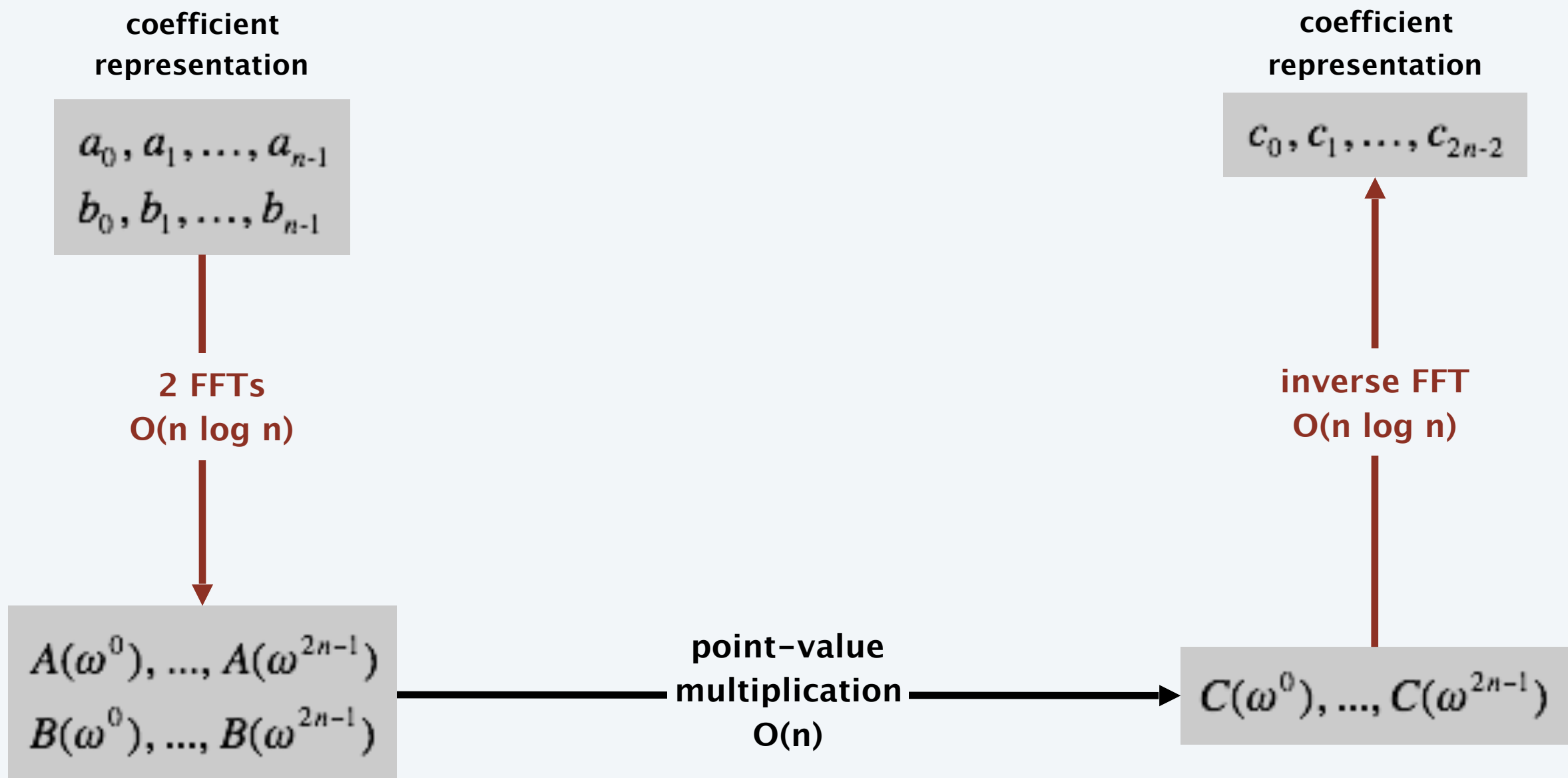


Polynomial multiplication

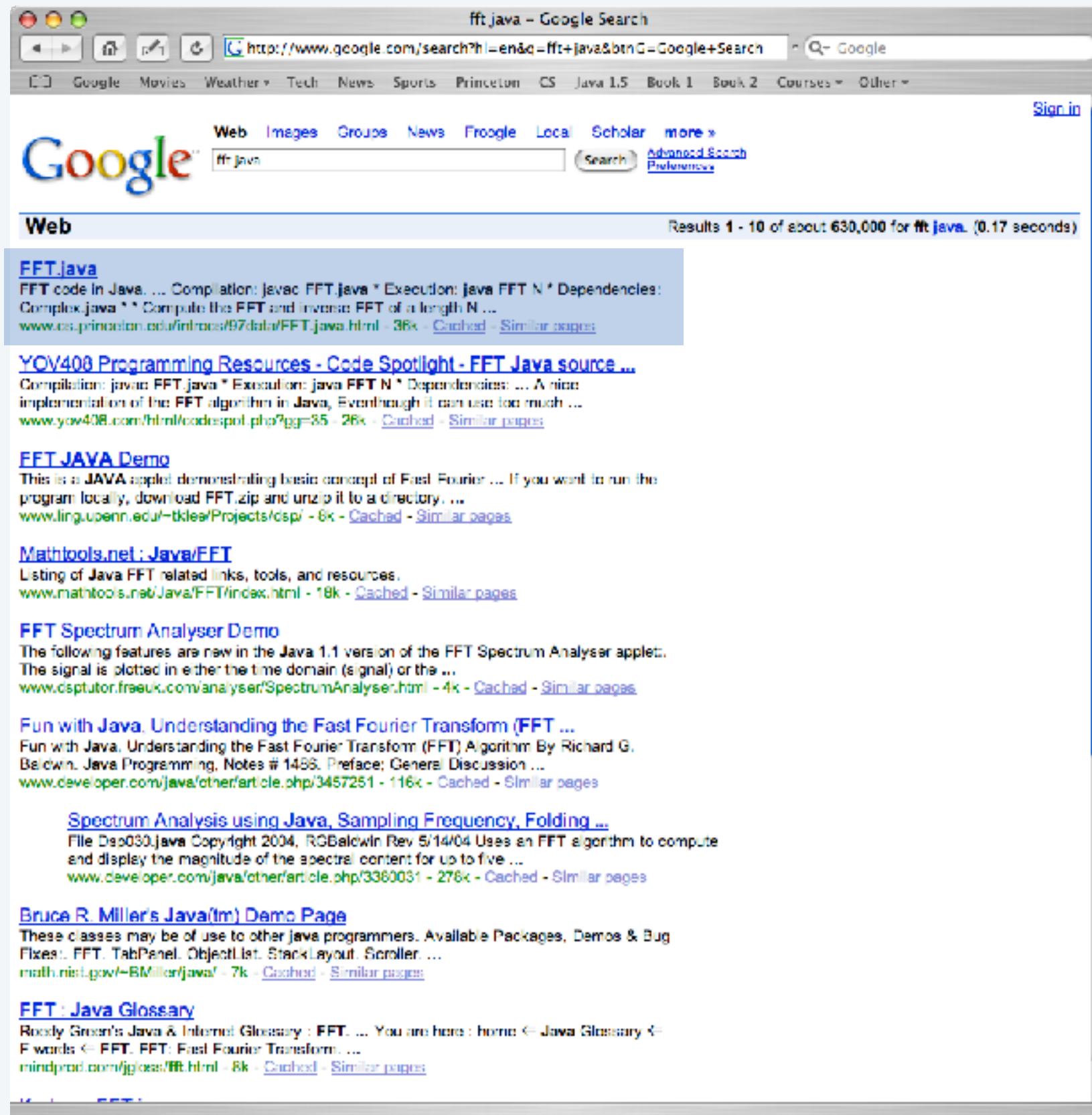
Theorem. Can multiply two degree $n - 1$ polynomials in $O(n \log n)$ steps.

Pf.

pad with 0s to make n a power of 2



FFT in practice ?



FFT in practice

Fastest Fourier transform in the West. [Frigo and Johnson]

- Optimized C library.
- Features: DFT, DCT, real, complex, any size, any dimension.
- Won 1999 Wilkinson Prize for Numerical Software.
- Portable, competitive with vendor-tuned code.

Implementation details.

- Core algorithm is nonrecursive version of Cooley-Tukey.
- Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
- Runs in $O(n \log n)$ time, even when n is prime.
- Multidimensional FFTs.



<http://www.fftw.org>

Integer multiplication, redux

Integer multiplication. Given two n -bit integers $a = a_{n-1} \dots a_1 a_0$ and $b = b_{n-1} \dots b_1 b_0$, compute their product $a \cdot b$.

Convolution algorithm.

- Form two polynomials.
- Note: $a = A(2)$, $b = B(2)$.
- Compute $C(x) = A(x) \cdot B(x)$.
- Evaluate $C(2) = a \cdot b$.
- Running time: $O(n \log n)$ complex arithmetic operations.

$$A(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$$

$$B(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1}$$

Theory. [Schönhage-Strassen 1971] $O(n \log n \log \log n)$ bit operations.

Theory. [Fürer 2007] $n \log n 2^{O(\log^* n)}$ bit operations.

Practice. [GNU Multiple Precision Arithmetic Library]

It uses brute force, Karatsuba, and FFT, depending on the size of n .