5.1.2 换元积分法与分部积分法

一、第一换元积分法(凑微分法)

二、第二换元积分法 (变量代换法)

三、分部积分法

换元积分法基本思路：

设 $\underline{F'(u)=f(u)}$，$\underline{u=\varphi(x)}$ 可导，则有

$$\mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

$$\therefore\quad \int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C=F(u)+C\Big|_{u=\varphi(x)}$$

$$=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

$$\int f[\varphi(x)]\varphi'(x)\,\mathrm{d}x \underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}} \int f(u)\,\mathrm{d}u$$

一、第一类换元法

定理1 设 $f(u)$ 有原函数, $u=\varphi(x)$ 可导, **则有换元公式**

$$\int f[\varphi(x)]\underline{\varphi'(x)\mathrm{d}x}=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

即 $$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f(\varphi(x))\mathrm{d}\,\varphi(x)$$

(也称配元法, 凑微分法)

例1. 求 $\int (ax+b)^m \mathrm{d}x \quad (m \neq -1)$.

解: 令 $u = ax + b$,则 $\mathrm{d}u = a\mathrm{d}x$, 故

$$\text{原式} = \int u^m \frac{1}{a}\mathrm{d}u = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C$$

$$= \frac{1}{a(m+1)}(ax+b)^{m+1} + C$$

注: 当 $m = -1$ 时

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

例2. 求 $\int \frac{\mathrm{d}x}{a^2+x^2}$.

利用公式

$$\int \frac{\mathrm{d}u}{1+u^2} = \arctan u + C$$

解: $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u=\frac{1}{a}\mathrm{d}x$

$$= \frac{1}{a}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{a}\arctan u + C$$

$$= \frac{1}{a}\arctan(\frac{x}{a}) + C$$

例3. 求 $\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} \ (a > 0).$

解: $\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{\mathrm{d}x}{a\sqrt{1 - (\frac{x}{a})^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1 - (\frac{x}{a})^2}}$

$$= \arcsin \frac{x}{a} + C$$

利用公式 $\int \frac{\mathrm{d}u}{\sqrt{1 - u^2}} = \arcsin u + C$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$$ **(直接配元)**

例4. 求 $\int \tan x \mathrm{d}x$.

解: $\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$

$= -\ln\left|\cos x\right| + C$

类似地

$\int \cot x \mathrm{d}x = \int \frac{\cos x \,\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$

$= \ln\left|\sin x\right| + C$

例5. 求 $\int \frac{\mathrm{d}x}{x^2-a^2}$.

解:

$$\because \quad \frac{1}{x^2-a^2} = \frac{1}{2a}\,\frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

$$\therefore \text{ 原式} = \frac{1}{2a}\left[\int \frac{\mathrm{d}x}{x-a} - \int \frac{\mathrm{d}x}{x+a}\right]$$

$$= \frac{1}{2a}\left[\int \frac{\mathrm{d}(x-a)}{x-a} - \int \frac{\mathrm{d}(x+a)}{x+a}\right]$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

常用的几种配元形式:

$$(1)\ \int f(ax+b)\mathrm{d}x=\frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$$

$$\left.\begin{aligned}&(2)\ \int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\ \mathrm{d}x^n\\&(3)\ \int f(x^n)\frac{1}{x}\mathrm{d}x=\frac{1}{n}\int f(x^n)\ \frac{1}{x^n}\ \mathrm{d}x^n\end{aligned}\right\}\text{凑幂法}$$

$$(4)\ \int f(\sin x)\cos x\mathrm{d}x=\int f(\sin x)\mathrm{d}\sin x$$

$$(5)\ \int f(\cos x)\sin x\mathrm{d}x=-\int f(\cos x)\ \mathrm{d}\cos x$$

$$(6)\ \int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\ \mathrm{d}\tan x$$

$$(7)\ \int f(e^x)e^x\mathrm{d}x = \int f(e^x)\ \mathrm{d}e^x$$

$$(8)\ \int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\ \mathrm{d}\ln x$$

例6. 求 $\displaystyle\int\frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $\displaystyle = \int\frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int\frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C$$

例7. 求 $\int \sec^6 x\mathrm{d}x$.

解: 原式 $= \int (\tan^2 x+1)^2 \cdot \sec^2 x dx$

$$= \int (\tan^2 x+1)^2 d\tan x$$

$$= \int (\tan^4 x+2\tan^2 x+1)\mathrm{d}\tan x$$

$$= \frac{1}{5}\tan^5 x+\frac{2}{3}\tan^3 x+\tan x+C$$

例8. 求 $\int \frac{\mathrm{d}x}{1+e^x}$.

解法1

$$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x}\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+e^x)}{1+e^x}$$

$$= x - \ln(1+e^x) + C$$

解法2

$$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+e^{-x})}{1+e^{-x}}$$

$$= -\ln(1+e^{-x}) + C$$

$-\ln(1+e^{-x}) = -\ln[e^{-x}(e^x+1)]$ 两法结果一样

例9. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2} \int \left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x} \right] \mathrm{d}\sin x$$

$$= \frac{1}{2} \left[\ln\left|1+\sin x\right| - \ln\left|1-\sin x\right| \right] + C$$

$$= \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

解法 2 $$\int \sec x \mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \mathrm{d}x$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln\left| \sec x + \tan x \right| + C$$

同样可证

$$\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$$

$$\int \sec x \mathrm{d}x = \ln\left| \sec x + \tan x \right| + C$$

例10. 求 $\int \cos^4 x \, dx$.

解: $\because \cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}(1 + 2\cos 2x + \frac{1+\cos 4x}{2})$$

$$= \frac{1}{4}(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)$$

$$\therefore \quad \int \cos^4 x \, dx = \frac{1}{4}\int (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x) \, dx$$

$$= \frac{1}{4}\left[\frac{3}{2}\int dx + \int \cos 2x \, d(2x) + \frac{1}{8}\int \cos 4x \, d(4x)\right]$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

例11. 求 $\int \sin^2 x \cos^2 3x \, dx$.

解: $\because \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x \sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1-\cos 4x)$$

$\therefore$ 原式 $= \frac{1}{4}\int dx - \frac{1}{64}\int \cos 8x \, d(8x)$

$$- \frac{1}{2}\int \sin^2 2x \, d(\sin 2x) - \frac{1}{32}\int \cos 4x \, d(4x)$$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$$

例12. 求 $\int \frac{x+1}{x(1+xe^x)}\mathrm{d}x$.

解: 原式= $\int \frac{(x+1)e^x}{xe^x(1+xe^x)}\mathrm{d}x = \int\left(\frac{1}{xe^x}-\frac{1}{1+xe^x}\right)\mathrm{d}(xe^x)$

$$= \ln\left|xe^x\right| - \ln\left|1+xe^x\right| + C$$

$$= x + \ln|x| - \ln\left|1+xe^x\right| + C$$

分析: $\frac{1}{xe^x(1+xe^x)} = \frac{1+xe^x-xe^x}{xe^x(1+xe^x)} = \frac{1}{xe^x} - \frac{1}{1+xe^x}$

$$(x+1)e^x\,\mathrm{d}x = xe^x\,\mathrm{d}x + e^x\,\mathrm{d}x = \mathrm{d}(xe^x)$$

例13. 求 $\displaystyle\int\left[\frac{f(x)}{f'(x)}-\frac{f''(x)f^2(x)}{f'^3(x)}\right]\mathrm{d}\,x.$

解：原式 $\displaystyle=\int\frac{f(x)}{f'(x)}\left[1-\frac{f''(x)f(x)}{f'^2(x)}\right]\mathrm{d}x$

$$=\int\frac{f(x)}{f'(x)}\cdot\underline{\frac{f'^2(x)-f''(x)f(x)}{f'^2(x)}}\,\mathrm{d}x$$

$$=\int\frac{f(x)}{f'(x)}\,\mathrm{d}\left(\frac{f(x)}{f'(x)}\right)$$

$$=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$$

小结 常用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1=\sin^2 x+\cos^2 x \text{等}$$

(2) 降低幂次: 利用倍角公式, 如

$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x); \qquad \sin^2 x=\tfrac{1}{2}(1-\cos 2x);$$

万能凑幂法
$$\begin{cases} \int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n \\ \int f(x^n)\frac{1}{x}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}\,x^n \end{cases}$$

(3) 统一函数: 利用三角公式; 配元方法

(4) 巧妙换元或配元

思考与练习　1. 下列各题求积方法有何不同?

(1) $\int \frac{\mathrm{d}x}{4+x} = \int \frac{\mathrm{d}(4+x)}{4+x}$

(2) $\int \frac{\mathrm{d}x}{4+x^2} = \frac{1}{2}\int \frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2}$

(3) $\int \frac{x}{4+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(4+x^2)}{4+x^2}$

(4) $\int \frac{x^2}{4+x^2}\mathrm{d}x = \int [1-\frac{4}{4+x^2}]\mathrm{d}\,x$

(5) $\int \frac{\mathrm{d}x}{4-x^2} = \frac{1}{4}\int [\frac{1}{2-x}+\frac{1}{2+x}]\mathrm{d}x$

(6) $\int \frac{\mathrm{d}x}{\sqrt{4x-x^2}} = \int \frac{\mathrm{d}(x-2)}{\sqrt{4-(x-2)^2}}$

2. 求 $\int \frac{\mathrm{d}x}{x(x^{10}+1)}$.

提示:

法1 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)}\mathrm{d}x$

法2 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \frac{1}{10}\int \frac{\mathrm{d}\ x^{10}}{x^{10}(x^{10}+1)}$

法3 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{\mathrm{d}\ x}{x^{11}(1+x^{-10})} = \frac{-1}{10}\int \frac{\mathrm{d}\ x^{-10}}{1+x^{-10}}$

二、第二类换元法(变量代换法)

第一类换元法解决的问题

$$\underset{\text{难求}}{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x} = \underset{\text{易求}}{\int f(u)\mathrm{d}u}\Big|_{u=\varphi(x)}$$

若所求积分 $\int f(u)\mathrm{d}u$ 难求，

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求，

则得第二类换元积分法 .

定理2.设 $x=\psi(t)$ **是单调可导函数,且** $\psi'(t)\neq 0$,

$f[\psi(t)]\psi'(t)$**具有原函数,则有换元公式**

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $\underline{x=\psi(t)}$ 的反函数.

$$\boxed{\Phi'(t)=f[\psi(t)]\psi'(t)}$$

证: 设 $f[\psi(t)]\psi'(t)$ 的原函数为 $\Phi(t)$, 令

$$F(x)=\Phi[\psi^{-1}(x)]$$

则

$$F'(x)=\frac{\mathrm{d}\Phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=f[\psi(t)]\cancel{\psi'(t)}\cdot\frac{1}{\cancel{\psi'(t)}}=f(x)$$

$$\therefore\quad\int f(x)\,\mathrm{d}x=F(x)+C=\Phi[\psi^{-1}(x)]+C$$

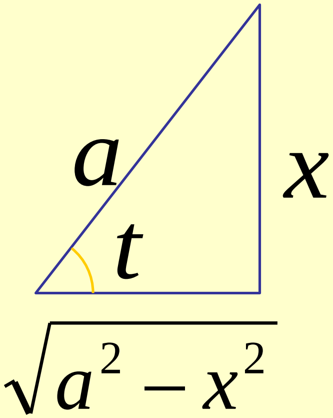
$$=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

例1. 求 $\int \sqrt{a^2 - x^2}\,\mathrm{d}x \ \ (a > 0)$.

解: 令 $x = a\sin t,\ \ t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, 则

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 t} = a\cos t$$

$$\mathrm{d}x = a\cos t\,\mathrm{d}t$$



$\therefore$ 原式 $= \int a\cos t \cdot a\cos t\,\mathrm{d}t = a^2 \int \cos^2 t\,\mathrm{d}t$

$$= a^2\left(\frac{t}{2} + \frac{\sin 2t}{4}\right) + C$$

$$\sin 2t = 2\sin t\cos t = 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C$$

例2. 求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0).$

解: 令 $x = a\tan t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$，则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

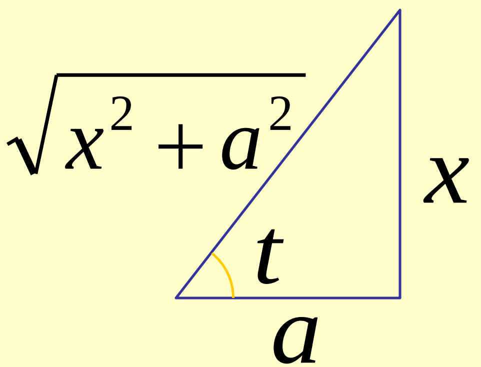
$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore \text{原式} = \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$

$$= \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right] + C_1$$

$$= \ln\left[x + \sqrt{x^2+a^2}\right] + C \qquad (C = C_1 - \ln a)$$



例3. 求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0).$

解: 当$x>a$时, 令 $x=a\sec t,\ t\in(0,\frac{\pi}{2})$, 则

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2 t-a^2}=a\tan t$$

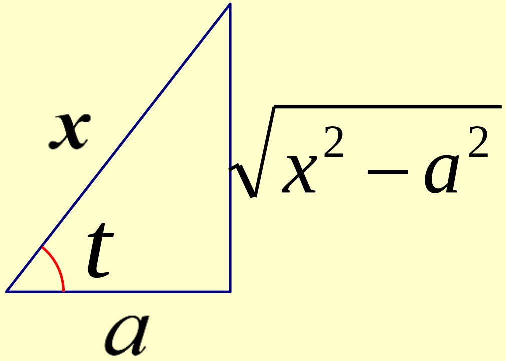
$$\mathrm{d}x=a\sec t\tan t\,\mathrm{d}t$$

$$\therefore\ \text{原式}=\int\frac{a\sec t\tan t}{a\tan t}\mathrm{d}t=\int\sec t\,\mathrm{d}t$$

$$=\ln\left|\sec t+\tan t\right|+C_1$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\qquad(C=C_1-\ln a)$$



当$x<-a$时, 令 $x=-u$, 则$u>a$, 于是

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=-\int\frac{\mathrm{d}u}{\sqrt{u^2-a^2}}=-\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$=-\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$=-\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\quad(C=C_1-2\ln a)$$

$x>a$时, $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

说明:

被积函数含有 $\sqrt{x^2+a^2}$ 或 $\sqrt{x^2-a^2}$ 时, 除采用三角代换外, 还可利用公式

$$\operatorname{ch}^2 t-\operatorname{sh}^2 t=1$$

采用双曲代换

$$x=a\operatorname{sh} t \quad \text{或} \quad x=a\operatorname{ch} t$$

消去根式, 所得结果一致.

例4. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\,\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$,则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}\,t$

$$\text{原式}=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}\,t=-\int (a^2t^2-1)^{\frac{1}{2}}\,|\,t\,|\,\mathrm{d}\,t$$

当 $x>0$ 时,

$$\text{原式}=-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\,\mathrm{d}(a^2t^2-1)$$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时, 类似可得同样结果 .

小结

1. 第二类换元法常见类型:

$$\left.\begin{array}{ll}(1)\ \int f(x,\sqrt[n]{ax+b})\,\mathrm{d}x, & 令\ t=\sqrt[n]{ax+b}\\ (2)\ \int f\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x, & 令\ t=\sqrt[n]{\frac{ax+b}{cx+d}}\end{array}\right\}$$

$$(3)\ \int f(x,\sqrt{a^2-x^2})\,\mathrm{d}x, 令\ x=a\sin t\ 或\ x=a\cos t$$

$$(4)\ \int f(x,\sqrt{a^2+x^2})\,\mathrm{d}x, 令\ x=a\tan t\ 或\ x=a\,\mathrm{sh}\,t$$

$$(5)\ \int f(x,\sqrt{x^2-a^2})\,\mathrm{d}x, 令\ x=a\sec t\ 或\ x=a\,\mathrm{ch}\,t$$

(6) $\int f(a^x)\,\mathrm{d}x$，令 $t = a^x$

(7) 分母中因子次数较高时, 可试用倒代换

2. 常用基本积分公式的补充

(16) $\int \tan x \,\mathrm{d}x = -\ln\left|\cos x\right| + C$

(17) $\int \cot x \mathrm{d}x = \ln\left|\sin x\right| + C$

(18) $\int \sec x \mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$

(19) $\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$

$$(20)\quad \int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a}+C$$

$$(21)\quad \int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

$$(22)\quad \int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a}+C$$

$$(23)\quad \int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln(x+\sqrt{x^2+a^2})+C$$

$$(24)\quad \int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

例5. 求 $\int \frac{\mathrm{d}x}{x^2+2x+3}$.

解: 原式 $= \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\mathrm{d}(x+1)$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$

(公式 (20))

$$\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a}+C$$

例21. 求 $I = \int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解: $I = \frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$

(公式 (23)) $\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \ln(x+\sqrt{x^2+a^2})+C$

例6. 求 $\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解：原式 $= \int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}} = \arcsin\frac{2x-1}{\sqrt{5}} + C$

(公式 (22))

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$$

例23. 求 $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

解：原式 $= -\int \frac{\mathrm{d}\,e^{-x}}{\sqrt{1-e^{-2x}}} = -\arcsin e^{-x} + C$

(公式 (22))

例7. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$, 得

$$\text{原式}=-\int \frac{t}{\sqrt{a^2t^2+1}}\,\mathrm{d}t$$

$$=-\frac{1}{2a^2}\int \frac{\mathrm{d}\,(a^2t^2+1)}{\sqrt{a^2t^2+1}}=-\frac{1}{a^2}\sqrt{a^2t^2+1}+C$$

$$=-\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

例8. 求 $\displaystyle\int\frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解: 原式 $\displaystyle=\int\frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$

$\boxed{\text{令 } x+1=\frac{1}{t}}$

$$=\int\frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t=-\int\frac{t^2}{\sqrt{1-t^2}}\mathrm{d}t$$

$$=\int\frac{(1-t^2)-1}{\sqrt{1-t^2}}\mathrm{d}t=\boxed{\int\sqrt{1-t^2}\,\mathrm{d}t}-\int\frac{1}{\sqrt{1-t^2}}\mathrm{d}t$$

$$=\frac{1}{2}t\sqrt{1-t^2}+\frac{1}{2}\arcsin t-\arcsin t+C$$

$\boxed{\text{例1}}$

$$=\frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2}-\frac{1}{2}\arcsin\frac{1}{x+1}+C$$

思考与练习

1. 下列积分应如何换元才使积分简便？

(1) $\int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$

令 $t=\sqrt{1+x^2}$

(2) $\int \frac{\mathrm{d}x}{\sqrt{1+e^x}}$

令 $t=\sqrt{1+e^x}$

(3) $\int \frac{\mathrm{d}x}{x(x^7+2)}$

令 $t=\frac{1}{x}$

2. 已知 $\int x^5 f(x)\,\mathrm{d}x = \sqrt{x^2-1}+C$, 求 $\int f(x)\,\mathrm{d}x$.

解: 两边求导, 得 $x^5 f(x) = \dfrac{x}{\sqrt{x^2-1}}$, 则

$$\int f(x)\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^4\sqrt{x^2-1}} \quad (令\ t=\frac{1}{x})$$

$$= \int \frac{-t^3\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{1}{2}\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\,\mathrm{d}t^2$$

$$= \frac{-1}{2}\int (1-t^2)^{\frac{1}{2}}\,\mathrm{d}(1-t^2) + \frac{1}{2}\int (1-t^2)^{-\frac{1}{2}}\,\mathrm{d}(1-t^2)$$

$$= \frac{-1}{3}(1-t^2)^{\frac{3}{2}} + (1-t^2)^{\frac{1}{2}} + C = \cdots$$

(代回原变量)

例题1. 求下列积分:

1) $$\int x^2 \frac{1}{\sqrt{x^3+1}}\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{x^3+1}}\mathrm{d}(x^3+1)$$

$$= \frac{2}{3}\sqrt{x^3+1}+C$$

2) $$\int \frac{2x+3}{\sqrt{1+2x-x^2}}\mathrm{d}x = \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\mathrm{d}x$$

$$= -\int \frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5\int \frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$= -2\sqrt{1+2x-x^2} + 5\arcsin\frac{x-1}{\sqrt{2}} + C$$

2. 求不定积分 $\int \frac{2\sin x\cos x\sqrt{1+\sin^2 x}}{2+\sin^2 x}\,\mathrm{d}x.$

解: 利用凑微分法，得

$$原式 = \int \frac{\sqrt{1+\sin^2 x}}{2+\sin^2 x}\,\mathrm{d}(1+\sin^2 x)$$

令 $t=\sqrt{1+\sin^2 x}$

$$= \int \frac{2t^2}{1+t^2}\,\mathrm{d}t = 2\int (1-\frac{1}{1+t^2})\,\mathrm{d}t$$

$$= 2t - 2\arctan t + C$$

$$= 2[\sqrt{1+\sin^2 x} - \arctan\sqrt{1+\sin^2 x}\,] + C$$

3. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\mathrm{d}x.$

解: 令 $x=\sin t$, $1+x^2=1+\sin^2 t$, $\mathrm{d}x=\cos t\,\mathrm{d}t$

$$\text{原式}=\int \frac{\cos t}{(1+\sin^2 t)\cos t}\mathrm{d}t=\int \frac{1}{1+\sin^2 t}\mathrm{d}t$$

$\downarrow$ 分子分母同除以 $\cos^2 t$

$$=\int \frac{\sec^2 t}{\sec^2 t+\tan^2 t}\mathrm{d}t=\int \frac{1}{1+2\tan^2 t}\mathrm{d}\tan t$$

$$=\frac{1}{\sqrt{2}}\int \frac{1}{1+(\sqrt{2}\tan t)^2}\mathrm{d}\sqrt{2}\tan t$$

$$=\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan t)+C=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C$$

三、分部积分法

由导数公式 $(uv)' = u'v + uv'$

积分得: $uv = \int u'v\,\mathrm{d}x + \int uv'\,\mathrm{d}x$

$$\Longrightarrow \left.\begin{aligned} &\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x \\ \text{或}\quad &\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \end{aligned}\right\} \text{分部积分公式}$$

选取 u 及 v' (或 $\mathrm{d}v$) 的原则:

1) v 容易求得 ;

2) $\int u'v\,\mathrm{d}x$ 比 $\int uv'\,\mathrm{d}x$ 容易计算 .

例1. 求 $\int x\cos x\,\mathrm{d}x$.

解: 令 $u = x,\ v' = \cos x,$

则 $u' = 1,\ v = \sin x$

$\therefore$ 原式 $= \int x\,\mathrm{d}(\sin x) = x\sin x - \int \sin x\,\mathrm{d}x$

$= x\sin x + \cos x + C$

思考: 如何求 $\int x^2 \sin x\,\mathrm{d}x$?

提示: 令 $u = x^2,\ v' = \sin x,$ 则

原式 $= -\int x^2\,\mathrm{d}(\cos x)$

$= -x^2\cos x + 2\int x\cos x\,\mathrm{d}x = \cdots$

例2. 求 $\int x \ln x \, \mathrm{d}x$.

解: 令 $u = \ln x,\ v' = x$

则 $u' = \frac{1}{x}, \quad v = \frac{1}{2}x^2$

$\therefore$ 原式 $= \int \ln x \ \mathrm{d}(\frac{1}{2}x^2)$.

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, \mathrm{d}x$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

例3. 求 $\int x\arctan x\,\mathrm{d}x$.

解: 令 $u=\arctan x,\ v'=x$ 则

$$\text{原式}=\int \arctan x\,\mathrm{d}(\frac{1}{2}x^2)$$

$$=\frac{1}{2}x^2\arctan x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,\mathrm{d}x$$

$$=\frac{1}{2}x^2\arctan x-\frac{1}{2}\int(1-\frac{1}{1+x^2})\,\mathrm{d}x$$

$$=\frac{1}{2}x^2\arctan x-\frac{1}{2}(x-\arctan x)+C$$

例4. 求 $\int e^x \sin x\,\mathrm{d}x$.

解: 令 $u=\sin x,\ v'=e^x$，则

$$\int e^x \sin x\,\mathrm{d}x = \int \sin x\,\mathrm{d}e^x$$

$$= e^x \sin x - \underline{\int e^x \cos x\,\mathrm{d}x}$$

$$= e^x \sin x - \int \cos x\,\mathrm{d}e^x$$

$$= e^x \sin x - e^x \cos x - \underline{\int e^x \sin x\,\mathrm{d}x}$$

故 $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$

说明: 也可设 $u=e^x, v'$ 为三角函数，但两次所设类型必须一致.

用分部积分法求不定积分的常见类型:

(1) 形如 $\int P(x)\sin axdx;\quad \int P(x)\cos axdx,$

(其中 $P(x)$为多项式函数) 的不定积分.

(2) 反三角函数的不定积分:

(1) $\int \arctan xdx;$ (2) $\int \arccos xdx;$

(3) $\int \arcsin xdx;$ (4) $\int \operatorname{arc}\cot xdx.$

(3) 形如 $\int P(x)\ln(ax)dx;\quad \int P(x)a^{bx}dx,$

(其中 $P(x)$为多项式函数) 的不定积分.

(4) 形如 $I_1 = \int e^{ax}\cos bxdx;\quad I_2 = \int e^{ax}\sin bxdx$

的不定积分.

技巧: 选取 u 及 v' 的一般方法：

把被积函数视为两个函数之积，按 " 反对幂指三" 的顺序，前者为 u 后者为 v'.

反：反三角函数
对：对数函数
幂：幂函数
指：指数函数
三：三角函数

例5. 求 $\int \arccos x \, dx$.

解：令 $u = \arccos x$, $v' = 1$, 则

$$\text{原式} = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \arccos x - \frac{1}{2} \int (1-x^2)^{-1/2} d(1-x^2)$$

$$= x \arccos x - \sqrt{1-x^2} + C$$

例6. 求 $\int \frac{\ln\cos x}{\cos^2 x}\,\mathrm{d}x$.

解: 令 $u = \ln\cos x,\ v' = \dfrac{1}{\cos^2 x}$ ，则

$$u' = -\tan x,\quad v = \tan x$$

$$\text{原式} = \tan x \cdot \ln\cos x + \int \tan^2 x\,\mathrm{d}x$$

$$= \tan x \cdot \ln\cos x + \int (\sec^2 x - 1)\,\mathrm{d}x$$

$$= \tan x \cdot \ln\cos x + \tan x - x + C$$

例7. 求 $\int e^{\sqrt{x}}\,\mathrm{d}x$.

解: 令 $\sqrt{x}=t$, 则 $x=t^2$, $\mathrm{d}x=2t\,\mathrm{d}t$

$$\text{原式}=2\int t\,e^{t}\,\mathrm{d}t=2\int t\,\mathrm{d}e^{t}$$

$$=2te^{t}-2\int e^{t}\,\mathrm{d}t$$

$$=2(t\,e^{t}-e^{t})+C$$

$$=2e^{\sqrt{x}}(\sqrt{x}-1)+C$$

例8. 求 $\int \sqrt{x^2+a^2}\,\mathrm{d}x \ (a>0)$.

解: 令 $u=\sqrt{x^2+a^2}$, $v'=1$, 则 $u'=\dfrac{x}{\sqrt{x^2+a^2}}$, $v=x$

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x$$

$$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,\mathrm{d}x + a^2\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}$$

$\therefore$ 原式 $= \dfrac{1}{2}x\sqrt{x^2+a^2} + \dfrac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + C$

例9. 求 $I_n = \int \frac{\mathrm{d}x}{(x^2+a^2)^n}$.

解: 令 $u = \frac{1}{(x^2+a^2)^n}$, $v' = 1$, 则 $u' = \frac{-2nx}{(x^2+a^2)^{n+1}}$, $v = x$

$$\therefore\ I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\mathrm{d}x$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}}\mathrm{d}x$$

$$= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1}$$

得递推公式 $I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n$

$$I_n = \int \frac{\mathrm{d}x}{(x^2+a^2)^n}$$

递推公式 $I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n$

说明: 已知 $I_1 = \frac{1}{a}\arctan\frac{x}{a} + C$ 利用递推公式可求得 I_n.

例如,

$$I_3 = \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2} I_2$$

$$= \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}\left(\frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^2} I_1\right)$$

$$= \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4}\frac{x}{x^2+a^2} + \frac{3}{8a^5}\arctan\frac{x}{a} + C$$

例10. 证明递推公式

$$I_n=\int\tan^n x\,\mathrm{d}x=\frac{\tan^{n-1}x}{n-1}-I_{n-2}\quad(n\geq 2)$$

证: $I_n=\int\tan^{n-2}x\,(\sec^2 x-1)\,\mathrm{d}x$

$$=\int\tan^{n-2}x\,\mathrm{d}(\tan x)-I_{n-2}$$

$$=\frac{\tan^{n-1}x}{n-1}-I_{n-2}$$

注: $I_n\to\cdots\to I_0$ 或 I_1

$$I_0=x+C,\qquad I_1=-\ln|\cos x|+C$$

说明:

分部积分题目的类型:

1) 直接分部化简积分 ;

2) 分部产生循环式 , 由此解出积分式 ;

(注意: 两次分部选择的 u , v 函数类型不变 ,
解出积分后加 C)

3) 对含自然数 n 的积分, 通过分部积分建立递推公式 .

例11. 已知 $f(x)$ 的一个原函数是 $\dfrac{\cos x}{x}$，求 $\int x f'(x)\,\mathrm{d}x$.

解:
$$\begin{aligned}\int x f'(x)\,\mathrm{d}x &= \int x\,\mathrm{d}f(x)\\ &= x f(x) - \int f(x)\,\mathrm{d}x\\ &= x\left(\frac{\cos x}{x}\right)' - \frac{\cos x}{x} + C\\ &= -\sin x - 2\frac{\cos x}{x} + C\end{aligned}$$

说明: 此题若先求出 $f'(x)$ 再求积分反而复杂.

$$\int x f'(x)\,\mathrm{d}x = \int\left(-\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2}\right)\mathrm{d}x$$

例12. 求 $I=\int \dfrac{e^{\arctan x}}{(1+x^2)^{3/2}}\,\mathrm{d}x.$

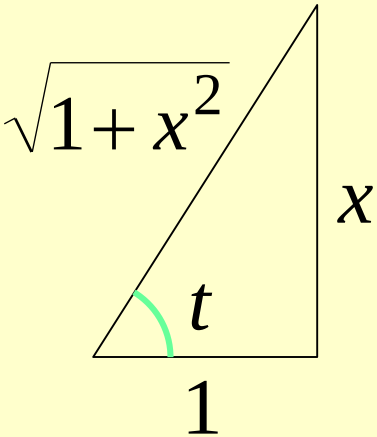
解法1　先换元后分部

令 $t=\arctan x$, 即 $x=\tan t$, 则

$$I=\int \frac{e^t}{\sec^3 t}\cdot \sec^2 t\,\mathrm{d}t=\int e^t\cos t\,\mathrm{d}t$$

$$=e^t\sin t-\int e^t\sin t\,\mathrm{d}t$$

$$=e^t\sin t+e^t\cos t-\int e^t\cos t\,\mathrm{d}t$$



故 $I=\dfrac{1}{2}(\sin t+\cos t)e^t+C$

$$=\frac{1}{2}\left[\frac{x}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right]e^{\arctan x}+C$$

解法2　用分部积分法

$$I = \int \frac{e^{\arctan x}}{(1+x^2)^{3/2}}\,\mathrm{d}x$$

$$I = \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}\,e^{\arctan x}$$

$$= \frac{1}{\sqrt{1+x^2}}e^{\arctan x} + \int \frac{x\,e^{\arctan x}}{(1+x^2)^{3/2}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{1+x^2}}e^{\arctan x} + \int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}e^{\arctan x}$$

$$= \frac{1}{\sqrt{1+x^2}}e^{\arctan x}(1+x) - I$$

$$\therefore \quad I = \frac{1+x}{2\sqrt{1+x^2}}e^{\arctan x} + C$$

内容小结

分部积分公式 $\int u v' \mathrm{d}x = u v - \int u' v \mathrm{d}x$

1. 使用原则： v易求出, $\int u' v \mathrm{d}x$易积分

2. 使用经验：“反对幂指三” , 前 $\boldsymbol{u}$ 后v'

3. 题目类型：

分部化简； 循环解出; 递推公式

例13. 求 $I=\int \sin(\ln x)\,\mathrm{d}x$

解: 令 $t=\ln x$, 则 $x=e^t$, $\mathrm{d}\,x=e^t\,\mathrm{d}\,t$

$$\therefore\quad I=\int e^t\sin t\,\mathrm{d}\,t=e^t\sin t-\int e^t\cos t\,\mathrm{d}\,t$$

$$=e^t(\sin t-\cos t)-\int e^t\sin t\,\mathrm{d}\,t$$

$$=e^t(\sin t-\cos t)-I$$

$$\therefore\quad I=\frac{1}{2}e^t(\sin t-\cos t)+C$$

$$=\frac{1}{2}x[\sin(\ln x)-\cos(\ln x)]+C$$

例14. 求 $\int x^3(\ln x)^4\,\mathrm{d}x$.

解: 令 $u=\ln x$, 则 $x=e^u$, $\mathrm{d}x=e^u\,\mathrm{d}u$

原式 $=\int e^{3u}u^4\cdot e^u\,\mathrm{d}u=\int u^4e^{4u}\,\mathrm{d}u$

$$=\frac{1}{4}\int u^4\,\mathrm{d}e^{4u}=\frac{1}{4}u^4e^{4u}-\int u^3e^{4u}\mathrm{d}u$$

$$=\cdots$$

$$=\frac{1}{4}e^{4u}\left(u^4-u^3+\frac{3}{4}u^2-\frac{3}{8}u+\frac{3}{32}\right)+C$$

$$=\frac{1}{4}x^4\left(\ln^4 x-\ln^3 x+\frac{3}{4}\ln^2 x-\frac{3}{8}\ln x+\frac{3}{32}\right)+C$$

思考与练习

1. 下述运算错在哪里? 应如何改正?

$$\int \frac{\cos x}{\sin x}\,\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{\sin x} = \frac{\sin x}{\sin x} - \int \left(\frac{1}{\sin x}\right)' \sin x\,\mathrm{d}x$$

$$= 1 - \int \frac{-\cos x}{\sin^2 x} \sin x\,\mathrm{d}x = 1 + \int \frac{\cos x}{\sin x}\,\mathrm{d}x$$

$$\therefore \quad \int \frac{\cos x}{\sin x}\,\mathrm{d}x - \int \frac{\cos x}{\sin x}\,\mathrm{d}x = 1, \quad \text{得 } \mathbf{0 = 1}$$

$$\boxed{= \ln|\sin x| + C}$$

答: 不定积分是原函数族 , 相减不应为 0 .

求此积分的正确作法是用换元法 .

2. 求不定积分 $\displaystyle\int \frac{xe^x}{\sqrt{e^x-1}}\,\mathrm{d}x.$

解：方法1 (先分部，再换元)

$$\int \frac{xe^x}{\sqrt{e^x-1}}\,\mathrm{d}x = \int \frac{x}{\sqrt{e^x-1}}\,\mathrm{d}(e^x-1)$$

$$= 2\int x\,\mathrm{d}\sqrt{(e^x-1)} = 2x\sqrt{e^x-1} - 2\int \sqrt{e^x-1}\,\mathrm{d}x$$

令 $u=\sqrt{e^x-1}$, 则 $\mathrm{d}x = \dfrac{2u}{1+u^2}\,\mathrm{d}u$

$$= 2x\sqrt{e^x-1} - 4\int \frac{u^2+1-1}{1+u^2}\,\mathrm{d}u \quad \boxed{-4(u-\arctan u)+C}$$

$$= 2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C$$

方法2 (先换元,再分部)

令 $u=\sqrt{e^x-1}$, 则 $x=\ln(1+u^2)$, $\mathrm{d}x=\dfrac{2u}{1+u^2}\mathrm{d}u$

故 $$\int\frac{xe^x}{\sqrt{e^x-1}}\mathrm{d}x=\int\frac{(1+u^2)\ln(1+u^2)}{u}\cdot\frac{2u}{1+u^2}\mathrm{d}u$$

$$=2\int\ln(1+u^2)\,\mathrm{d}u$$

$$=2u\ln(1+u^2)-4\int\frac{1+u^2-1}{1+u^2}\mathrm{d}u$$

$$=2u\ln(1+u^2)-4u+4\arctan u+C$$

$$=2x\sqrt{e^x-1}-4\sqrt{e^x-1}+4\arctan\sqrt{e^x-1}+C$$