3.2 偏导数与全微分

3.2.4 复合函数的偏导数和全微分

3.2.5 一阶全微分形式的不变性

3.2.6 隐函数的微分法

3.2.4 多元复合函数的偏导数与全微分

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

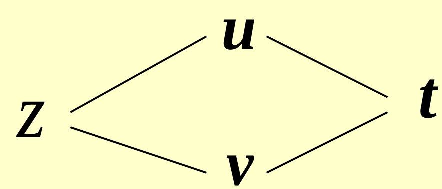
微分法则 $\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$

本节内容: 多元复合函数求导的链式法则

一、多元复合函数求导的链式法则

定理 若函数 $u=\varphi(t),\ v=\psi(t)$ 在点 t 可导，$z=f(u,v)$ 在点 (u,v) 处可微，则复合函数 $z=f(\varphi(t),\psi(t))$ 在点 t 可导，且有链式法则

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



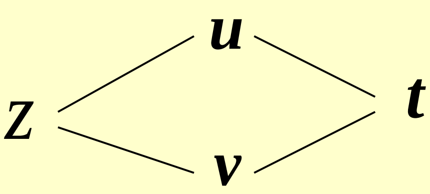
证：设 t 取增量 Δt，则相应中间变量有增量 Δu，Δv，

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\quad\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}\quad(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\,)$$

令 $\Delta t\to 0$, 则有 $\Delta u\to 0$, $\Delta v\to 0$,

$$\frac{\Delta u}{\Delta t}\to\frac{\mathrm{d}u}{\mathrm{d}t},\qquad \frac{\Delta v}{\Delta t}\to\frac{\mathrm{d}v}{\mathrm{d}t}$$



$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\to 0$$

(△t<0 时, 根式前加 “–”号)

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}}$$

法则:

连线相乘，分线相加

(全导数公式)

注: 若定理中 $f(u,v)$ 在点 (u,v) 可微减弱为

偏导数存在, 则定理结论不一定成立.

例如: $z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$

$u = t, \quad v = t$

易知: $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)} = f_u(0,0) = 0$, $\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)} = f_v(0,0) = 0$

但复合函数 $z = f(t,t) = \dfrac{t}{2}$

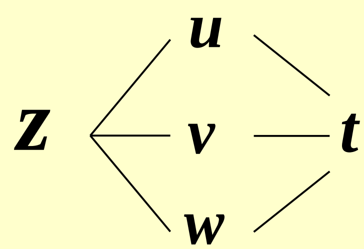
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = 0\cdot 1 + 0\cdot 1 = 0$$

推广: 设下面所涉及的函数都可微 .

1) 中间变量多于两个的情形. 例如, $z = f(u, v, w)$,

$u = \varphi(t), v = \psi(t), w = \omega(t)$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

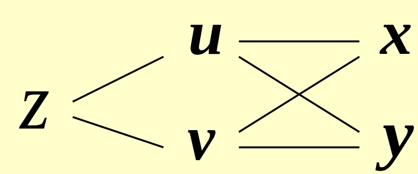


$$= f_1'\varphi' + f_2'\psi' + f_3'\omega'$$

2) 中间变量是多元函数的情形.例如,

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1'\varphi_1' + f_2'\psi_1'$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = f_1'\varphi_2' + f_2'\psi_2'$$

又如, $z = f(x,v)$, $v = \psi(x,y)$

$$\begin{tikzcd}[column sep=small] z \arrow[rr, dash] \arrow[dr, dash] & & x \\ & v \arrow[ur, dash] \arrow[r, dash] & y \end{tikzcd}$$

当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}} = \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1' + f_2' \psi_1'$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = f_2' \psi_2'$$

注意: 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同,

$\frac{\partial z}{\partial x}$ 表示固定复合函数 $z = f(x,\psi(x,y))$ 中的 y 对 x 求导

$\frac{\partial f}{\partial x}$ 表示固定 $f(x,v)$ v 对 x 求导

注：在使用链式法则时，必须注意外层函数可微这个条件.

一般地, 若 $f(u_1,\cdots,u_m)$ 在点 $(u_1,\cdots,u_m)$ 可微, 函数组

$$u_k = g_k(x_1,\cdots,x_n)\ (k=1,2,\cdots,m)$$

在点 $(x_1,\cdots,x_n)$ 具有对于 $x_i\ (i=1,2,\cdots,n)$ 的偏导数,

则复合函数

$$f(g_1(x_1,\cdots,x_n),g_2(x_1,\cdots,x_n),\cdots,g_m(x_1,\cdots,x_n))$$

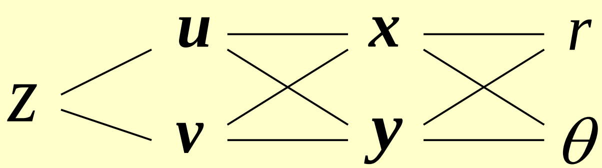
关于自变量 $x_i\ (i=1,2,\cdots,n)$ 的偏导数为

$$\frac{\partial f}{\partial x_i}=\sum_{k=1}^{m}\frac{\partial f}{\partial u_k}\cdot\frac{\partial u_k}{\partial x_i}\quad (i=1,2,\cdots,n).$$

3) 中间变量是多层函数的情形. 例如,

$$z = f(u,v), \quad u = u(x,y), \quad v = v(x,y),$$

$$x = x(r,\theta), \quad y = y(r,\theta)$$

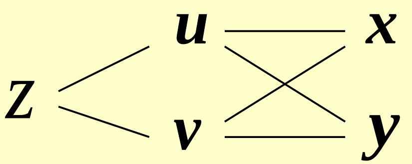


$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

例1. 设 $z = e^u \sin v$，$u = xy$，$v = x + y$，求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解: $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$



$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$

$$= e^{xy}[y \cdot \sin(x+y) + \cos(x+y)]$$

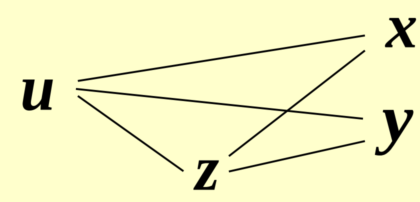
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^{xy}[x \cdot \sin(x+y) + \cos(x+y)]$$

例2. $u = f(x, y, z) = e^{x^2+y^2+z^2},\ z = x^2 \sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解: $$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$



$$= 2xe^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2} \cdot 2x \sin y$$

$$= 2x(1 + 2x^2 \sin^2 y)e^{x^2+y^2+x^4 \sin^2 y}$$

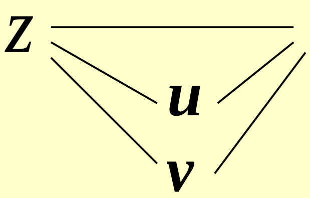
$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= 2ye^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2} \cdot x^2 \cos y$$

$$= 2(y + x^4 \sin y \cos y)e^{x^2+y^2+x^4 \sin^2 y}$$

例3. 设 $z = uv + \sin t$, $u = e^t$, $v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解:
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

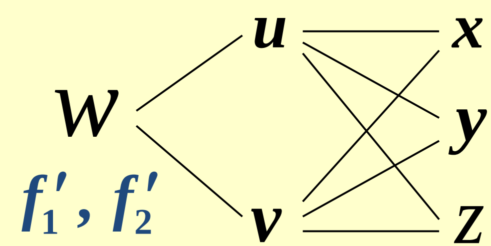


$$= v e^t - u \sin t + \cos t$$

$$= e^t(\cos t - \sin t) + \cos t$$

注意：多元抽象复合函数求导在偏微分方程变形与验证解的问题中经常遇到, 下列两个例题有助于掌握这方面问题的求导技巧与常用导数符号.

例4. 设 $w = f(x+y+z, xyz)$, f 具有二阶连续偏导数, 求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.



解: 令 $u = x+y+z$, $v = xyz$, 则

$$w = f(u, v)$$

$$\frac{\partial w}{\partial x} = f_1' \cdot 1 + f_2' \cdot yz$$

$$= f_1'(x+y+z, xyz) + yz\, f_2'(x+y+z, xyz)$$

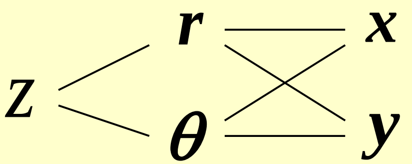
$$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' \cdot 1 + f_{12}'' \cdot xy + y f_2' + yz[\, f_{21}'' \cdot 1 + f_{22}'' \cdot xy]$$

$$= f_{11}'' + y(x+z) f_{12}'' + xy^2 z\, f_{22}'' + y\, f_2'$$

例5. 设 $u=f(x,y)$ 二阶偏导数连续, 求下列表达式在极坐标系下的形式 (1) $(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2$, (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

解: 已知 $x=r\cos\theta$, $y=r\sin\theta$, 则

$$r=\sqrt{x^2+y^2},\ \theta=\arctan\frac{y}{x}$$

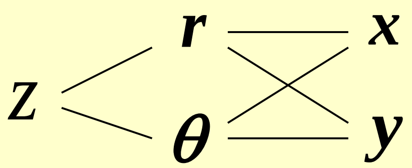


(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{x^2+y^2}$$

$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$



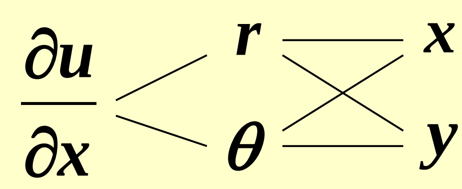
$$\boxed{\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}}$$

$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

$$\therefore\quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

已知 $\frac{\partial u}{\partial x}=\boxed{\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}}$



$$(2)\quad \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$$

注意利用已有公式

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta \cancel{- 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$\cancel{+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta \cancel{+ 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$\cancel{-\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$= \frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right]$$

3.2.5 一阶全微分形式的不变性

设函数 $z=f(u,v),u=\varphi(x,y),v=\psi(x,y)$ 都可微,

则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x})\mathrm{d}x+(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y})\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y)+\frac{\partial z}{\partial v}(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y)$$

$$=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

可见无论 $\boldsymbol{u},\boldsymbol{v}$ 是自变量还是中间变量, 其全微分表达形式都一样, 这个性质叫做一阶全微分形式的不变性.

由一阶全微分形式的不变性可得：

微分法则： 设u、v是可微函数，f 有连续偏导数，则

$$1.\ d(u \pm v) = du \pm dv$$

$$2.\ d(ku) = k\,du$$

$$3.\ d(uv) = v\,du + u\,dv$$

$$4.\ d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

$$5.\ df(u,v) = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv$$

一阶全微分形式不变性

利用一阶全微分形式不变性解题

例5. 设 $f(u,v)$ 可微，求 $z=f(\frac{x}{y},\frac{y}{x})$ 的偏导数.

解：由一阶全微分形式不变性，有

$$dz=f_1 d(\frac{x}{y})+f_2 d(\frac{y}{x})$$

$$=f_1\frac{ydx-xdy}{y^2}+f_2\frac{xdy-ydx}{x^2}$$

$$=(\frac{1}{y}f_1-\frac{y}{x^2}f_2)dx+(-\frac{x}{y^2}f_1+\frac{1}{x}f_2)dy$$

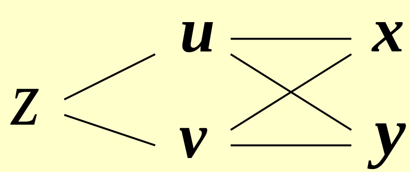
$$\therefore\quad \frac{\partial z}{\partial x}=\frac{1}{y}f_1-\frac{y}{x^2}f_2,\quad \frac{\partial z}{\partial y}=-\frac{x}{y^2}f_1+\frac{1}{x}f_2.$$

内容小结

复合函数求导的链式法则　**“连线相乘，分线相加”**

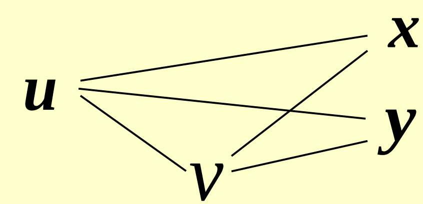
$$z = f(u,v)\,,\quad u = \varphi(x,y),\quad v = \psi(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1'\varphi_1' + f_2'\psi_1'$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = f_1'\varphi_2' + f_2'\psi_2'$$

例:其它变形　$u = f(x,y,v),\, v = \varphi(x,y),$



$$\frac{\partial u}{\partial x} = f_1' + f_3'\cdot\varphi_1'\,;\qquad \frac{\partial u}{\partial y} = f_2' + f_3'\cdot\varphi_2'$$

思考与练习

例1. $z=\arctan\frac{x}{y}, x=u+v, y=u-v$

验证：$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{u-v}{u^2+v^2}$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}\cdot 1+\frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2})\cdot(-1)$$

$$=\frac{y+x}{x^2+y^2}=\frac{u}{u^2+v^2} \qquad \cdots\cdots$$

例 2. $u = f(\frac{x}{y}, \frac{y}{z})$

$$\frac{\partial u}{\partial x} = f_1' \cdot \frac{1}{y} = \frac{1}{y} f_1'$$

$$\frac{\partial u}{\partial y} = f_1' \cdot (-\frac{x}{y^2}) + f_2' \cdot \frac{1}{z} = -\frac{x}{y^2} f_1' + \frac{1}{z} f_2'$$

$$\frac{\partial u}{\partial z} = f_2' \cdot (-\frac{y}{z^2}) = -\frac{y}{z^2} f_2'$$

例3. $z = f(u, x, y),\ u = xe^y$

$$\frac{\partial z}{\partial x} = f_1' \cdot e^y + f_2'$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^y \cdot f_1' + x e^{2y} \cdot f_{11}'' + e^y \cdot f_{13}'' + x e^y \cdot f_{21}'' + f_{23}''$$

例题

1. 已知 $f(x,y)\big|_{y=x^2}=1$, $f_1'(x,y)\big|_{y=x^2}=2x$, 求 $f_2'(x,y)\big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

2. 设函数 $z=f(x,y)$ 在点$(1,1)$处可微，且

$$f(1,1)=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$\varphi(x)=f(x,f(x,x))$，求 $\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. **(2001考研)**

解: 由题设 $\varphi(1)=f(1,f(1,1))=f(1,1)=1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}=3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$=3\Big[\,f_1'(x,f(x,x))$$

$$\left.+f_2'(x,f(x,x))\big(f_1'(x,x)+f_2'(x,x)\big)\Big]\right|_{x=1}$$

$$=3\cdot\big[2+3\cdot(2+3)\big]=51$$

思考题

设$z = f(u, v, x)$，而$u = \phi(x)$，$v = \psi(x)$，

则$\dfrac{dz}{dx} = \dfrac{\partial f}{\partial u}\dfrac{du}{dx} + \dfrac{\partial f}{\partial v}\dfrac{dv}{dx} + \dfrac{\partial f}{\partial x}$，

试问$\dfrac{dz}{dx}$与$\dfrac{\partial f}{\partial x}$是否相同？为什么？

思考题解答

不相同.

等式左端的z是作为一个自变量x 的函数，

而等式右端最后一项f 是作为u, v, x的三元函数，

写出来为

$$\left.\frac{dz}{dx}\right|_{x} = \left.\frac{\partial f}{\partial u}\right|_{(u,v,x)} \cdot \left.\frac{du}{dx}\right|_{x} + \left.\frac{\partial f}{\partial v}\right|_{(u,v,x)} \cdot \left.\frac{dv}{dx}\right|_{x} + \left.\frac{\partial f}{\partial x}\right|_{(u,v,x)}.$$

练 习 题

一、填空题:

1、设$z=\dfrac{x\cos y}{y\cos x}$, 则$\dfrac{\partial z}{\partial x}=$____________________;

$\dfrac{\partial z}{\partial y}=$____________________.

2、设$z=\dfrac{x^2\ln(3x-2y)}{y^2}$, 则$\dfrac{\partial z}{\partial x}=$____________________;

$\dfrac{\partial z}{\partial y}=$____________________.

3、设$z=e^{\sin t-2t^3}$, 则$\dfrac{dz}{dt}=$____________________.

二、设$z=ue^{\frac{v}{u}}$, 而$u=x^2+y^2, v=xy$, 求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

三、设$z=\arctan(xy)$, 而$y=e^{x}$, 求$\dfrac{dz}{dx}$.

四、设 $z=f(x^{2}-y^{2},e^{xy})$, (其中f具 有一阶连续偏导数), 求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

五、设$u=f(x+xy+xyz)$, (其中f具 有一阶连续偏导数), 求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}$.

六、设$z=f(x,\dfrac{x}{y})$, (其中f具 有二阶连续偏导数), 求$\dfrac{\partial^{2}z}{\partial x^{2}},\dfrac{\partial^{2}z}{\partial x\partial y},\dfrac{\partial^{2}z}{\partial y^{2}}$.

七、设$z = \dfrac{y}{f(x^2 - y^2)}$,其中为可导函数,

验证:$\dfrac{1}{x}\dfrac{\partial z}{\partial x} + \dfrac{1}{y}\dfrac{\partial z}{\partial y} = \dfrac{z}{y^2}$.

八、设$z = \phi[x + \varphi(x - y), y]$,其中$\phi, \varphi$具有二阶导数, 求

$\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}$.

练习题答案

一、1、$\dfrac{\cos y(\cos x+x\sin x)}{y\cos^2 x},-\dfrac{x\cos x(y\sin y+\cos y)}{y^2\cos^2 x}$；

2、$\dfrac{2x}{y^2}\ln(3x-2y)+\dfrac{3x^2}{(3x-2y)y^2}$，

$$-\frac{2x^2}{y^3}\ln(3x-2y)-\frac{2x^2}{(3x-2y)y^2};$$

3、$\dfrac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$.

二、$\dfrac{\partial z}{\partial x}=[2x+y-\dfrac{2x^2y}{(x^2+y^2)y^2}]e^{\frac{xy}{x^2+y^2}}$，

$$\frac{\partial z}{\partial y}=[2y+x-\frac{2y^2x}{(x^2+y^2)}]e^{\frac{xy}{(x^2+y^2)}}.$$

三、$\dfrac{dz}{dx}=\dfrac{e^x(1+x)}{1+x^2e^{2x}}.$

四、$\dfrac{\partial z}{\partial x}=2xf_1'+ye^{xy}f_2',\dfrac{\partial z}{\partial y}=-2yf_1'+xe^{xy}f_2'.$

五、$\dfrac{\partial u}{\partial x}=f'(1+y+yz),\dfrac{\partial u}{\partial y}=f'(x+xz),\dfrac{\partial u}{\partial z}=xyf'.$

六、$\dfrac{\partial^2 z}{\partial x^2}=f_{11}''+\dfrac{2}{y}f_{12}''+\dfrac{1}{y^2}f_{22}'',$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{x}{y^2}(f_{12}''+\frac{1}{y}f_{22}'')-\frac{1}{y^2}f_2',$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{2x}{y^3}f_2'+\frac{x^2}{y^4}f_{22}''.$$

八、$\dfrac{\partial^2 z}{\partial x^2} = \phi_{11}(1+\varphi')^2 + \phi_1\varphi'',$

$$\frac{\partial^2 z}{\partial y^2} = \phi_{11}(\varphi')^2 - \phi_{12}\varphi' + \phi_1\varphi'' - \phi_{21}\varphi' + \phi_{22}.$$

练习题

1. 选择题

函数 $z = f(x, y)$ 在 (x_0, y_0) 可微的充分条件是(D)

(A) $f(x, y)$ 在 (x_0, y_0) 连续;

(B) $f'_x(x, y)$, $f'_y(x, y)$ 在 (x_0, y_0) 的某邻域内存在;

(C) $\Delta z - f'_x(x, y)\Delta x - f'_y(x, y)\Delta y$

当 $\sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$ 时是无穷小量;

(D) $\dfrac{\Delta z - f'_x(x, y)\Delta x - f'_y(x, y)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$

当 $\sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$ 时是无穷小量.

2. 证明函数

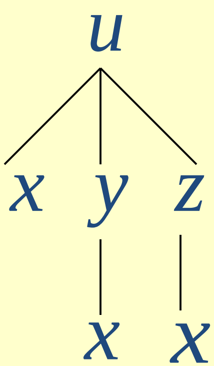
$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

在点 (0,0) 连续且偏导数存在，但偏导数在点 (0,0) 不连续，而 $f(x,y)$ 在点 (0,0) 可微 .

3. 设 $u=f(x,y,z)$ 有连续的一阶偏导数，又函数 $y=y(x)$ 及 $z=z(x)$ 分别由下列两式确定：

$$e^{xy}-xy=2,\; e^{x}=\int_0^{x-z}\frac{\sin t}{t}\,\mathrm{d}t,\text{ 求 }\frac{\mathrm{d}u}{\mathrm{d}x}.$$

解: 两个隐函数方程两边对 x 求导, 得



$$\begin{cases} e^{xy}(y+xy')-(y+xy')=0 \\ e^{x}=\dfrac{\sin(x-z)}{x-z}(1-z') \end{cases}$$

解得 $$y'=-\frac{y}{x},\quad z'=1-\frac{e^{x}(x-z)}{\sin(x-z)}$$

因此 $$\frac{\mathrm{d}u}{\mathrm{d}x}=f_1'-\frac{y}{x}f_2'+\left[1-\frac{e^{x}(x-z)}{\sin(x-z)}\right]f_3'$$

4. 设 $y = y(x)$, $z = z(x)$ 是由方程 $z = x f(x+y)$ 和 $F(x,y,z) = 0$ 所确定的函数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解法1 分别在各方程两端对 x 求导，得

$$\begin{cases} z' = f + x \cdot f' \cdot (1+y') \\ F_x + F_y \cdot y' + F_z \cdot z' = 0 \end{cases} \Longrightarrow \begin{cases} -x f' \cdot y' + z' = f + x f' \\ F_y \cdot y' + F_z \cdot z' = -F_x \end{cases}$$

$$\therefore \quad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\begin{vmatrix} -x f' & f + x f' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -x f' & 1 \\ F_y & F_z \end{vmatrix}} = \frac{(f + x f')F_y - x f' \cdot F_x}{F_y + x f' \cdot F_z}$$

$$(F_y + x f' \cdot F_z \neq 0)$$

解法2 微分法.

$$z = xf(x+y),\ \ F(x,y,z)=0$$

对各方程两边分别求微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + xf'\cdot(\mathrm{d}x+\mathrm{d}y) \\ F_1'\mathrm{d}x + F_2'\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

化简得

$$\begin{cases} (f+xf')\,\mathrm{d}x + x\,f'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1'\,\mathrm{d}x + F_2'\,\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

消去$\mathrm{d}\,y$可得 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

例 4 解偏微分方程

$$\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\frac{\partial z}{\partial y}, \quad y > 0. \tag{1}$$

解 所谓解偏微分方程(1)，就是要求函数 $z = z(x, y)$，使之满足(1)，作变量代换

$$\xi = x - 2\sqrt{y}, \quad \eta = x + 2\sqrt{y}. \tag{2}$$

在这个代换之下，x，y 的函数变成新变量 ξ，η 的函数，记为

$$z = z(x, y) = f(\xi, \eta).$$

现在来计算，在变量代换(2)之下(1)变成什么样子.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta},$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{\partial f}{\partial \xi}y^{-\frac{1}{2}} + \frac{\partial f}{\partial \eta}y^{-\frac{1}{2}},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial \xi^2} + 2\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2},$$

$$\frac{\partial^2 z}{\partial y^2} = -\left[\left(\frac{\partial^2 f}{\partial \xi^2}(-y^{-\frac{1}{2}}) + \frac{\partial^2 f}{\partial \xi \partial \eta} y^{-\frac{1}{2}}\right) y^{-\frac{1}{2}} - \frac{1}{2} y^{-\frac{3}{2}} \frac{\partial f}{\partial \xi}\right]$$

$$+\left(-\frac{\partial^2 f}{\partial \xi \partial \eta} y^{-\frac{1}{2}} + \frac{\partial^2 f}{\partial \eta^2} y^{-\frac{1}{2}}\right) y^{-\frac{1}{2}} - \frac{1}{2} y^{-\frac{3}{2}} \frac{\partial f}{\partial \eta}$$

$$=\left(\frac{\partial^2 f}{\partial \xi^2} - 2\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2}\right) y^{-1} + \frac{1}{2} y^{-\frac{3}{2}}\left(\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta}\right).$$

把这些表达式代入(1)即得

$$\varphi \frac{\partial^2 f}{\partial \xi \partial \eta} = 0. \tag{3}$$

这就是说，引入新变量 ξ，η 后，(1)变成了(3)，而(3)是很容易求解的．把(3)写成

$$\frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \xi}\right) = 0,$$

即$\frac{\partial f}{\partial \xi}$中不含有 η，可写$\frac{\partial f}{\partial \xi} = g(\xi)$，$g$ 是任一可微分的函数．由此得

$$f(\xi, \eta) = \int g(\xi)\, d\xi + \psi(\eta) = \varphi(\xi) + \psi(\eta).$$

还回到原来的变量，即得(1)的解为

$$z = \varphi(x - 2\sqrt{y}) + \psi(x + 2\sqrt{y})$$

这里 φ，ψ 是任意两个具有二阶连续导数的单变量函数． □

1. 求解偏微分方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z.$$

2. 设 a，b，c 满足 $b^2-ac>0$，λ_1，λ_2 是二次方程 $cx^2+2bx+a=0$ 的两个根.

试引进新变量

$$\xi=x+\lambda_1 y,\quad \eta=x+\lambda_2 y$$

解二阶偏微分方程

$$a\frac{\partial^2 u}{\partial x^2}+2b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=0.$$

3. 求解二阶偏微分方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$