5. DIVIDE AND CONQUER I

▸ *mergesort*

▸ *counting inversions*

▸ *closest pair of points*

▸ *randomized quicksort*

▸ *median and selection*

Divide-and-conquer paradigm

Divide-and-conquer.

- Divide up problem into several subproblems.

- Solve each subproblem recursively.

- Combine solutions to subproblems into overall solution.
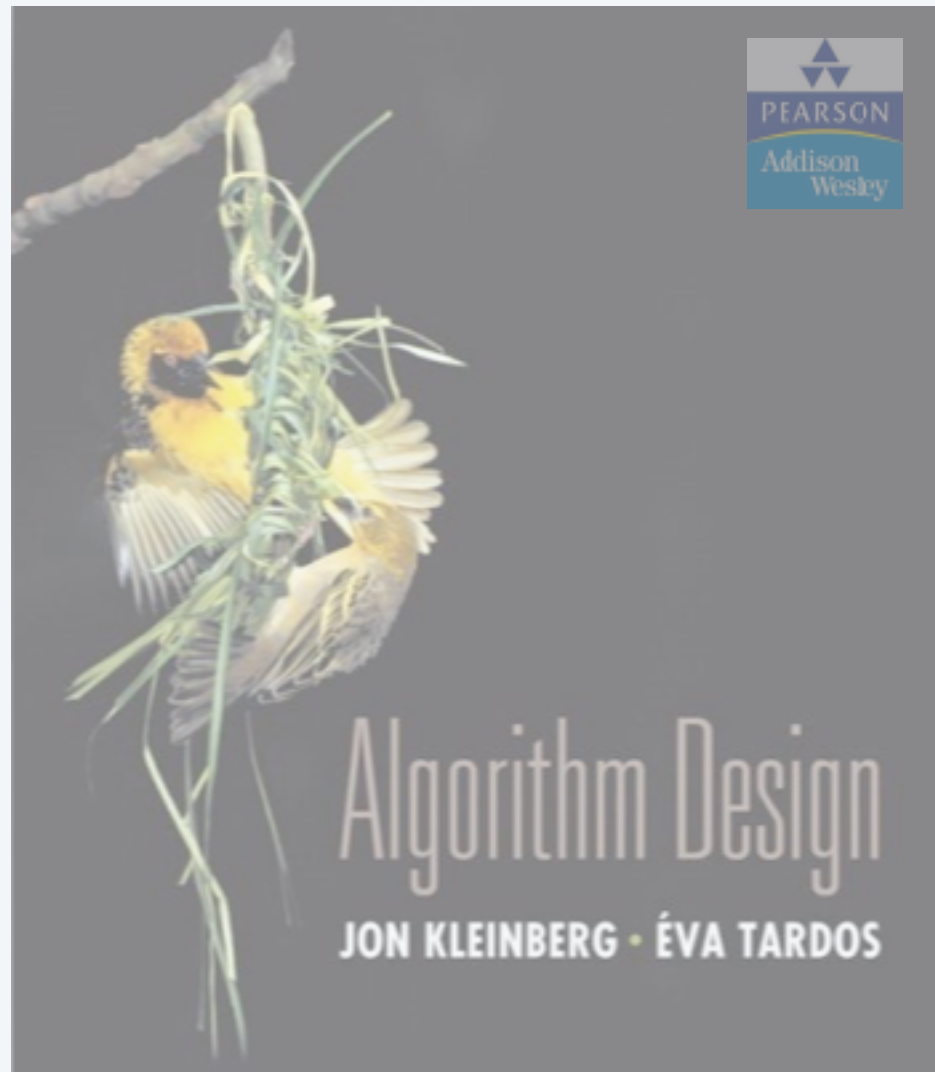
Most common usage.

- Divide problem of size n into two subproblems of size $n/2$ in linear time.

- Solve two subproblems recursively.

- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: $\Theta(n^2)$.

- Divide-and-conquer: $\Theta(n \log n)$.

attributed to Julius Caesar

5. DIVIDE AND CONQUER

‣ *mergesort*

Algorithm Design

JON KLEINBERG · ÉVA TARDOS

Sorting problem

Problem. Given a list of n elements from a totally-ordered universe, rearrange them in ascending order.

Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.

- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Minimum spanning trees (Kruskal's algorithm).
- Scheduling to minimize maximum lateness or average completion time.
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.

First Draft
of a
Report on the
EDVAC

John von Neumann

input

| A | L | G | O | R | I | T | H | M | S |

sort left half

| A | G | L | O | R | | I | T | H | M | S |

sort right half

| A | G | L | O | R | | H | I | M | S | T |

merge results

| A | G | H | I | L | M | O | R | S | T |
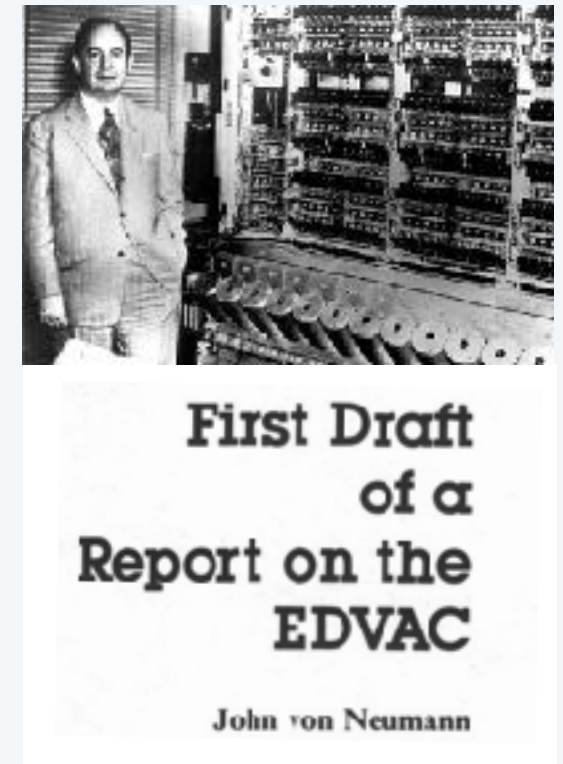
Merging

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).

sorted list A

3	7	10	a_i	18

sorted list B

2	11	b_j	17	23

5   2

merge to form sorted list C

2	3	7	10	11					
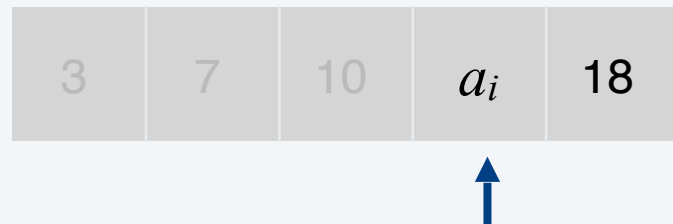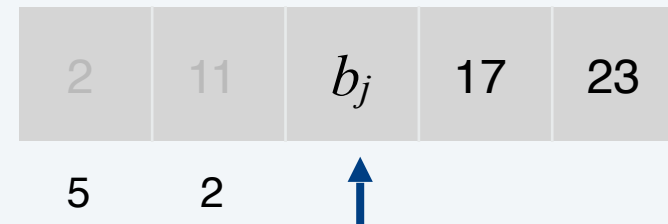
A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of size $\leq n$.

Note. $T(n)$ is monotone nondecreasing.

Mergesort recurrence.

$$
T(n) \leq
\begin{cases}
0 & \text{if } n = 1 \\
T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + n & \text{otherwise}
\end{cases}
$$

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence.

Initially we assume n is a power of 2 and replace $\leq$ with $=$.

Divide-and-conquer recurrence: proof by recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n\,/\,2) + n & \text{otherwise} \end{cases}$$

assuming n
is a power of 2

Pf 1.

$T(n)$ n $= n$

$T(n\,/\,2)$ $T(n\,/\,2)$ $2\,(n/2)$ $= n$

$T(n\,/\,4)$ $T(n\,/\,4)$ $T(n\,/\,4)$ $T(n\,/\,4)$ $4\,(n/4)$ $= n$

$\log_2 n$

$T(n\,/\,8)$ $T(n\,/\,8)$ $T(n\,/\,8)$ $T(n\,/\,8)$ $T(n\,/\,8)$ $T(n\,/\,8)$ $T(n\,/\,8)$ $T(n\,/\,8)$ $8\,(n/8)$ $= n$

$T(n) = n \lg n$

9
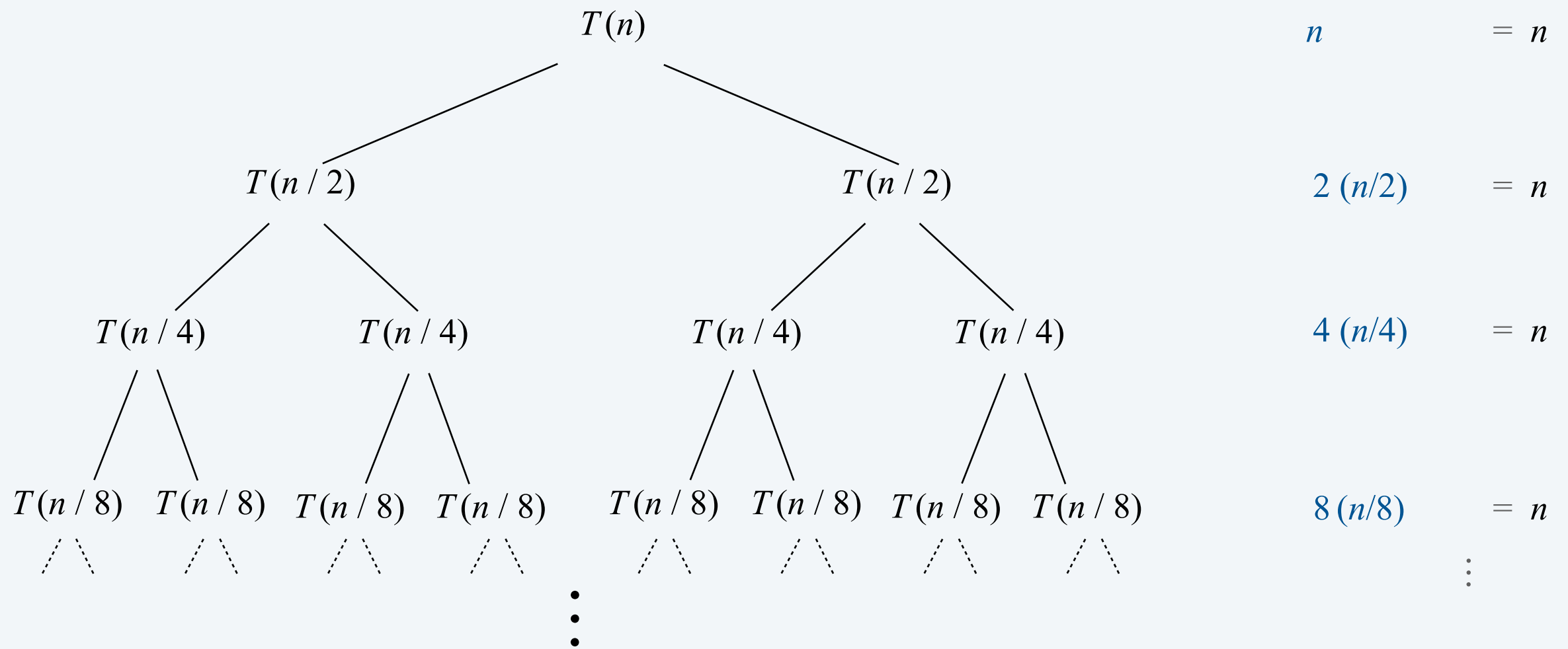
Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$
\begin{aligned}
T(2n) &= 2\,T(n) + 2n \\
      &= 2\,n \log_2 n + 2n \\
      &= 2\,n\,(\log_2 (2n) - 1) + 2n \\
      &= 2\,n \log_2 (2n). \quad \blacksquare
\end{aligned}
$$

Analysis of mergesort recurrence
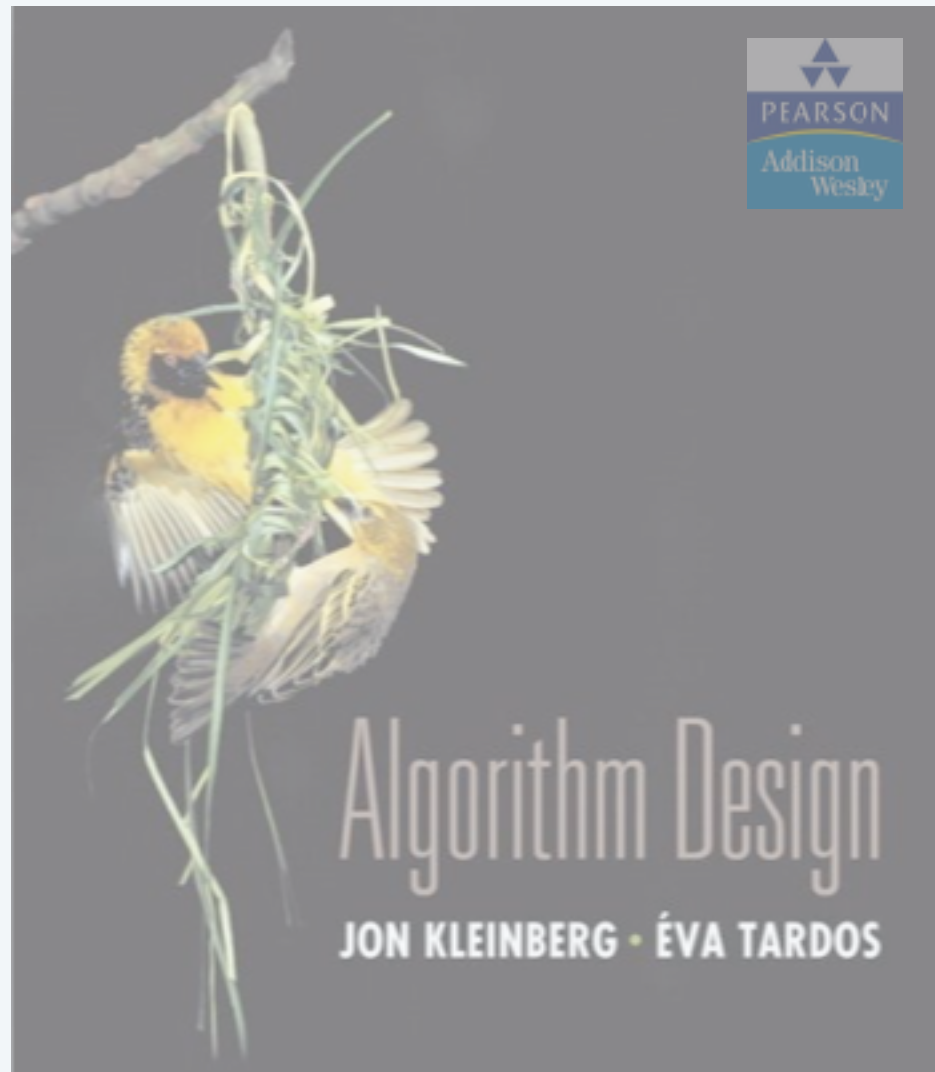
Claim. If $T(n)$ satisfies the following recurrence, then $T(n) \leq n \lceil \log_2 n \rceil$.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Pf. [by strong induction on n]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.
- Induction step: assume true for $1, 2, \ldots, n-1$.

$$
\begin{aligned}
T(n) \quad & \leq T(n_1) + T(n_2) + n \\
& \leq n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\
& \leq n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\
& = n \lceil \log_2 n_2 \rceil + n \\
& \leq n (\lceil \log_2 n \rceil - 1) + n \\
& = n \lceil \log_2 n \rceil. \quad \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
n_2 &= \lceil n/2 \rceil \\
&\leq \lceil 2^{\lceil \log_2 n \rceil} / 2 \rceil \\
&= 2^{\lceil \log_2 n \rceil} / 2
\end{aligned}
$$

$$\log_2 n_2 \leq \lceil \log_2 n \rceil - 1$$

5. DIVIDE AND CONQUER

Counting inversions

Music site tries to match your song preferences with others.

- You rank n songs.

- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: $1, 2, …, n$.

- Your rank: $a_1, a_2, …, a_n$.

- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

Think and analyze: is there a greedy algorithm?

Guess: bubble sort

	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 inversions: 3–2, 4–2

Brute force: check all $\Theta(n^2)$ pairs.

Counting inversions: applications

- Voting theory.

- Collaborative filtering.

- Measuring the "sortedness" of an array.

- Sensitivity analysis of Google's ranking function.

- Rank aggregation for meta-searching on the Web.

- Nonparametric statistics (e.g., Kendall's tau distance).

Rank Aggregation Methods for the Web

Cynthia Dwork·   Ravi Kumar[†]   Moni Naor[‡]   D. Sivakumar[§]

ABSTRACT

We consider the problem of combining ranking results from various sources. In the context of the Web, the main applications include building meta-search engines, combining ranking functions, selecting documents based on multiple criteria, and improving search precision through word associations. We develop a set of techniques for the rank aggregation problem and compare their performance to that of well-known methods. A primary goal of our work is to design rank aggregation techniques that can effectively combat "spam," a serious problem in Web searches. Experiments show that our methods are simple, efficient, and effective.

Keywords: rank aggregation, ranking functions, meta-search, multi-word queries, spam

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

input

| 1 | 5 | 4 | 8 | 10 | 2 | 6 | 9 | 3 | 7 |

count inversions in left half A

| 1 | 5 | 4 | 8 | 10 |

5-4

count inversions in right half B

| 2 | 6 | 9 | 3 | 7 |

6-3 9-3 9-7

count inversions (a, b) with a ∈ A and b ∈ B

| 1 | 5 | 4 | 8 | 10 |

| 2 | 6 | 9 | 3 | 7 |

4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10-3 10-6 10-7 10-9

output 1 + 3 + 13 = 17

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.

- For each element $b \in B$,

    - binary search in A to find how many elements in A are greater than b.

list A

7	10	18	3	14

list B

17	23	2	11	16

sort A

3	7	10	14	18

sort B

2	11	16	17	23

binary search to count inversions (a, b) with a $\in$ A and b $\in$ B

3	7	10	14	18

2	11	16	17	23

5	2	1	1	0

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

count inversions (a, b) with a ∈ A and b ∈ B

3	7	10	a_i	18

2	11	b_j	17	23

5   2

merge to form sorted list C

2	3	7	10	11

Input. List L.

Output. Number of inversions in L and sorted list of elements L'.

SORT-AND-COUNT (L)

IF list L has one element

RETURN $(0, L)$.

DIVIDE the list into two halves A and B.

$(r_A , A) \leftarrow$ SORT-AND-COUNT(A).

$(r_B , B) \leftarrow$ SORT-AND-COUNT(B).

$(r_{AB} , L') \leftarrow$ MERGE-AND-COUNT(A, B).

RETURN $(r_A + r_B + r_{AB} , L')$.

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Pf. The worst-case running time $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Divide the list into two halves

 A contains the first $\lceil n/2 \rceil$ elements

 B contains the remaining $\lfloor n/2 \rfloor$ elements

5. DIVIDE AND CONQUER

‣ *mergesort*

‣ *counting inversions*

‣ **closest pair of points**

‣ *randomized quicksort*

‣ *median and selection*

Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Project Candidate

- Minimum Steiger Tree
- Time consuming Routing Algorithm considering overplaying, congestion
- ML algorithm based length estimation for priority decision
- ? How to deal with Spacial/temporal correlation
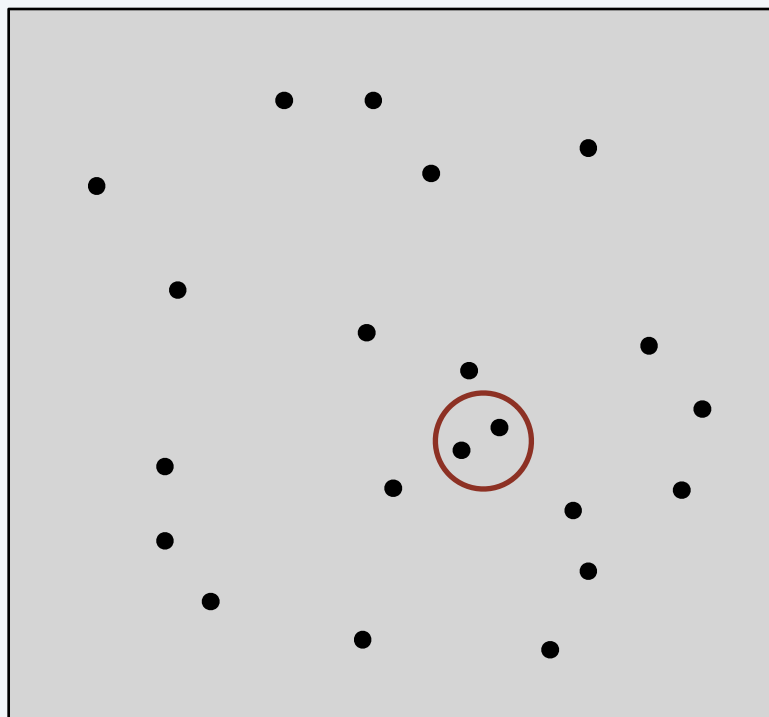- ? How to use routing algorithm as priori

Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.

1d version. Easy $O(n \log n)$ algorithm if points are on a line.

Nondegeneracy assumption. No two points have the same x-coordinate.
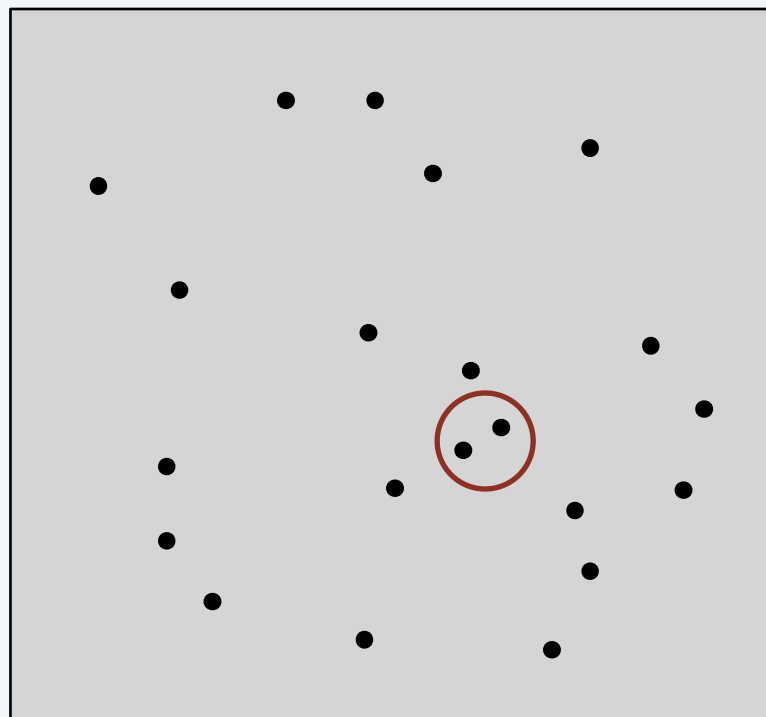
Closest pair of points: first attempt

Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.

Can we ?

Closest pair of points: first attempt

Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.
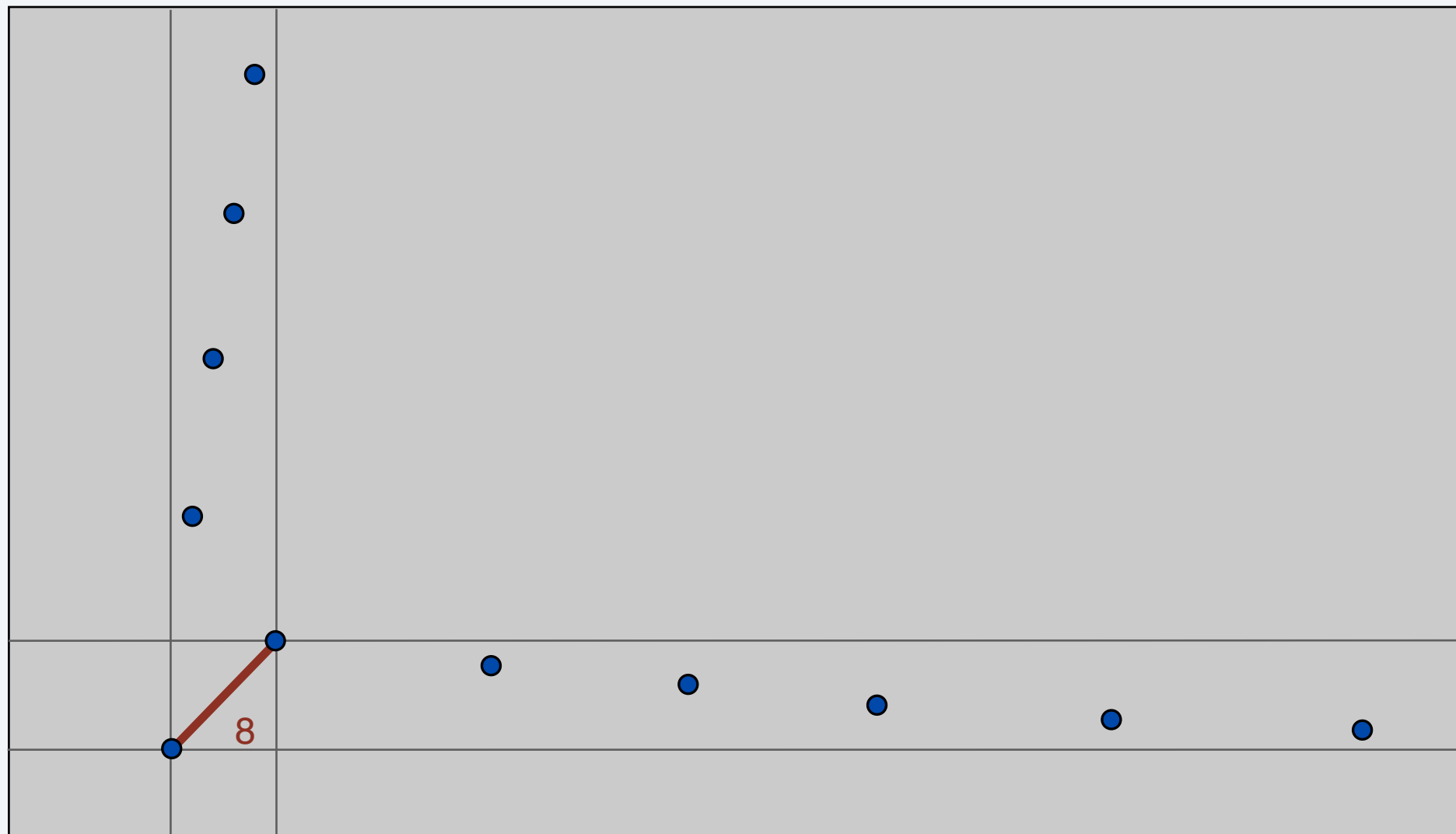
Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.
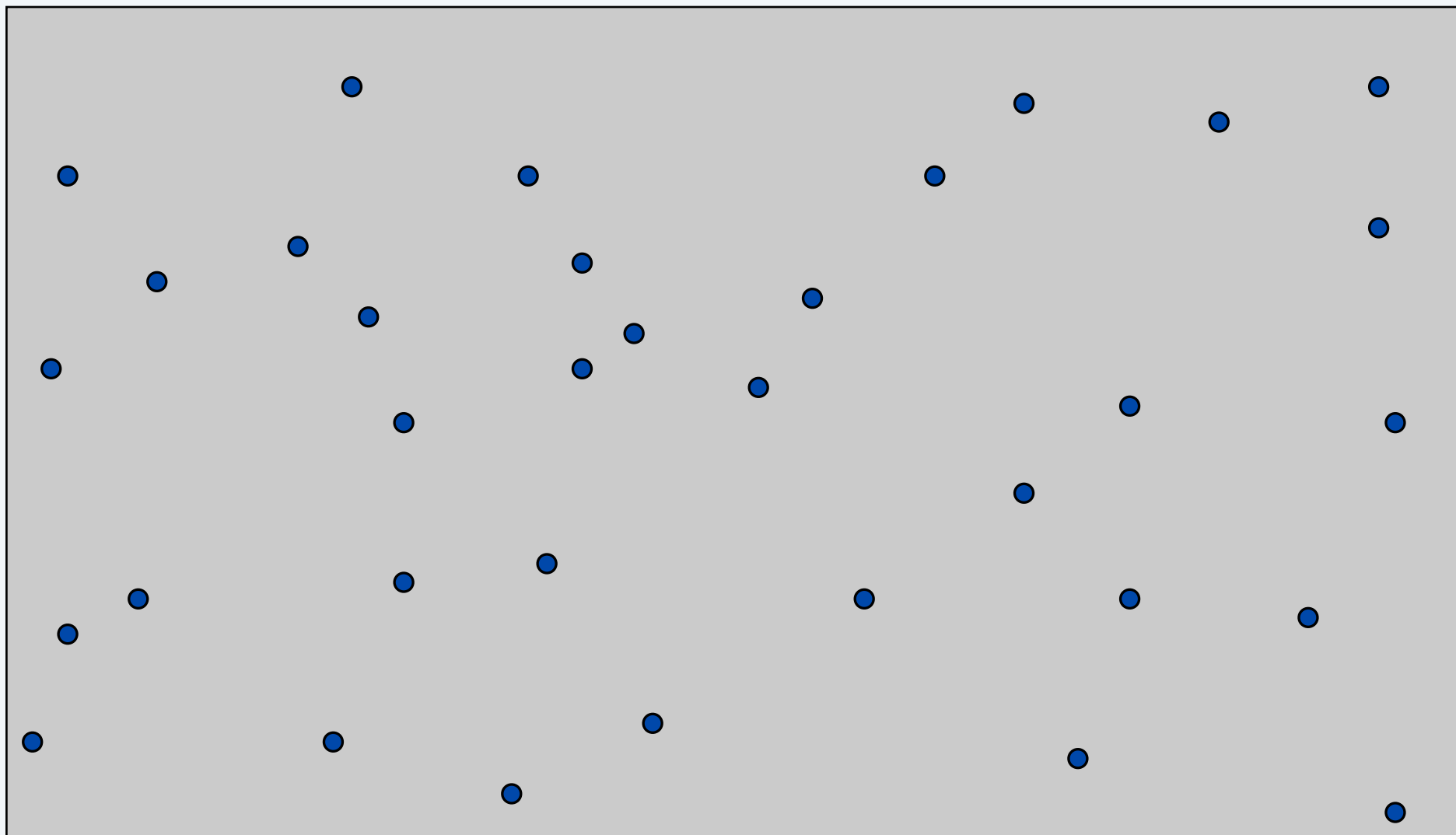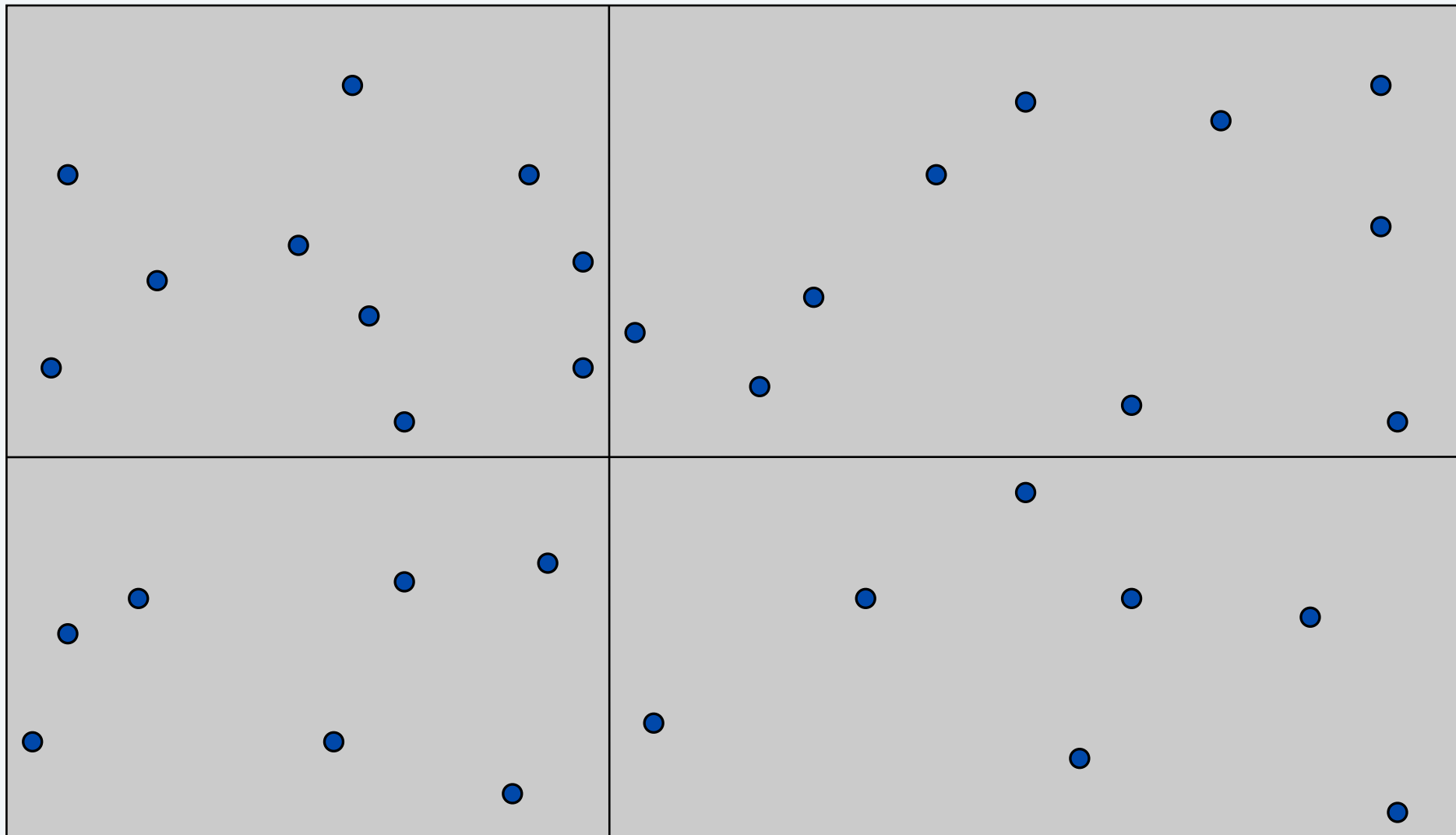
Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.

Obstacle. Impossible to ensure $n/4$ points in each piece.
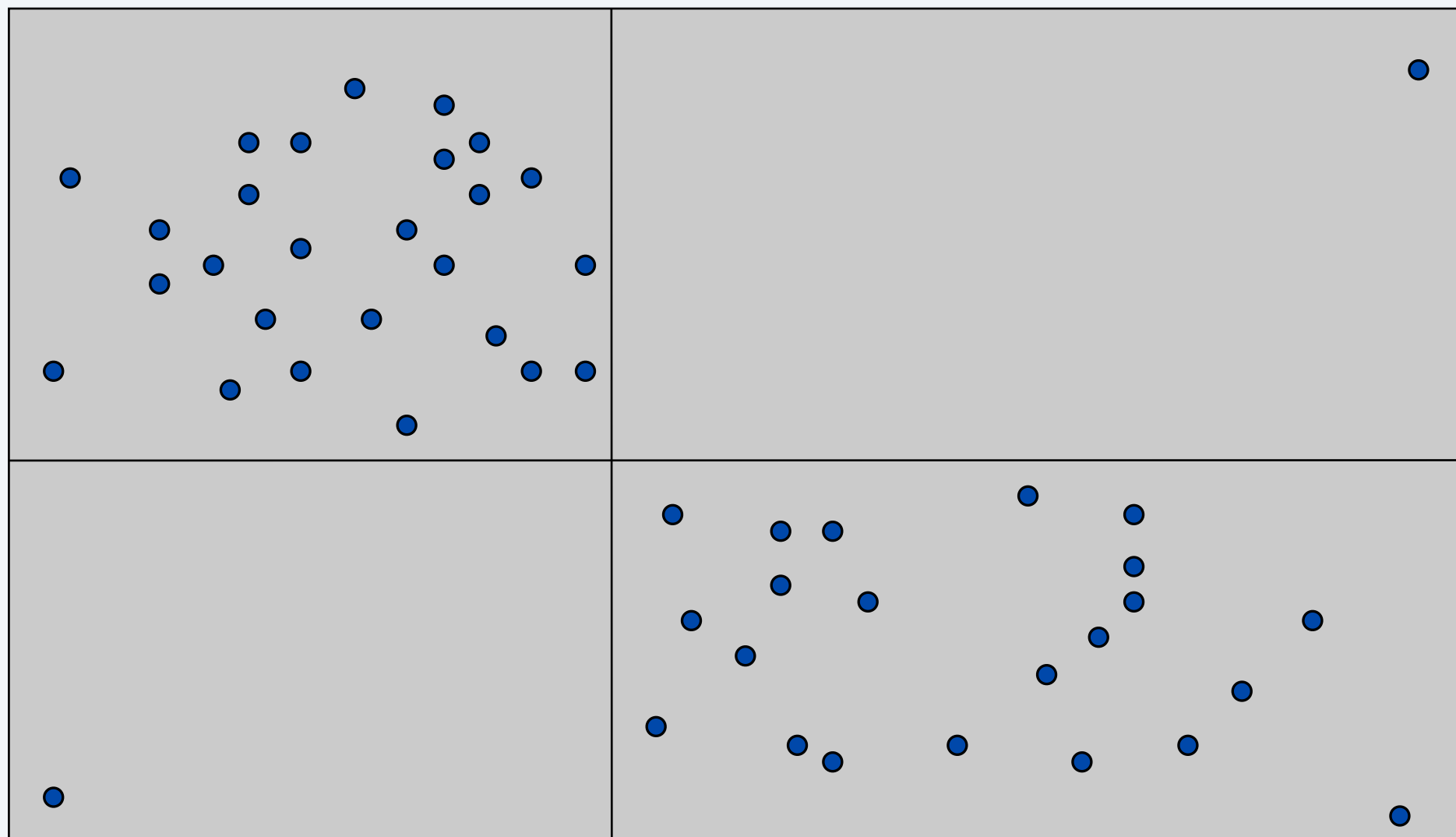
Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

seems like $\Theta(N^2)$
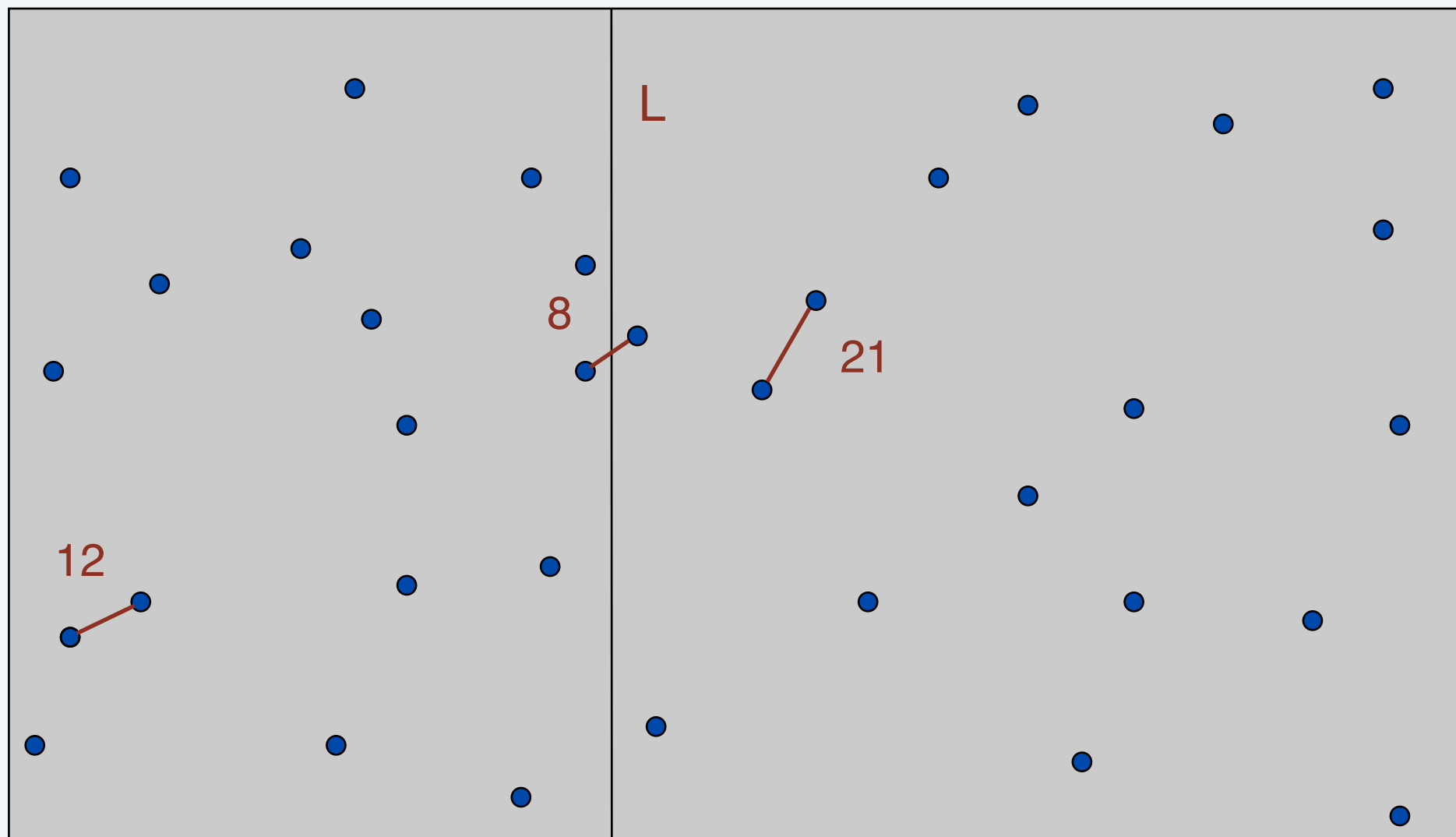
How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance $< \delta$.

- Observation: only need to consider points within δ of line L.

Find closest pair with one point in each side, assuming that distance $< \delta$.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y-coordinate.
- Only check distances of those within 11 positions in sorted list!

why 11?
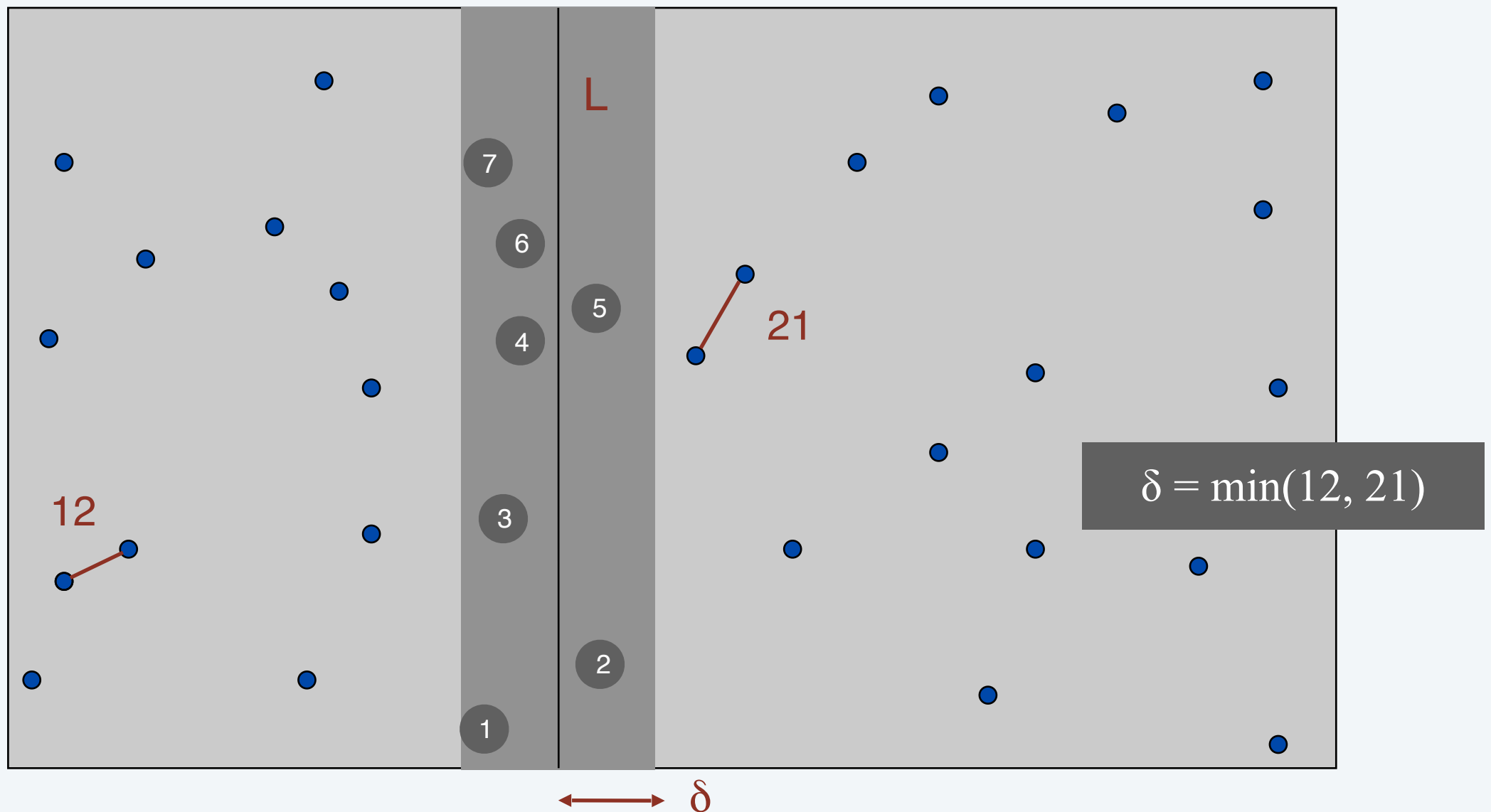
Def. Let s_i be the point in the 2δ-strip, with the ith smallest y-coordinate.

Claim. If $|i-j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2\,(\frac{1}{2}\,\delta)$. ∎

Fact. Claim remains true if we replace 12 with 7.

?

2 rows

$\frac{1}{2}\delta$

$\frac{1}{2}\delta$

$\frac{1}{2}\delta$

δ δ
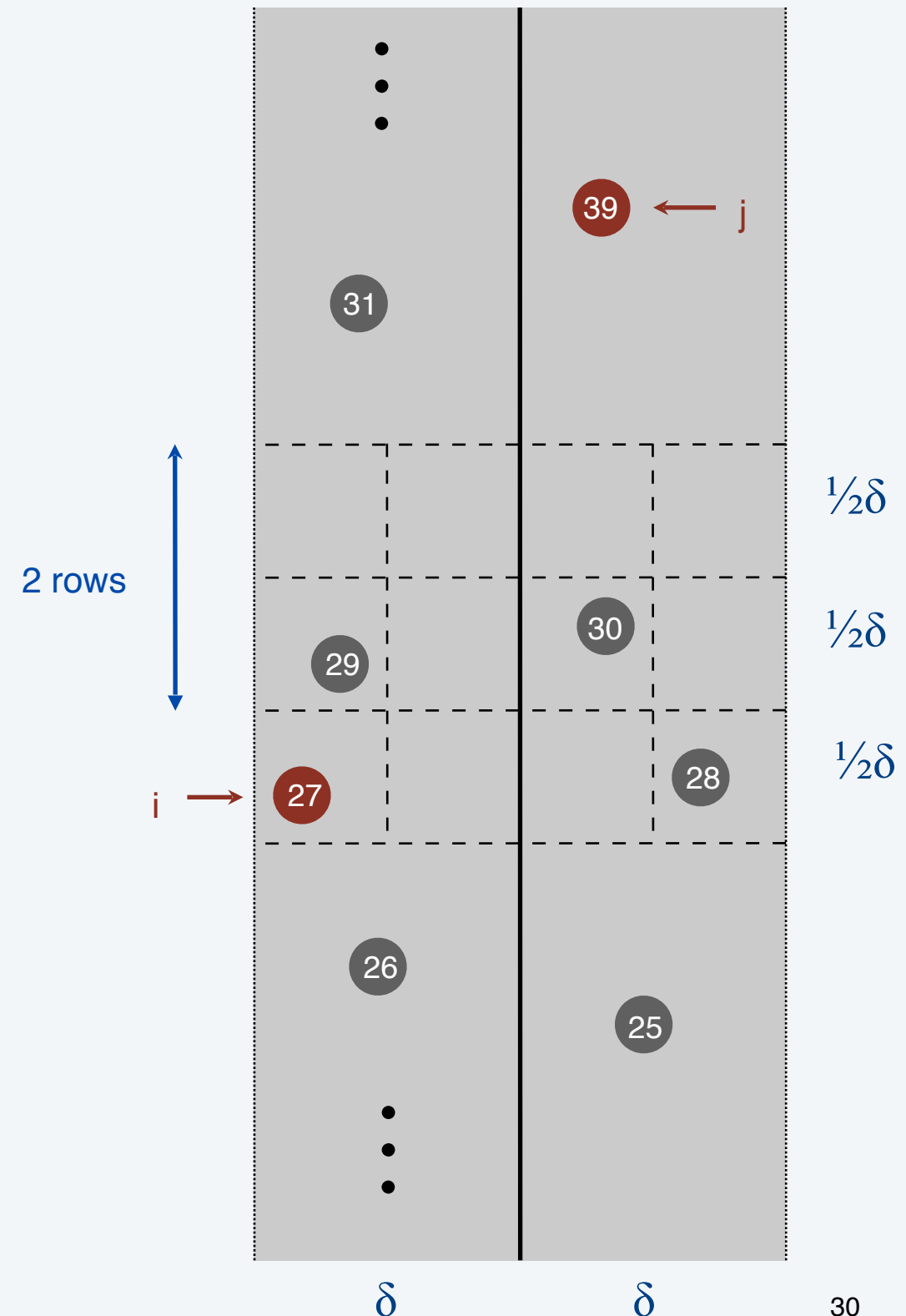
Closest pair of points: divide-and-conquer algorithm

CLOSEST-PAIR $(p_1, p_2, \ldots, p_n)$

Compute separation line L such that half the points are on each side of the line. — $O(n \log n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR (points in left half).

$\delta_2 \leftarrow$ CLOSEST-PAIR (points in right half). — $2\,T(n\,/\,2)$

$\delta \leftarrow \min \{ \delta_1, \delta_2 \}$.

Delete all points further than δ from line L. — $O(n)$

Sort remaining points by y-coordinate. — $O(n \log n)$

Scan points in y-order and compare distance between each point and next 11 neighbors. If any of these distances is less than δ, update δ. — $O(n)$

RETURN δ.

Closest pair of points: analysis

Theorem. The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log^2 n)$ time.

$$
T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n \log n) & \text{otherwise} \end{cases}
$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2$$

Lower bound. In quadratic decision tree model, any algorithm for closest pair (even in 1D) requires $\Omega(n \log n)$ quadratic tests.

Improved closest pair algorithm

Q. How to improve to $O(n \log n)$?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by x-coordinate, and all points sorted by y-coordinate.
- Sort by merging two pre-sorted lists.

Theorem. [Shamos 1975] The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log n)$ time.

Pf.
$$
T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}
$$

Note. See SECTION 13.7 for a randomized $O(n)$ time algorithm.
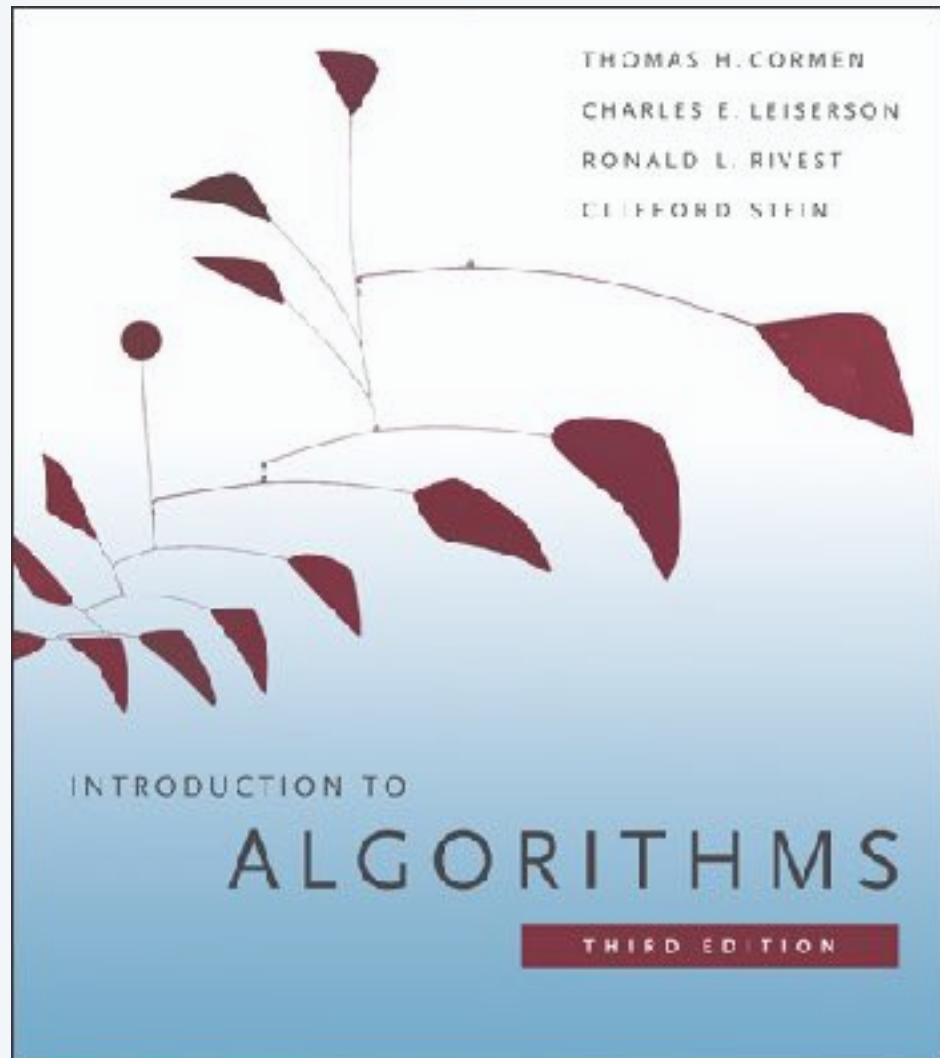
5. Divide and Conquer

- ‣ *mergesort*
- ‣ *counting inversions*
- ‣ *closest pair of points*
- ‣ **randomized quicksort**
- ‣ *median and selection*

CHAPTER 7

Randomized quicksort

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.

Recur in both left and right subarrays.

the array A

7	6	12	3	11	8	9	1	4	10	2

p

the partitioned array A

3	1	4	2	6	7	12	11	8	9	10

$\longleftarrow L \longrightarrow$ M $\longleftarrow \qquad R \longrightarrow$

RANDOMIZED-QUICKSORT (A)

IF list A has zero or one element

   RETURN.

Pick pivot $p \in A$ uniformly at random.

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, a_i). $\longleftarrow$ 3-way partitioning can be done in-place (using n–1 compares)

RANDOMIZED-QUICKSORT(L).

RANDOMIZED-QUICKSORT(R).

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array
of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

the original array of elements A

7	6	12	3	11	8	9	1	4	10	2	13	5

first partitioning element
(chosen uniformly at random)

first partitioning
element in
left subarray

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

- An element is compared with only its ancestors and descendants.

first partitioning
element in
left subarray

first partitioning element
(chosen uniformly at random)

**3 and 6 are compared
(when 3 is partitioning element)**

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array
of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.
- An element is compared with only its ancestors and descendants.

first partitioning
element in
left subarray

first partitioning element
(chosen uniformly at random)

**2 and 8 are not compared
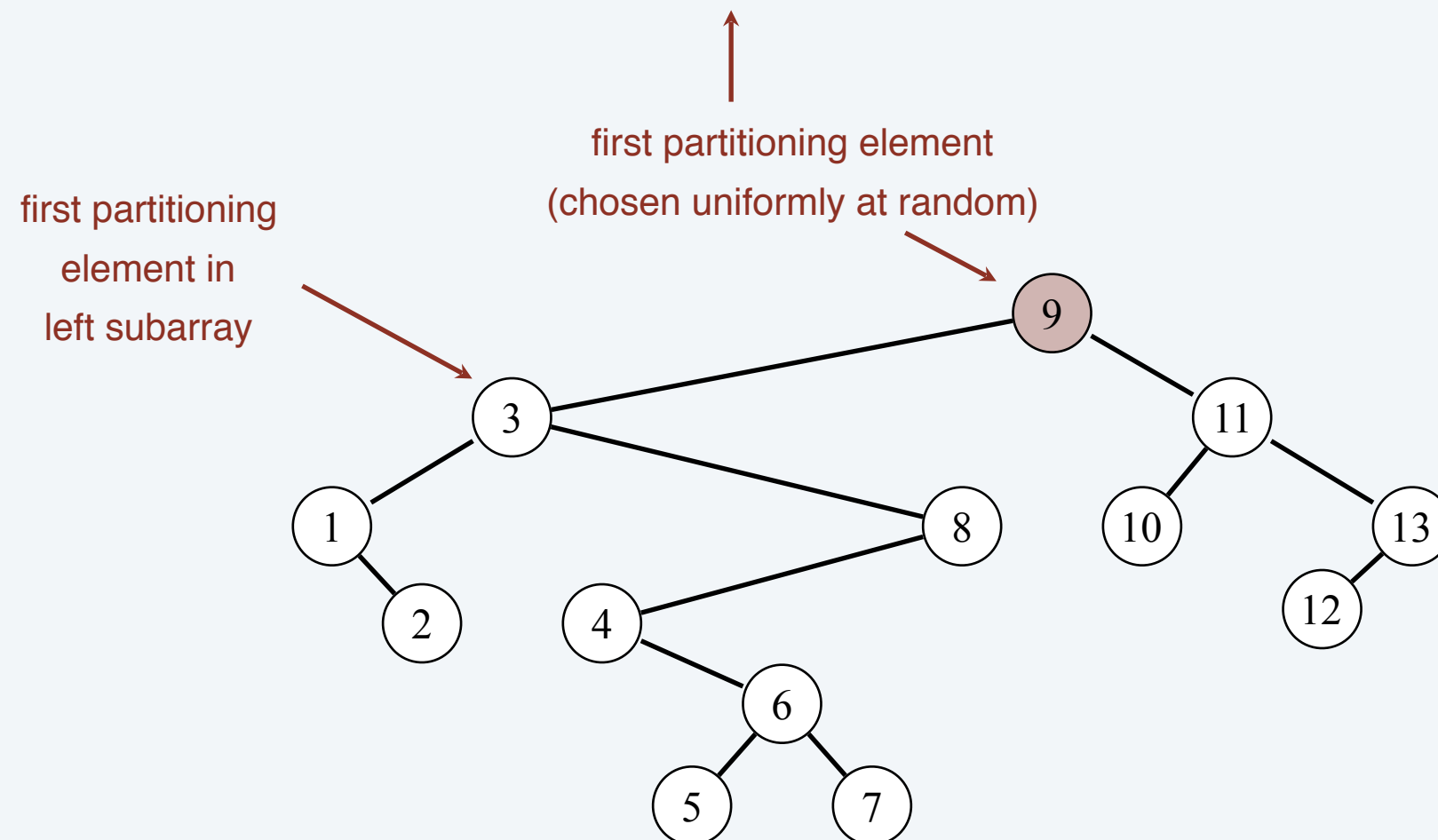(because 3 partitions them)**

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array
of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.
- An element is compared with only its ancestors and descendants.
- **Pr** [a_i and a_j are compared] $= 2 \ / \ |j - i + 1|$.

first partitioning
element in
left subarray

first partitioning element
(chosen uniformly at random)

**Pr[2 and 8 compared] = 2/7
(compared if either 2 or 8 are chosen
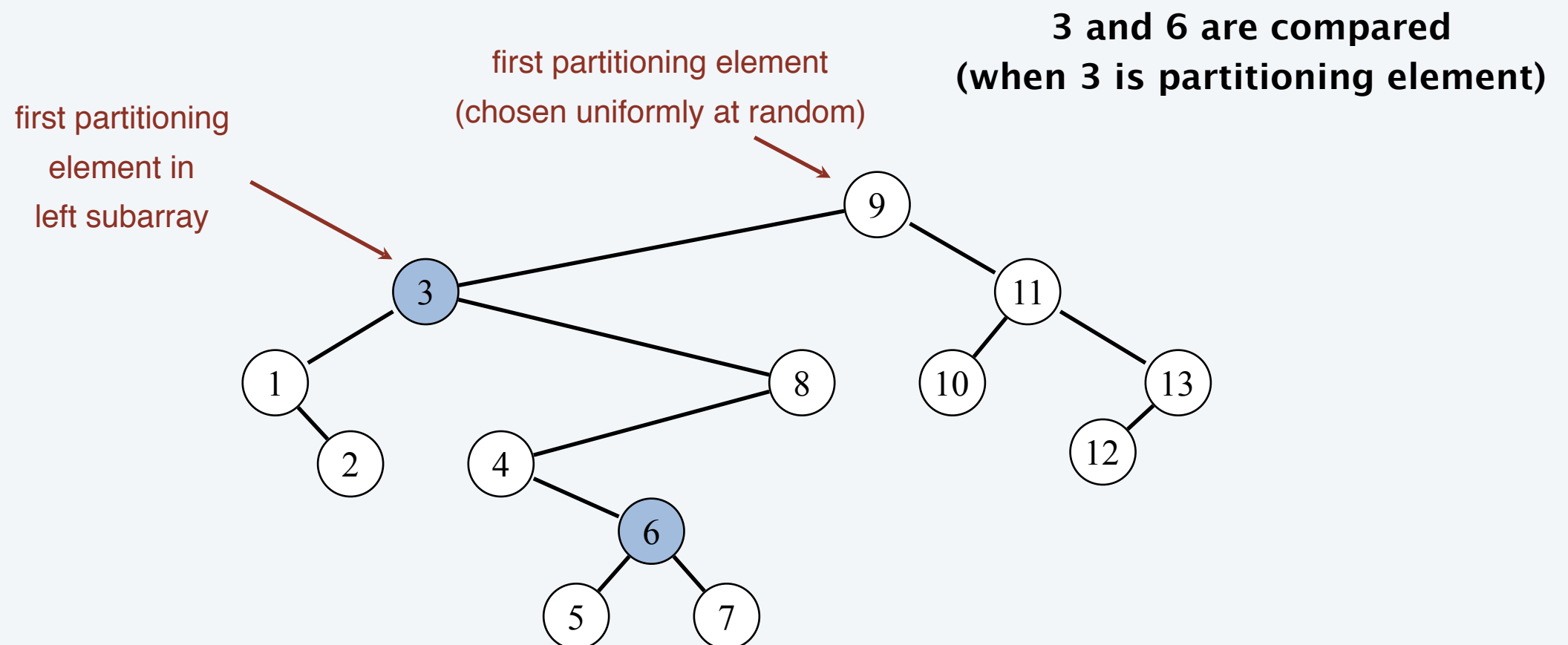as partition before 3, 4, 5, 6 or 7)**

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.
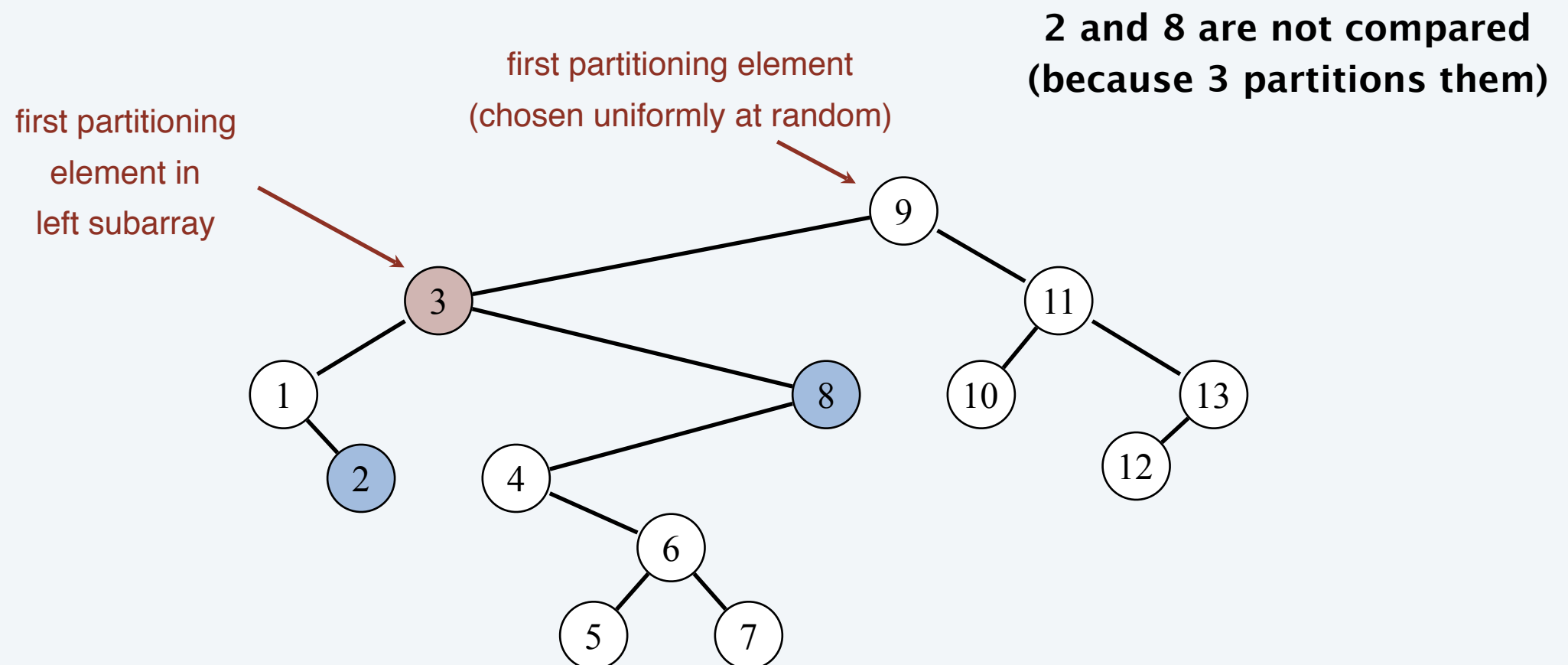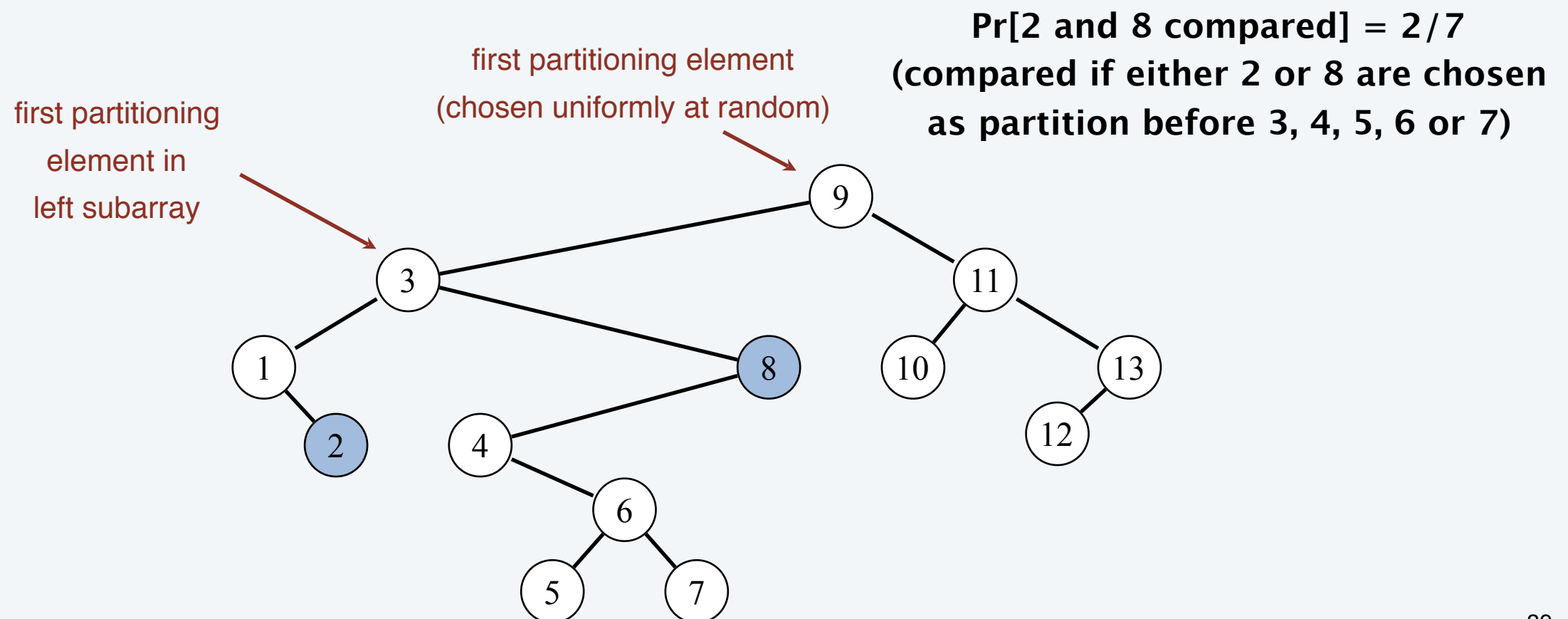- An element is compared with only its ancestors and descendants.
- **Pr** [a_i and a_j are compared] $= 2 \ / \ |j - i + 1|$.

- Expected number of compares $=$

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{2}{j - i + 1} = 2 \sum_{i=1}^{N} \sum_{j=2}^{N-i+1} \frac{1}{j}$$

$$\leq \ 2N \sum_{j=1}^{N} \frac{1}{j}$$

$$\sim \ 2N \int_{x=1}^{N} \frac{1}{x} \, dx$$

$$= \ 2N \ln N$$

all pairs i and j

Remark. Number of compares only decreases if equal elements.
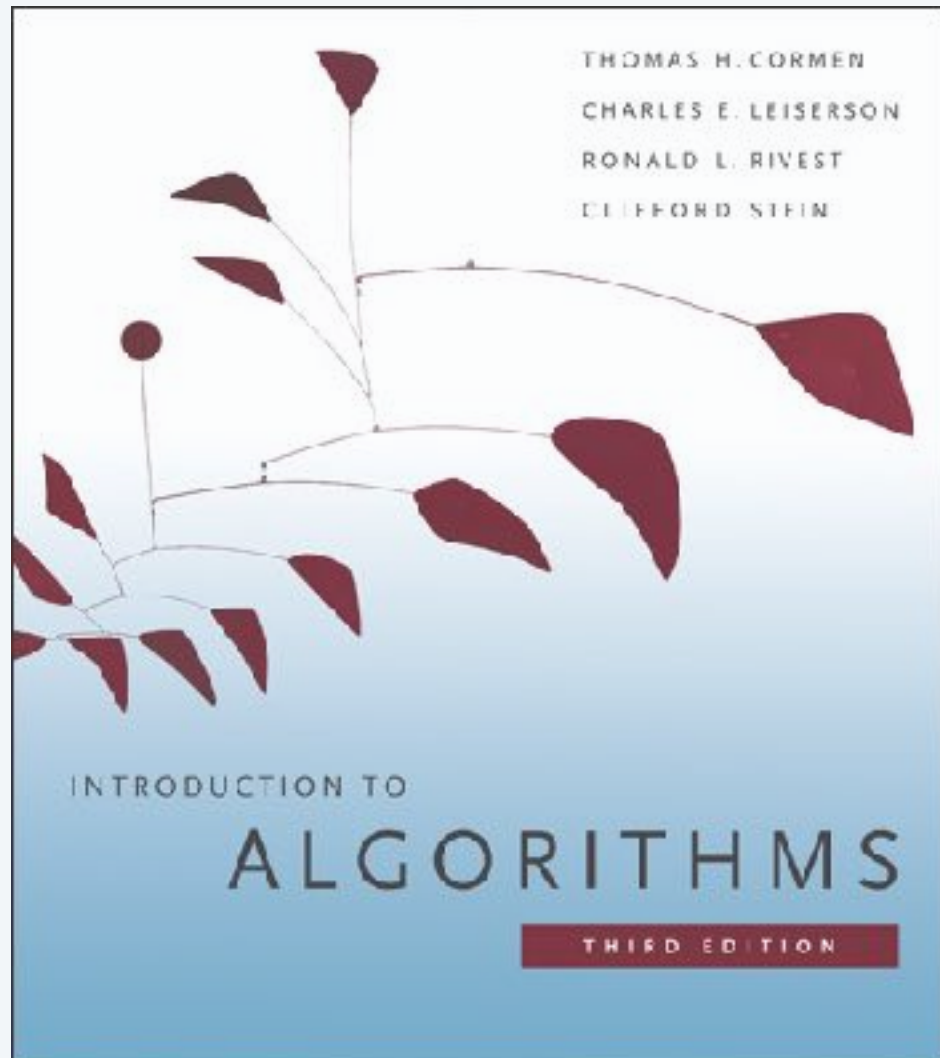
5. DIVIDE AND CONQUER

▸ *mergesort*

▸ *counting inversions*

▸ *closest pair of points*

▸ *randomized quicksort*

▸ **median and selection**

CHAPTER 9

Median and selection problems

Selection. Given n elements from a totally ordered universe, find k^{th} smallest.

- Minimum: $k = 1$; maximum: $k = n$.
- Median: $k = \lfloor (n + 1) / 2 \rfloor$.
- $O(n)$ compares for min or max.
- O($n \log n$) compares by sorting.
- $O(n \log k)$ compares with a binary heap.

Applications. Order statistics; find the "top k"; bottleneck paths, …

Q. Can we do it with $O(n)$ compares?

A. Yes! Selection is easier than sorting.

Quickselect

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.

Recur in one subarray—the one containing the kth smallest element.

QUICK-SELECT (A, k)

Pick pivot $p \in A$ uniformly at random.

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, p). ⟵ 3-way partitioning can be done in-place (using n–1 compares)

IF $k \leq |L|$ RETURN QUICK-SELECT (L, k).

ELSE IF $k > |L| + |M|$ RETURN QUICK-SELECT $(R, k - |L| - |M|)$

ELSE   RETURN p.

Quickselect analysis

Intuition. Split candy bar uniformly $\Rightarrow$ expected size of larger piece is ¾.

$$T(n) \ \leq \ T(\tfrac{3}{4}\,n) \ + n \quad \Rightarrow \quad T(n) \ \leq \ 4\,n$$

Def. $T(n, k)$ = expected # compares to select k^{th} smallest in an array of size n.

(2 + 2 ln 2) n = 3.3863 n compares to find the median

Proposition. $T(n, k) \ \sim \ 2n \ + 2\,k \ln(n\,/\,k) \ + 2\,(n - k) \ln(n\,/\,(n - k))$.

Pf. Rather technical. We'll prove a weaker but simpler version.

Quickselect analysis

Intuition. Split candy bar uniformly $\Rightarrow$ expected size of larger piece is ¾.

$$T(n) \le T(\tfrac{3}{4}\,n) + n \quad \Rightarrow \quad T(n) \le 4\,n$$

Def. $T(n, k)$ = expected # compares to select k^{th} smallest in an array of size $\le n$.

Def. $T(n) = \max_k T(n, k)$.

Proposition. $T(n) \le 4\,n$.

Pf. [by strong induction on n]

- Assume true for $1, 2, \ldots, n-1$.

- $T(n)$ satisfies the following recurrence:

can assume we always recur on largest subarray
since T(n) is monotonic and
we are trying to get an upper bound

$$T(n) \le n + 2/n\,[\ T(n/2) + \ldots + T(n-3) + T(n-2) + T(n-1)\ ]$$

$$\le n + 2/n\,[\ 4\,n/2 + \ldots + 4(n-3) + 4(n-2) + 4(n-1)\ ]$$

$$= n + 4\,(3/4\,n)$$

$$= 4\,n. \quad \blacksquare$$

tiny cheat: sum should start at $T(\lfloor n/2 \rfloor)$

Selection in worst case linear time

Goal. Find pivot element p that divides list of n elements into two pieces so that each piece is guaranteed to have $\leq 7/10\ n$ elements.

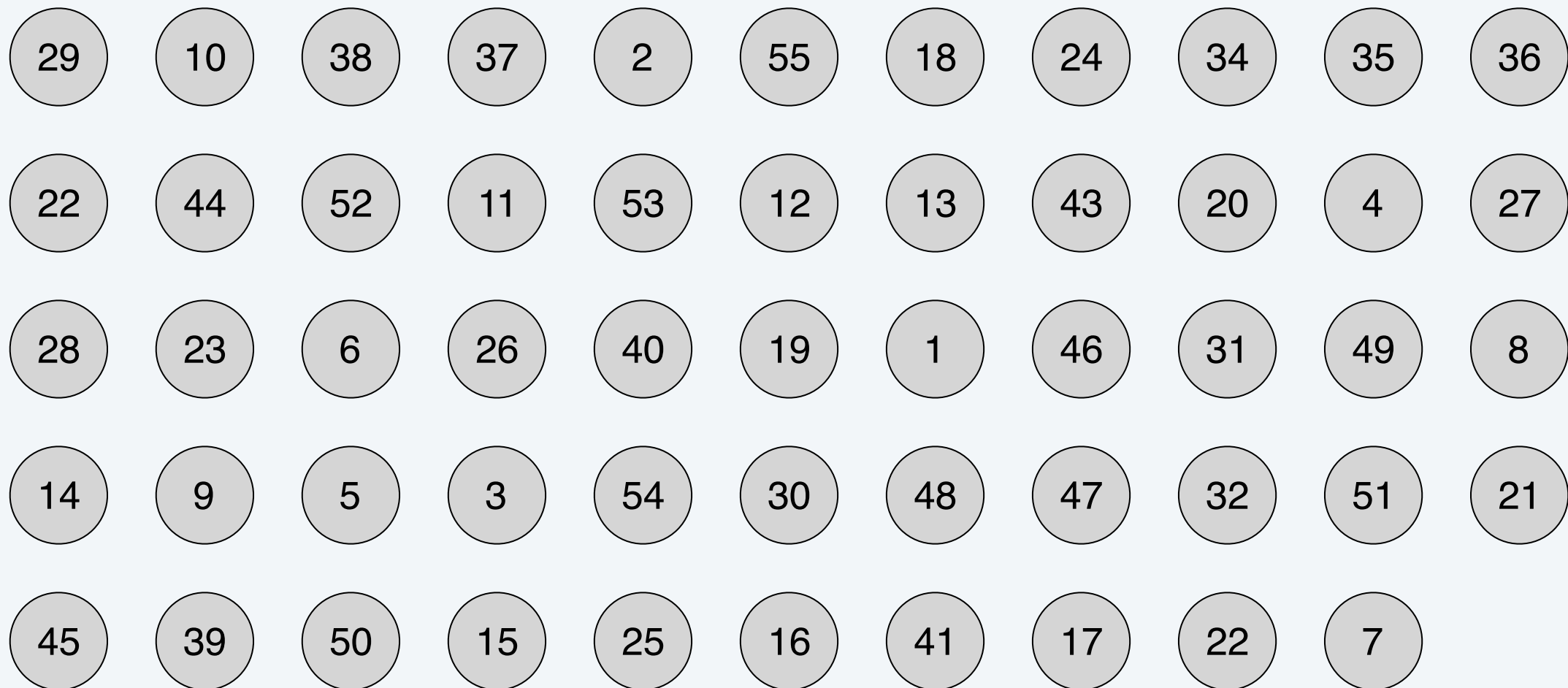Q. How to find approximate median in linear time?
A. Recursively compute median of sample of $\leq 2/10\ n$ elements.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(7/10\ n) + T(2/10\ n) + \Theta(n) & \text{otherwise} \end{cases}$$

two subproblems
of different sizes!

Choosing the pivot element

- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).

29	10	38	37	2	55	18	24	34	35	36
22	44	52	11	53	12	13	43	20	4	27
28	23	6	26	40	19	1	46	31	49	8
14	9	5	3	54	30	48	47	32	51	21
45	39	50	15	25	16	41	17	22	7	

N = 54

Choosing the pivot element

- Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).

medians

29	10	**38**	37	2	55	**18**	24	34	**35**	36
22	44	52	11	53	12	13	**43**	20	4	27
28	**23**	6	26	**40**	**19**	1	46	**31**	49	8
14	9	5	3	54	30	48	47	32	51	21
45	39	50	**15**	25	16	41	17	22	7	

N = 54
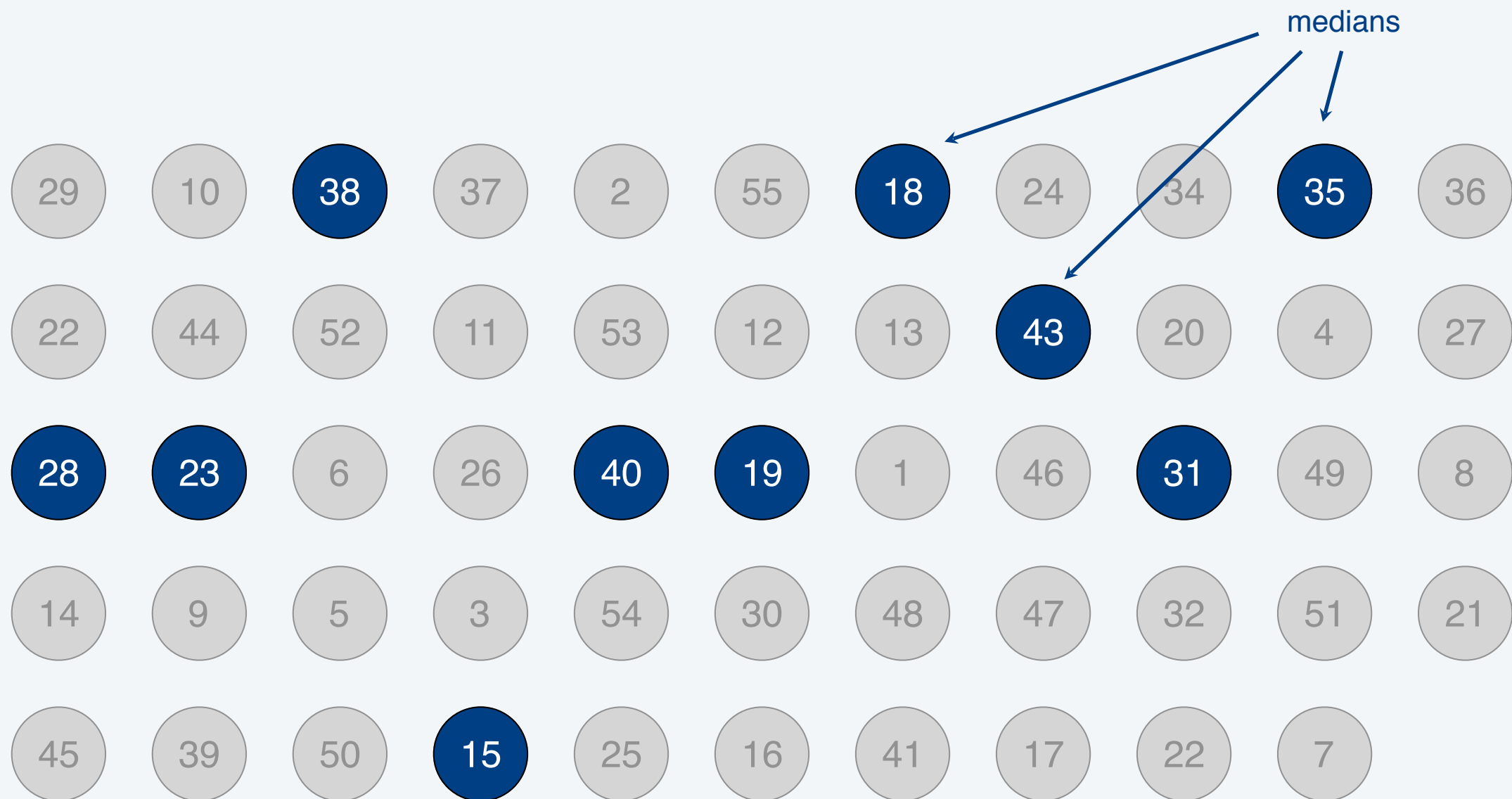
Choosing the pivot element
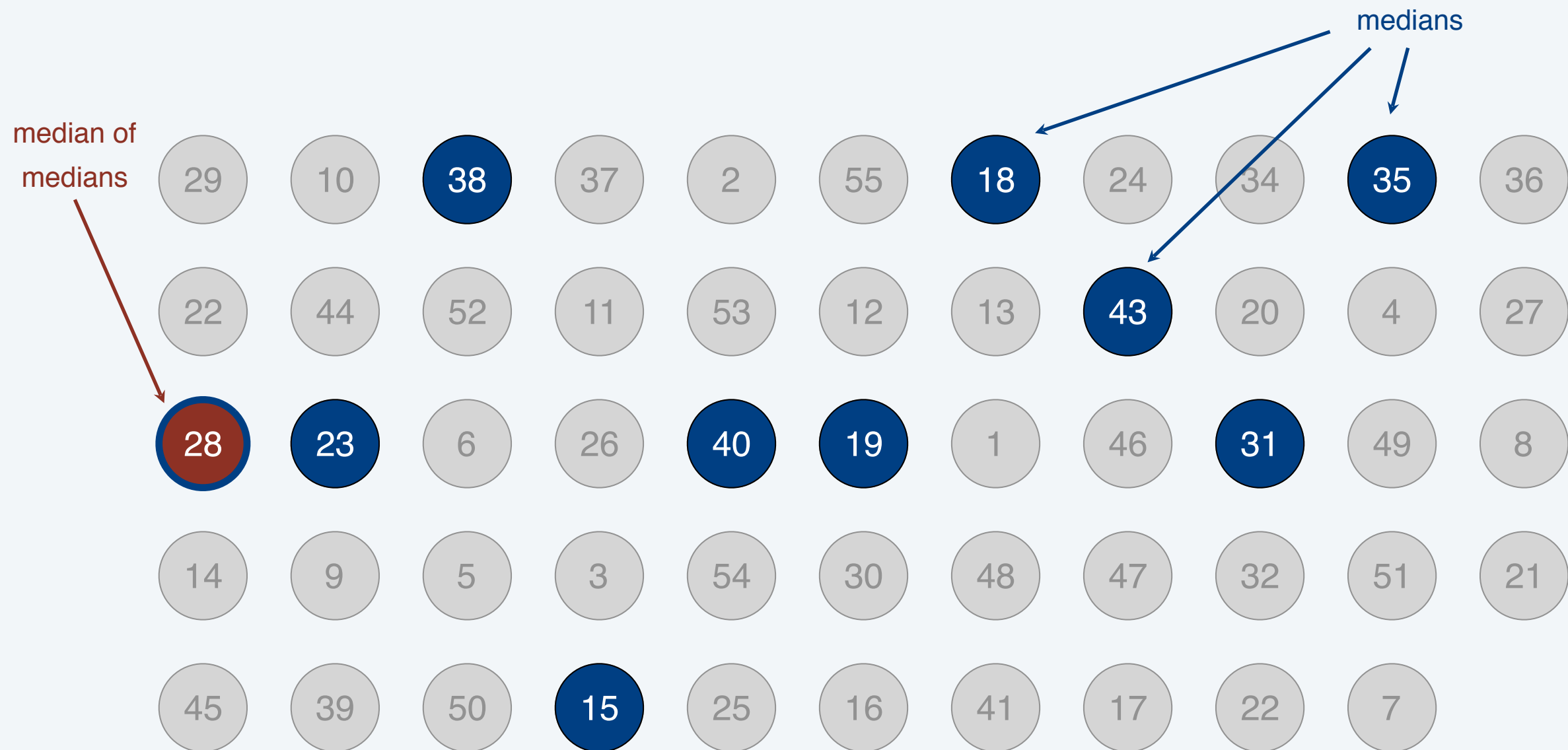
- Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).
- Find median of $\lfloor n/5 \rfloor$ medians recursively.
- Use median-of-medians as pivot element.

N = 54

Median-of-medians selection algorithm

MOM-SELECT (A, k)

$n \leftarrow |A|$.

IF $n < 50$ RETURN k^{th} smallest of element of A via mergesort.

Group A into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).

$B \leftarrow$ median of each group of 5.

$p \leftarrow$ MOM-SELECT$(B, \lfloor n / 10 \rfloor)$ $\longleftarrow$ median of medians

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, p).
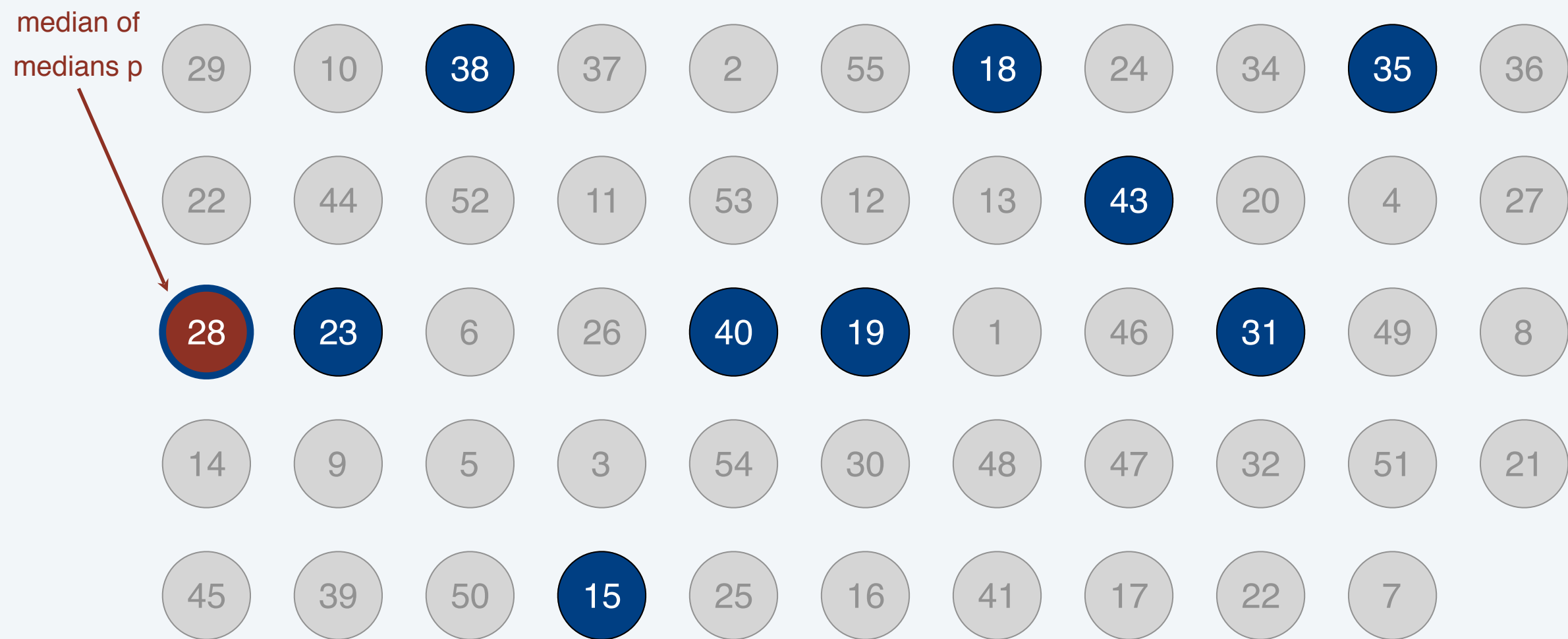
IF $k \leq |L|$ RETURN MOM-SELECT (L, k).

ELSE IF $k > |L| + |M|$ RETURN MOM-SELECT $(R, k - |L| - |M|)$

ELSE RETURN p.

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\leq p$.

median of medians p

29	10	38	37	2	55	18	24	34	35	36
22	44	52	11	53	12	13	43	20	4	27
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N = 54

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\leq p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\leq p$.

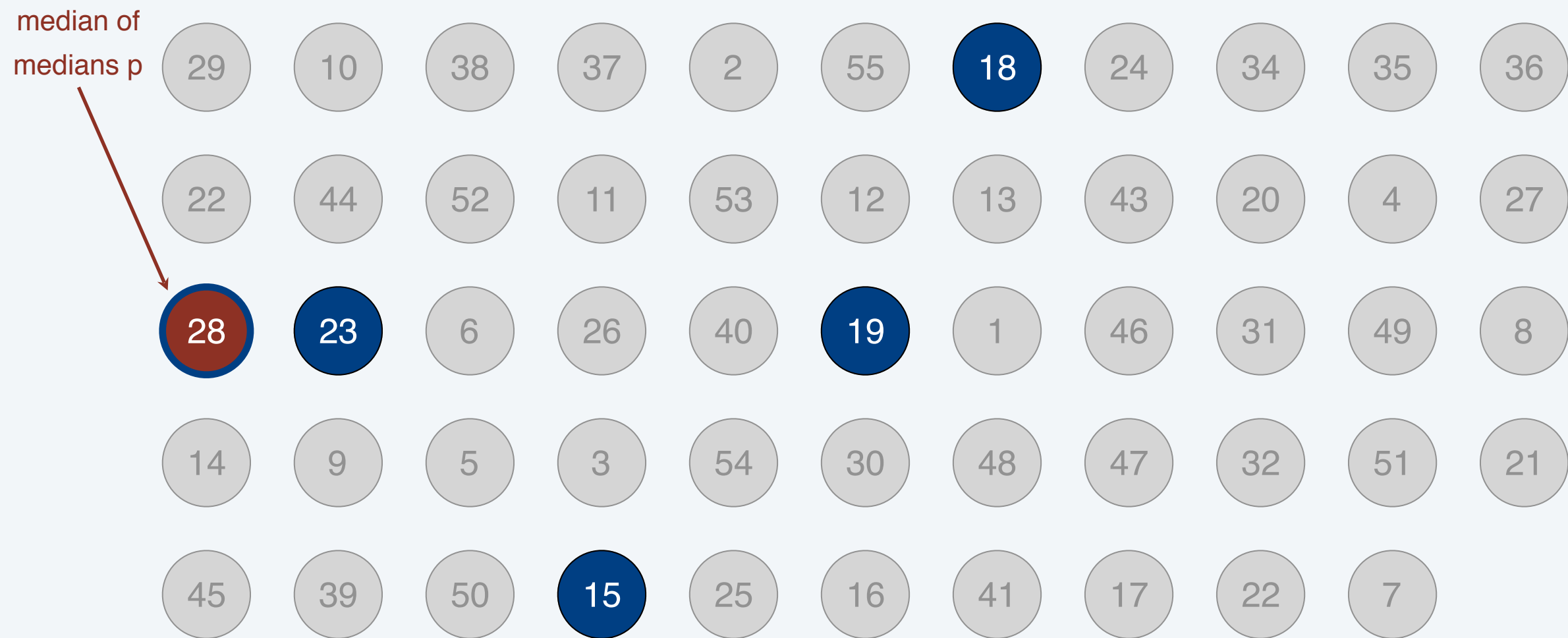median of medians p
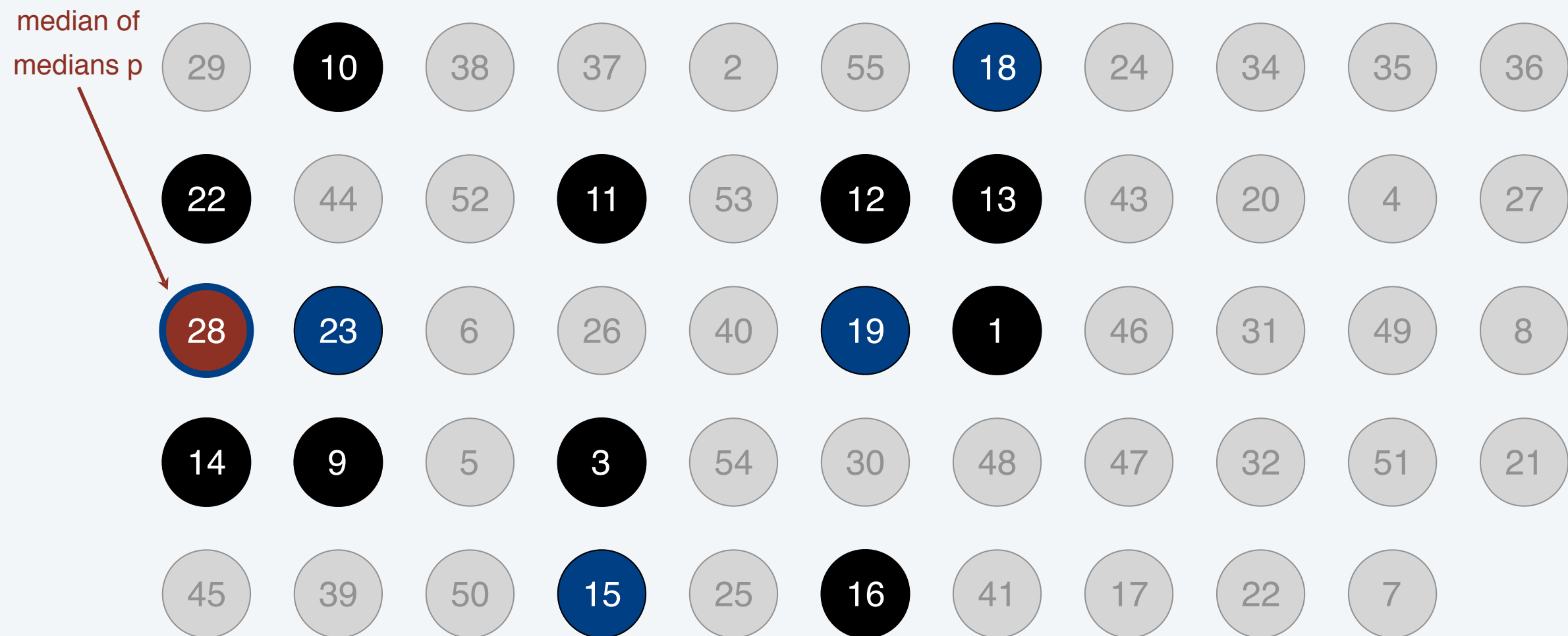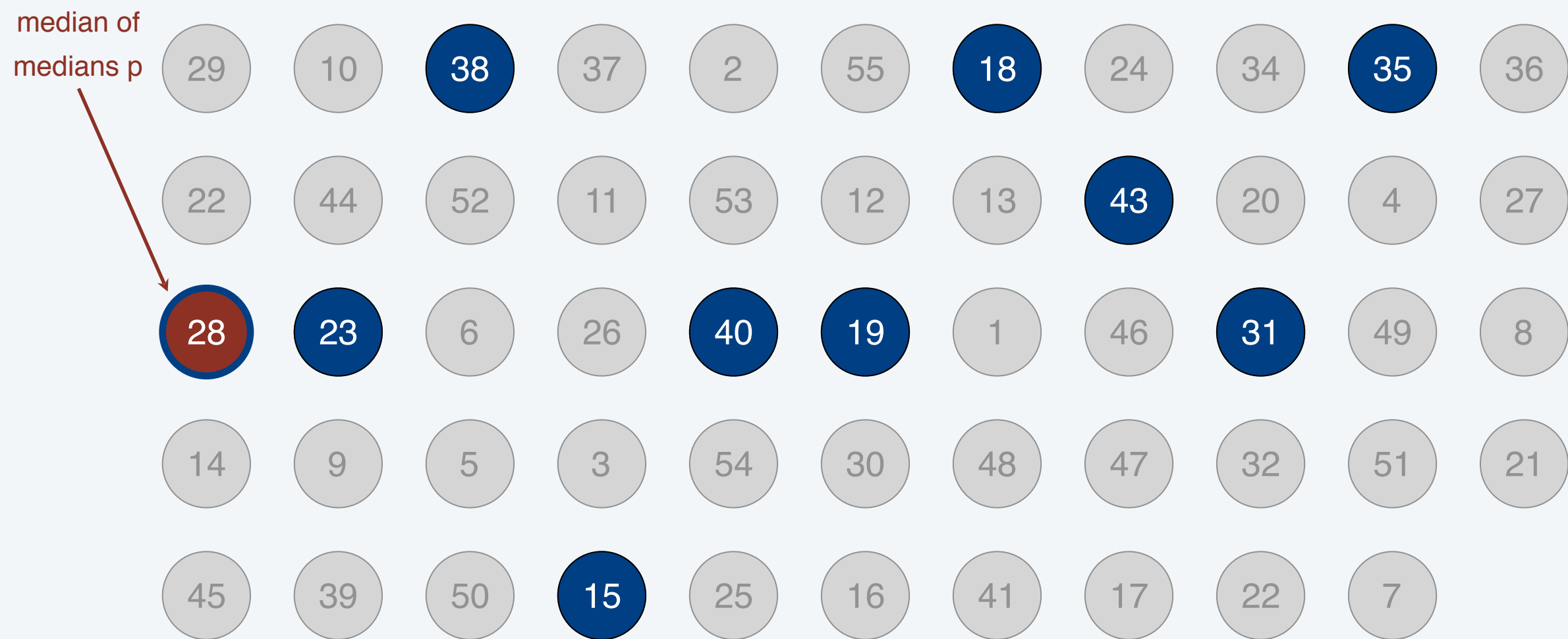
N = 54

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\leq p$.

- At least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ medians $\leq p$.

- At least $3 \lfloor n/10 \rfloor$ elements $\leq p$.
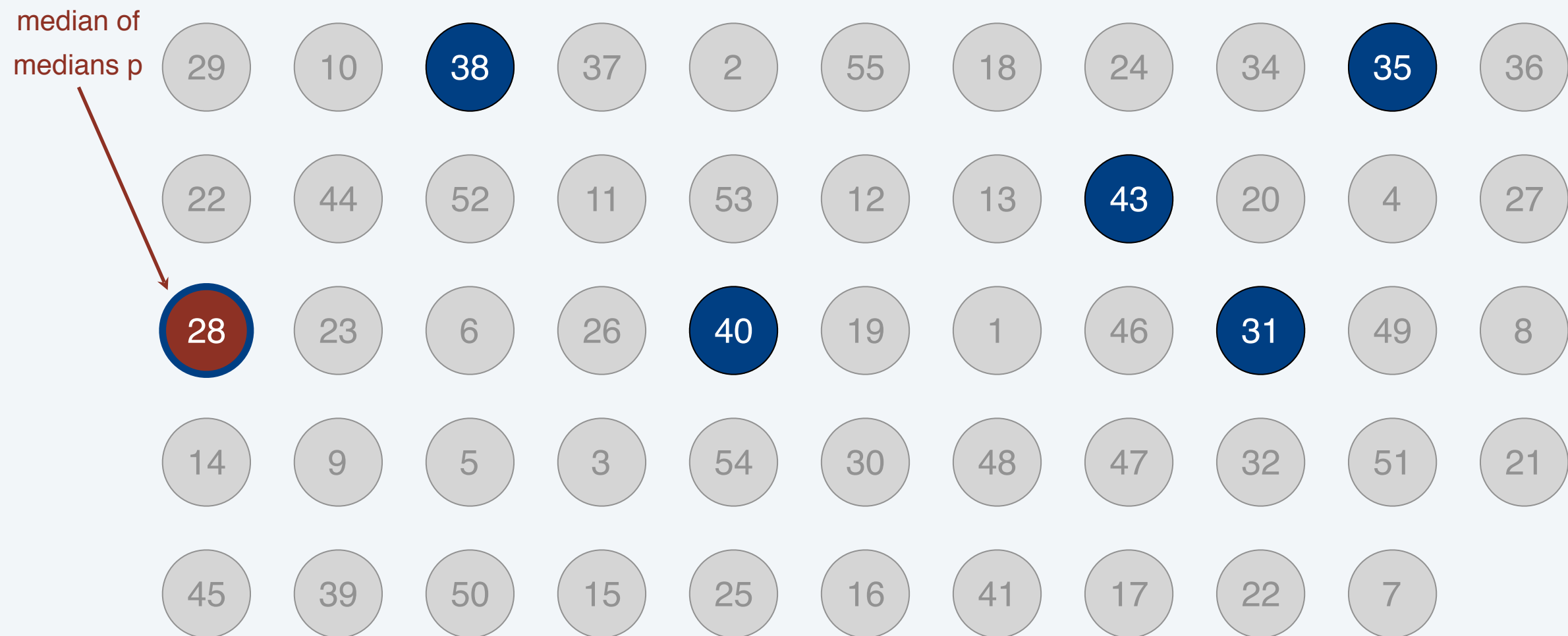

median of medians p

N = 54

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians ≥ p.

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\geq p$.
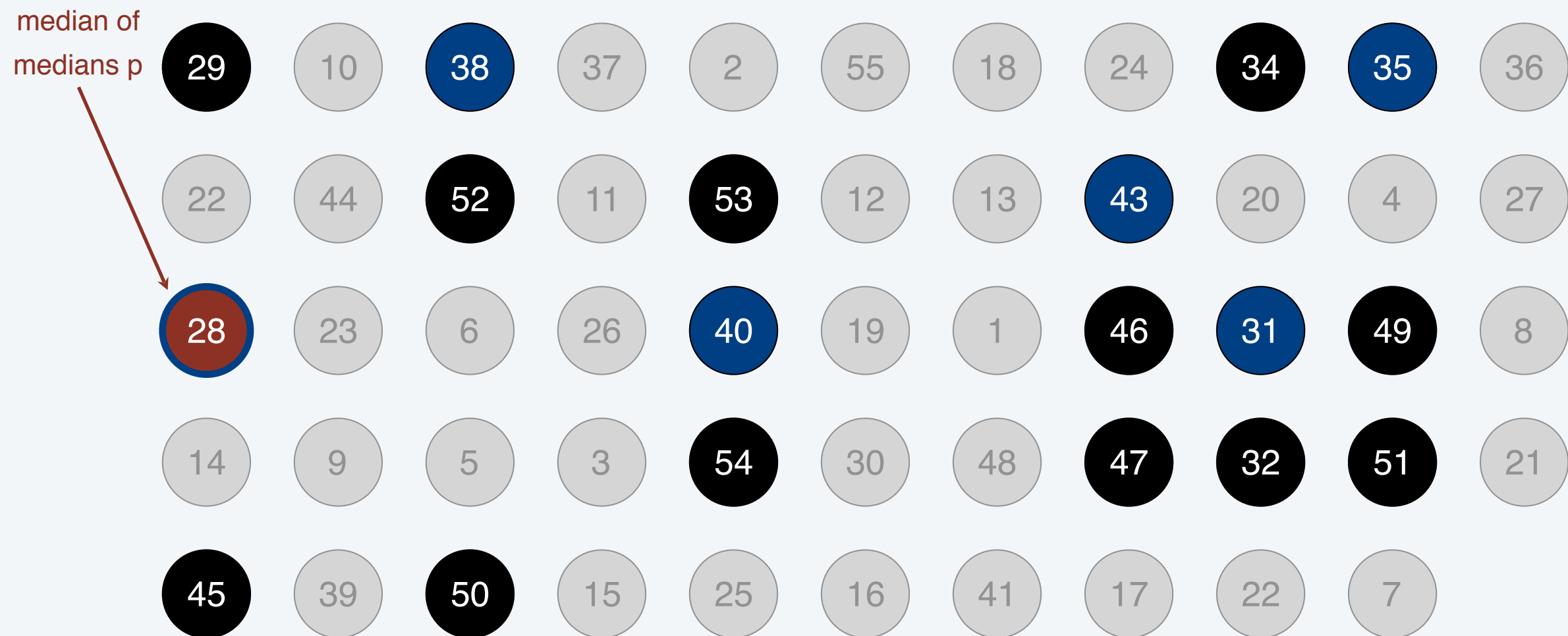- Symmetrically, at least $\lfloor n / 10 \rfloor$ medians $\geq p$.

N = 54

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians ≥ p.
- Symmetrically, at least $\lfloor n / 10 \rfloor$ medians ≥ p.
- At least $3 \lfloor n / 10 \rfloor$ elements ≥ p.

median of
medians p

N = 54

Median-of-medians selection algorithm recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p.
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # compares on an array of n elements.

$$C(n) \leq C(\lfloor n/5 \rfloor) + C(n - 3\lfloor n/10 \rfloor) + \tfrac{1}{5} n$$

median of medians   recursive select   computing median of 5 (6 compares per group)

partitioning (n compares)

Now, solve recurrence.

- Assume n is both a power of 5 and a power of 10?
- Assume $C(n)$ is monotone nondecreasing?

Median-of-medians selection algorithm recurrence

Analysis of selection algorithm recurrence.

- $T(n)$ = max # compares on an array of $\le n$ elements.
- $T(n)$ is monotone, but $C(n)$ is not!

$$T(n) \le \begin{cases} 6n & \text{if } n < 50 \\ T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n & \text{otherwise} \end{cases}$$
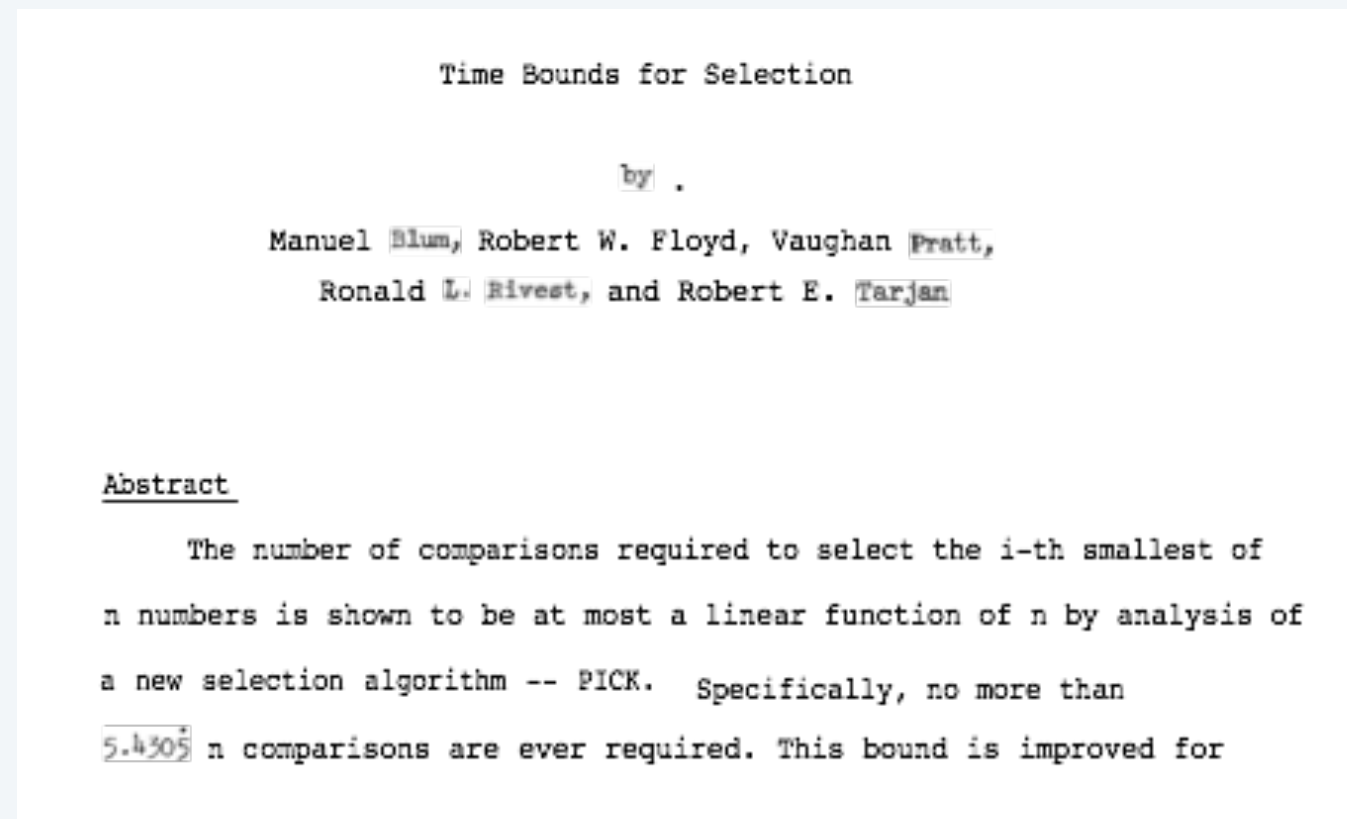
Claim. $T(n) \le 44\,n$.

- Base case: $T(n) \le 6\,n$ for $n < 50$ (mergesort).
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- Induction step: for $n \ge 50$, we have:

$$\begin{aligned} T(n) & \le T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5\,n \\[1mm] & \le 44\,(\lfloor n/5 \rfloor) + 44\,(n - 3\lfloor n/10 \rfloor) + 11/5\,n \\[1mm] & \le 44\,(n/5) + 44\,n - 44\,(n/4) + 11/5\,n \quad \longleftarrow \quad \text{for } n \ge 50,\ 3\lfloor n/10 \rfloor \ge n/4 \\[1mm] & = 44\,n. \quad \blacksquare \end{aligned}$$

Linear-time selection postmortem

Proposition. [Blum-Floyd-Pratt-Rivest-Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.

Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Theory.

- Optimized version of BFPRT: $\leq 5.4305\,n$ compares.
- Best known upper bound [Dor-Zwick 1995]: $\leq 2.95\,n$ compares.
- Best known lower bound [Dor-Zwick 1999]: $\geq (2 + \varepsilon)\,n$ compares.

Linear-time selection postmortem

Proposition. [Blum-Floyd-Pratt-Rivest-Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.
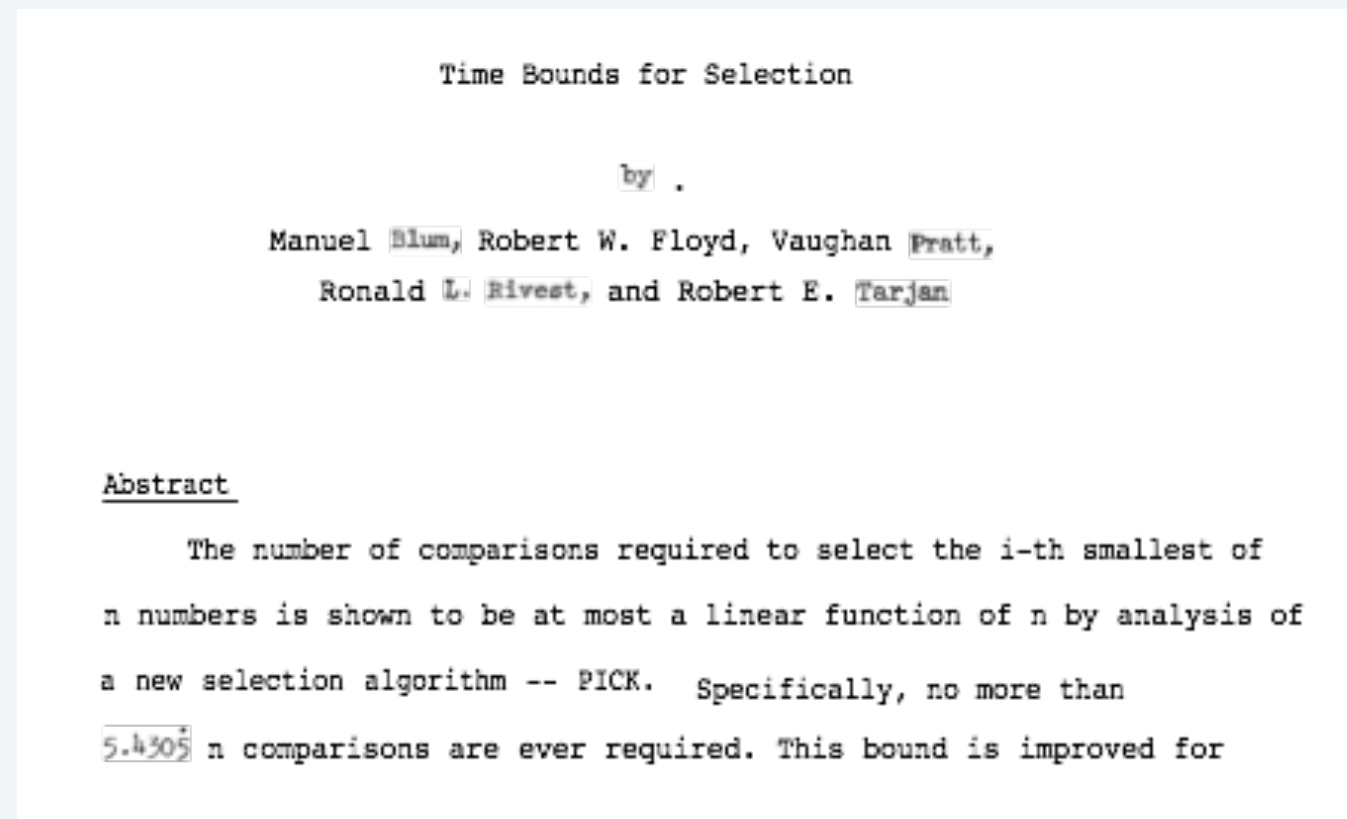

Practice. Constant and overhead (currently) too large to be useful.

Open. Practical selection algorithm whose worst-case running time is $O(n)$.