定积分习题选解

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5.2 定积分

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3. (1)由$f(x)$的定义可知$0 \leq f(x) < 1$, 且$f(x)$的不连续点为$x = 0$和$x = \frac{1}{n}(n = 1, 2\cdots)$. 因此, 对$\forall \varepsilon > 0$, 在区间$[\varepsilon, 1]$上$f(x)$只有有限个不连续点, 从而$f(x)$在$[\varepsilon, 1]$上可积. 因此存在划分$T$, 使$\sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon$. 现在划分$T$中增加分点$O$, 构成$[0, 1]$上的一个划分, 且在小区间$[0, \varepsilon]$上有$\omega_0 \Delta x_0 < \varepsilon$, 从而

$$\sum_{i=0}^{n} \omega_i \Delta x_i = \sum_{i=1}^{n} \omega_i \Delta x_i + \omega_0 \Delta x_0 < \varepsilon + \varepsilon = 2\varepsilon.$$

因此, 由推论5.5.1知$f(x)$在$[0, 1]$上可积.

(2)由$g(x)$的定义可知$-1 \leq g(x) \leq 1$, 且$g(x)$的不连续点为$x = 0$和$x = \frac{1}{n}(n = 1, 2\cdots)$. 因此, 对$\forall \varepsilon > 0$, 在区间$[\varepsilon, 1]$上, $g(x)$只有有限个不连续点, 从而$g(x)$在$[\varepsilon, 1]$上可积. 因此存在划分T, 使$\sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon$. 现在划分$T$中增加分点$O$, 构成$[0, 1]$上的一个划分, 且在小区间$[0, \varepsilon]$上有$\omega_0 \Delta x_0 < \varepsilon$, 从而

$$\sum_{i=0}^{n} \omega_i \Delta x_i = \sum_{i=1}^{n} \omega_i \Delta x_i + \omega_0 \Delta x_0 < \varepsilon + \varepsilon = 2\varepsilon.$$

因此$g(x)$在$[0,1]$上可积.

5. 反证法. 假设对$[0,1]$的任意闭子区间$[\alpha,\beta]$, 都存在$\eta\in[\alpha,\beta]$, 使得$f(\eta)\le 0$. 对$[0,1]$的任一分割:

$$0=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=1,$$

取$\eta_k\in[x_k,x_{k+1}]$, 使$f(x_k)\le 0$, $1\le k\le n$, 则Riemann和满足

$$\sum_{k=1}^{n}f(\eta_k)\Delta x_k\le 0.$$

由f可积, 得到

$$\lim_{\|T\|\to 0}\sum_{k=1}^{n}f(\eta_k)\Delta x_k=\int_0^1 f(x)dx\le 0,$$

与已知相矛盾. 故存在某个闭区间$[\alpha,\beta]$, $\forall x\in[\alpha,\beta]$, 有$f(x)>0$.

6. 因为$f(x)$在$[a,b]$上可积, 所以对$\forall\varepsilon>0$,存在一种划分, 使$\sum\limits_{i=1}^{n}\omega_i\Delta x_i<\varepsilon$, 其中$\omega_i=M_i-m_i$是$f(x)$在第$i$个区间上的振幅. 于是, $\dfrac{1}{f(x)}$在该区间上的振幅为

$$\eta_i=\frac{1}{m_i}-\frac{1}{M_i}=\frac{M_i-m_i}{m_iM_i}\le\frac{1}{\Lambda^2}\omega_i,$$

因此

$$\sum_{i=1}^{n} \eta_i \Delta x_i \leq \sum_{i=1}^{n} \frac{1}{\Lambda^2} \omega_i \Delta x_i < \frac{\varepsilon}{\Lambda^2},$$

即$\dfrac{1}{f(x)}$在$[a, b]$上也可积.

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1. 证明:　取$\varphi(x)=f(x)$, 则$\int_\alpha^\beta f^2(x)dx=0$. 假设存在$x_0\in[\alpha,\beta]$, 使$f(x_0)\neq 0$, 则由连续函数的局部保号性可知, 存在含$x_0$的区间$[a,b]\subset[\alpha,\beta]$, 使对任意的$x\in[a,b]$, 有$f^2(x)>f^2(x_0)/2>0$. 于是,

$$0=\int_\alpha^\beta f^2(x)dx\geq\int_a^b f^2(x)dx>\frac{f^2(x_0)}{2}(b-a)>0,$$

矛盾. 故在$[\alpha,\beta]$上, $f(x)\equiv 0$.

2. 由积分中值公式, 有

$$\begin{aligned}\lim_{n\to\infty}\int_{n^2}^{n^2+n}\frac{1}{\sqrt{x}}\,e^{-\frac{1}{x}}dx &= \lim_{n\to\infty}e^{-\frac{1}{\xi}}\int_{n^2}^{n^2+n}\frac{dx}{\sqrt{x}}\\ &= \lim_{n\to\infty}e^{-\frac{1}{\xi}}\frac{2n}{\sqrt{n^2+n}+n}=1,\end{aligned}$$

其中$n^2<\xi<n^2+n$, 从而$1/\xi\to 0\ (n\to\infty)$.

3.(4) $3\sqrt{e}=\int_e^{4e}\frac{1}{\sqrt{x}}dx<\int_e^{4e}\frac{\ln x}{\sqrt{x}}dx$. 又

$$\max_{x\in[e,4e]}\frac{\ln x}{\sqrt{x}}=\frac{\ln e^2}{\sqrt{e^2}}=\frac{2}{e},$$

所以

$$\int_e^{4e}\frac{\ln x}{\sqrt{x}}dx<3e\cdot\frac{2}{e}=6.$$

故,

$$3\sqrt{e}<\int_e^{4e}\frac{\ln x}{\sqrt{x}}dx<6.$$

6. 令$F(x)=\int_0^x f(\theta)\sin(\theta)d\theta, x\in[0,\pi]$, 则$F(x)\in C[0,\pi]$, 在$(0,\pi)$内可导，且$F(0)=F(\pi)=0$, 由Rolle定理可知，$\exists\alpha\in(0,\pi)$, 使得

$$F'(\alpha)=0,\Rightarrow f(\alpha)\sin(\alpha)=0.$$

因$\alpha\in(0,\pi)$, 故$\sin(\alpha)\neq 0$, 因此必有$f(\alpha)=0$.

往证$\exists\beta\in(0,\pi)(\beta\neq\alpha)$, 使得$f(\beta)=0$, 用反证法.

假设$f(x)$在$(0,\pi)$内只有唯一零点$x=\alpha$, 则$f(x)$在$(0,\alpha)$和(α,π)内必反号，否则不可能有$\int_0^\pi f(\theta)\sin(\theta)d\theta=0$. 而$\sin(\theta-\alpha)$在$(0,\alpha)$和$(\alpha,\pi)$内符号也相反，故$f(\theta)\sin(\theta-\alpha)$在这两个区间内必同号. 于是有$\int_0^\pi f(\theta)\sin(\theta-\alpha)d\theta>0$. 另一方面，由题设条件又有

$$\int_0^\pi f(\theta)\sin(\theta-\alpha)d\theta=\int_0^\pi f(\theta)(\sin(\theta)\cos(\alpha)-\cos(\theta)\sin(\alpha))d\theta$$
$$=\cos(\alpha)\int_0^\pi f(\theta)\sin(\theta)-\sin(\alpha)\int_0^\pi f(\theta)\cos(\theta)d\theta=0.$$

从而推出矛盾. 由此可得$f(x)$在$(0,\pi)$内至少有两个零点.

7.(2) 证法一: 由Cauchy-Schwarz不等式, 得

$$\begin{aligned}\int_\alpha^\beta[f(x)+g(x)]^2dx &= \int_\alpha^\beta f(x)[f(x)+g(x)]dx+\int_\alpha^\beta g(x)[f(x)+g(x)]dx\\ &\le \Big\{\int_\alpha^\beta f^2(x)dx\Big\}^{1/2}\cdot\Big\{\int_\alpha^\beta[f(x)+g(x)]^2dx\Big\}^{1/2}\\ &\quad+\Big\{\int_\alpha^\beta g^2(x)dx\Big\}^{1/2}\cdot\Big\{\int_\alpha^\beta[f(x)+g(x)]^2dx\Big\}^{1/2}.\end{aligned}$$

两端同除以$\Big\{\int_\alpha^\beta[f(x)+g(x)]^2dx\Big\}^{1/2}$即得Minkowski不等式.

8.

$$\Big|\int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^n f(\frac{i}{n})\Big| = \Big|\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x)dx - \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(\frac{i}{n})dx\Big|$$
$$= \Big|\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} [f(x) - f(\frac{i}{n})]dx\Big| \le \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big|f(x) - f(\frac{i}{n})\Big|dx \le \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} L\Big|x - \frac{i}{n}\Big|dx$$
$$\le \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \frac{L}{n}dx \le \frac{L}{n}.$$

9. 将$[0,1]$区间n等分，分点为$0, \frac{1}{n}, \cdots, \frac{n}{n}$. 因$f$在$[0,1]$上单调减, 所以对$\forall x \in [\frac{i-1}{n}, \frac{i}{n}](i = 1, 2, \cdots, n)$, $f(\frac{i-1}{n}) \ge f(x) \ge f(\frac{i}{n})$, 如上题，有

$$\int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^n f(\frac{i}{n}) = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x)dx - \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(\frac{i}{n})dx$$
$$= \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} [f(x) - f(\frac{i}{n})]dx \le \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big[f(\frac{i-1}{n}) - f(\frac{i}{n})\Big]dx$$
$$= \sum_{i=1}^n \frac{1}{n}[f(\frac{i-1}{n}) - f(\frac{i}{n})] = \frac{f(0) - f(1)}{n}.$$

10. 由$1 \le f(x) \le 2 \Rightarrow [f(x)-1][f(x-2)] \le 0 \Rightarrow f^2(x)-3f(x)+2 \le 0$

$$\Rightarrow f(x)-3+\frac{2}{f(x)} \le 0 \Rightarrow \int_0^1 f(x)dx - 3 + 2\int_0^1 \frac{1}{f(x)}dx \le 0$$

$$\Rightarrow 3 \ge \int_0^1 f(x)dx + 2\int_0^1 \frac{1}{f(x)}dx \ge 2\Big[2\int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx\Big]^{\frac{1}{2}}$$

$$\Rightarrow \int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx] \le \frac{9}{8}.$$

11. 因$g(x)$在$[\alpha,\beta]$上连续, 所以$\exists M, m(M \ge m > 0)$, 使对$\forall x \in [\alpha,\beta]$, 有$m \le g(x) \le M$. 设$f(x_0) = \max\limits_{\alpha \le x \le \beta} f(x)$, $x_0 \in (\alpha,\beta)$, 则对$\forall \varepsilon > 0, \exists \delta > 0$, 使得当$x \in [x_0-\delta, x_0+\delta] \subset [\alpha,\beta]$时, 有

$$f(x_0) - \frac{\varepsilon}{2} < f(x) \le f(x_0).$$

记$I_n = [\int_\alpha^\beta f^n(x)g(x)dx]^{\frac{1}{n}}$, 则

$$[2\delta(f(x_0)-\frac{\varepsilon}{2})^n m]^{\frac{1}{n}} \le [\int_{x_0-\delta}^{x_0+\delta} f^n(x)g(x)dx]^{\frac{1}{n}} \le I_n \le [f^n(x_0)M(\alpha-\beta)]^{\frac{1}{n}}.$$

令$n\to\infty$, 得

$$f(x_0)-\frac{\varepsilon}{2}\le\lim_{n\to\infty}I_n\le f(x_0).$$

由ε的任意性, 得

$$\lim_{n\to\infty}[\int_\alpha^\beta f^n(x)g(x)dx]^{\frac{1}{n}})=f(x_0)=\max_{\alpha\le x\le\beta}f(x).$$

当$x_0=a$, 或$x_0=b$时, 类似可证.

12. $\{\alpha_n\}$和$\{\alpha_n\}$如下：

$$\{\alpha_n\}=\int_0^1\max\{x,\beta_{n-1}\}dx,\quad\{\beta_n\}=\int_0^1\min\{x,\alpha_{n-1}\}dx,(n=2,3,\cdots)\tag{1}$$

由上式，有

$$\alpha_n\ge\int_0^1 xdx=\frac{1}{2},\quad\beta_n\le\int_0^1 xdx=\frac{1}{2},\quad(n=2,3,\cdots)\tag{2}$$

将（2）代人（1），有

$$\alpha_n\le\int_0^{\frac{1}{2}}\frac{1}{2}dx+\int_{\frac{1}{2}}^1 xdx=\frac{5}{8},\quad\beta_n\ge\int_0^{\frac{1}{2}}xdx+\int_{\frac{1}{2}}^1\frac{1}{2}dx=\frac{3}{8},\quad(n=2,3,\cdots)\tag{3}$$

$$\Rightarrow \frac{1}{2} \le \alpha_n \le \frac{5}{8}, \quad \frac{3}{8} \le \beta_n \le \frac{1}{2}, \quad (n = 2, 3, \cdots) \qquad (4)$$

由(4)及（1）可得

$$\begin{cases} 2\alpha_{n+1} = 2(\int_0^{\beta_n} \beta_n dx + \int_{\beta_n}^1 xdx) = 1 + \beta_n^2, \quad (5) \\ 2\beta_{n+1} = 2(\int_0^{\alpha_n} xdx + \int_{\alpha_n}^1 \alpha_n dx = 2\alpha_n - \alpha_n^2, \quad (6) \end{cases} \qquad (n = 2, 3, \cdots)$$

$$\Rightarrow \quad \alpha_{n+2} - \alpha_{n+1} = \frac{\beta_{n+1} - \beta_n}{2}(\beta_{n+1} - \beta_n),$$

$$\beta_{n+1} - \beta_n = \frac{2 - \alpha_n - \alpha_{n-1}}{2}(\alpha_n - \alpha_{n-1}), \quad (n = 2, 3, \cdots)$$

$$\Rightarrow \quad \alpha_{n+2} - \alpha_{n+1} = \frac{\beta_{n+1} + \beta_n}{2} \cdot \frac{2 - \alpha_n - \alpha_{n-1}}{2}(\alpha_n - \alpha_{n-1}),$$

由（4）$\Rightarrow |\alpha_{n+2} - \alpha_{n+1}| \le |\alpha_n - \alpha_{n-1}|, \quad n = 2, 3, \cdots$

反复用上式，得

$$|\alpha_{2m+2} - \alpha_{2m+1}| \le \frac{1}{4^m}|\alpha_2 - \alpha_1|, \quad |\alpha_{2m+3} - \alpha_{2m+2}| \le \frac{1}{4^m}|\alpha_3 - \alpha_2|, \quad m = 1, 2, \cdots$$

令$m \to \infty$, 得$\alpha_{2m+2} - \alpha_{2m+1} \to 0, \quad \alpha_{2m+3} - \alpha_{2m+2} \to 0$,, 从而可得$\lim\limits_{n\to\infty} \alpha_n$存在，同理可知，$\lim\limits_{n\to\infty} \beta_n$存在. 设$\lim\limits_{n\to\infty} \alpha_n = \alpha$，$\lim\limits_{n\to\infty} \beta_n = \beta$, 由（5）（6）可得

$$2\alpha = 1 + \beta^2, \quad 2\beta = 2\alpha - \alpha^2 \qquad (7)$$

解得: $\alpha=2-\sqrt{2},\quad \beta=\sqrt{2}-1.$

13. 证：$\Lambda_{n+1}=\int_\alpha^\beta \varphi(x)f^{n+1}(x)dx\le M\int_\alpha^\beta \varphi(x)f^n(x)dx=M\Lambda_n,$

其中$M=\max\limits_{\alpha\le x\le\beta} f(x).$

$\Rightarrow \dfrac{\Lambda_{n+1}}{\Lambda_n}\le M$, 即数列$\{\dfrac{\Lambda_{n+1}}{\Lambda_n}\}$有上界. 再由Cauchy-Schwartz不等式，有

$$\begin{aligned}\Lambda_{n+1}^2 &= (\int_\alpha^\beta \varphi(x)f^{n+1}(x)dx)^2=(\int_\alpha^\beta [\varphi(x)]^{\frac{1}{2}}[f(x)]^{\frac{n+2}{2}}[\varphi(x)]^{\frac{1}{2}}[f(x)]^{\frac{n}{2}}dx)^2\\ &\le \int_\alpha^\beta [\varphi(x)][f(x)]^{n+2}dx\int_\alpha^\beta [\varphi(x)][f(x)]^n dx=\Lambda_{n+2}\cdot\Lambda_n,\end{aligned}$$

于是，$\dfrac{\Lambda_{n+1}}{\Lambda_n}\le\dfrac{\Lambda_{n+2}}{\Lambda_{n+1}}$, 即数列$\{\dfrac{\Lambda_{n+1}}{\Lambda_n}\}$是单调增加的，故其极限存在. 再由命题（*）及第11题知，

$$\lim_{n\to\infty}\frac{\Lambda_{n+1}}{\Lambda_n}=\lim_{n\to\infty}\sqrt[n]{\Lambda_n}=\lim_{n\to\infty}(\int_\alpha^\beta[\varphi(x)][f(x)]^n dx)^{\frac{1}{n}}=\max_{\alpha\le x\le\beta}f(x).$$

注：命题（*）：若$x_n>0(n=1,2,\cdots,)$且$\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}$存在, 则$\lim\limits_{n\to\infty}\sqrt[n]{x_n}=\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}.$

1.(2) 由积分第一中值定理, 存在$\xi \in [n, n+1]$, 使得

$$\lim_{n\to\infty}\int_n^{n+1}\frac{\sin x}{x}\,dx=\lim_{n\to\infty}\sin\xi\cdot\int_n^{n+1}\frac{1}{x}\,dx=\lim_{n\to\infty}\sin\xi\cdot\ln\frac{n+1}{n}=0$$

(4) 令$r=t^2$, 则

$$\int_x^{x+1}\sin t^2\,dt=\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{\sin r}{\sqrt{r}}\,dr=-\frac{\cos r}{2\sqrt{r}}\Big|_{x^2}^{(x+1)^2}-\frac{1}{4}\int_{x^2}^{(x+1)^2}\frac{\cos r}{r\sqrt{r}}\,dr.$$

因为当$x>0$时, 有

$$\left|\frac{\cos x^2}{x}-\frac{\cos(x^2+1)}{x+1}\right|\le\frac{1}{x}+\frac{1}{x+1}\to 0\quad(x\to+\infty),$$

$$\left|\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{\cos r}{r\sqrt{r}}\,dr\right|\le\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{1}{r\sqrt{r}}\,dr=\frac{1}{x}-\frac{1}{x+1}\to 0\quad(x\to+\infty),$$

所以$\displaystyle\lim_{x\to+\infty}\int_x^{(x+1)}\sin t^2\,dt=0.$

(5)令$S(x)=\int_0^x|\cos t|dt$. 由于$|\cos t|$是以π为周期的周期函数, 故在任一周期长的区间上定积分值相同. 设$n\pi\le x<(n+1)\pi$(n为正整数), 则

$$\int_0^{n\pi}|\cos t|dt\le S(x)<\int_0^{(n+1)\pi}|\cos t|dt.$$

又

$$\int_0^{\pi}|\cos t|dt=\int_0^{\frac{\pi}{2}}\cos x dx-\int_{\frac{\pi}{2}}^{\pi}\cos x dx=2,$$

故$2n\le S(x)<2(n+1)$. 因此

$$\frac{2n}{(n+1)\pi}<\frac{S(x)}{x}<\frac{2(n+1)}{n\pi}.$$

因为当$x\to+\infty$时, $n\to\infty$且

$$\lim_{n\to\infty}\frac{2n}{(n+1)\pi}=\lim_{n\to\infty}\frac{2(n+1)}{n\pi}=\frac{2}{\pi},$$

故$\displaystyle\lim_{n\to\infty}\frac{S(x)}{x}=\frac{2}{\pi}$.

3. (2) 由page137的4（4）知

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{2}\sin(x+\pi/4)} dx$$

$$= \frac{1}{2\sqrt{2}} \ln|\csc(x+\pi/4) - \cot(x+\pi/4)|\Big|_0^{\pi/2} = \frac{\sqrt{2}}{2}\ln(\sqrt{2}+1).$$

(4) 令$\ln x = t$, 则$dx = e^t dt$. 当$x = 1$时, $t = 0$; 当$x = e$时, $t = 1$. 于是

$$\int_1^e \sin(\ln x) dx = \int_0^1 e^t \sin t dt = e^t \sin t\Big|_0^1 - \int_0^1 e^t \cos t dt,$$

其中

$$\begin{aligned}\int_0^1 e^t \cos t dt &= e^t \sin t\Big|_0^1 - \int_0^1 e^t \sin t dt \\ &= e\sin 1 + e^t \cos t\Big|_0^1 - \int_0^1 e^t \cos t dt \\ &= e\sin 1 + e\cos 1 - 1 - \int_0^1 e^t \cos t dt,\end{aligned}$$

于是

$$\int_0^1 e^t\cos t dt=\frac{1}{2}(e\sin 1+e\cos 1)-\frac{1}{2}.$$

故

$$\int_1^e \sin(\ln x)dx=e\sin 1-\frac{1}{2}e(\sin 1+\cos 1)+\frac{1}{2}=\frac{1}{2}e(\sin 1-\cos 1)+\frac{1}{2}.$$

(5) (该积分为反常积分，发散) 令$x=\tan t$, 则$dx=\sec^2 t dt$. 当$x=0$时, $t=0$; 当$x=1$时, $t=\frac{\pi}{4}$. 于是

$$原式=\int_0^{\frac{\pi}{4}}\frac{\sec t}{\tan t}dt=\int_0^{\frac{\pi}{4}}\csc t dt=\ln|\csc t-\cot t|\Big|_0^{\frac{\pi}{4}}=+\infty.$$

注：将原题积分区间改为$[1,2]$, 则

$$\int_1^2\frac{1}{x\sqrt{1+x^2}}dx=[\ln x-\ln(\sqrt{x^2+1}+1)]_1^2=\ln 2-\ln(\sqrt{5}+1)+\ln(\sqrt{3}+1).$$

(6) 原式$=\int_0^1\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\ dx=\int_0^1\frac{1}{(x-\frac{1}{x})^2+2}d(x-\frac{1}{x})$. 令$t=x-\frac{1}{x}$, 则当$x\to 0^+$时, $t\to-\infty$; 当$x=1$时, $t=0$. 于是

$$原式=\int_{-\infty}^0\frac{1}{t^2+2}\ dt=\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\Big|_{-\infty}^0=\frac{\sqrt{2}}{4}\pi.$$

注：上述方法需要用到反常积分, 可用常规的有理函数积分法求原函数.

(7) 记原积分为I_n, 则

$$I_n = \frac{1}{2}x^2\ln^n x\Big|_1^e - \frac{1}{2}\int_1^e x^2\cdot n\ln^{n-1}x\cdot\frac{1}{x}dx = \frac{1}{2}e^2 - \frac{n}{2}I_{n-1},$$

又$I_1 = \int_1^e x\ln x dx = \frac{1}{2}e^2 - \frac{1}{4}(e^2-1)$, 由递推得到

$$I_n = \frac{1}{2}e^2[1 - \frac{n}{2} + \frac{n(n-1)}{2^2} + \cdots + (-1)^{n-1}\frac{n!}{2^n}] + (-1)^{n+1}\frac{n!}{2^{n+1}}.$$

(11)当$\alpha \le 0$时, 原式$=\int_0^1 x(x-\alpha)dx = (\frac{x^3}{3} - \frac{ax^2}{2})\Big|_0^1 = \frac{1}{3} - \frac{\alpha}{2}$;

当$0 < \alpha < 1$时,

原式$= \int_0^\alpha x(\alpha - x)dx + \int_\alpha^1 x(x-\alpha)dx = (-\frac{x^3}{3} + \frac{ax^2}{2})\Big|_0^\alpha + (\frac{x^3}{3} - \frac{ax^2}{2})\Big|_\alpha^1 = \frac{\alpha^3}{3} - \frac{\alpha}{2} + \frac{1}{3}$;

当$\alpha \ge 1$时, 原式$= \int_0^1 x(\alpha - x)dx = (-\frac{x^3}{3} + \frac{ax^2}{2})\Big|_0^1 = \frac{\alpha}{2} - \frac{1}{3}$.

(12) 原式$= \int_0^{\ln 2} 1dx + \int_{\ln 2}^{\ln 3} 2dx + \int_{\ln 3}^{\ln 4} 3dx + \int_{\ln 4}^{\ln 5} 4dx + \int_{\ln 5}^{\ln 6} 5dx + \int_{\ln 6}^{\ln 7} 6dx + \int_{\ln 7}^{2} = 14 - \ln(7!)$.

5. 对于左端的不等式，注意到当$k-1<x<k$时，有$\sqrt{k}>\sqrt{x}$, 故有$\sqrt{k}>\int_{k-1}^{k}\sqrt{x}dx$, 从而得

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}>\int_0^n\sqrt{x}dx=\frac{2}{3}n^{3/2}.$$

对于右端不等式，因曲线$y=\sqrt{x}$在$(0,+\infty)$上是凸的，所以有

$$\frac{\sqrt{k-1}+\sqrt{k}}{2}<\int_{k-1}^{k}\sqrt{x}dx.$$

由此可得

$$\begin{aligned}\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}&=\frac{\sqrt{0}+\sqrt{1}}{2}+\frac{\sqrt{1}+\sqrt{2}}{2}+\cdots+\frac{\sqrt{n-1}+\sqrt{n}}{2}+\frac{\sqrt{n}}{2}\\&<\int_0^n\sqrt{x}dx+\frac{\sqrt{n}}{2}=\frac{4n+3}{6}\sqrt{n}.\end{aligned}$$

6. (原题有误)

$$\begin{aligned}\int_0^1 x^n f(x)dx&=\frac{1}{n+1}\int_0^1 f(x)dx^{n+1}=\frac{1}{n+1}x^{n+1}f(x)\Big|_0^1-\frac{1}{n+1}\int_0^1 x^{n+1}f'(x)dx\\&=\frac{f(1)}{n+1}-\frac{1}{(n+1)(n+2)}\int_0^1 f'(x)dx^{n+2}\end{aligned}$$

$$= \frac{f(1)}{n+1} - \frac{f'(1)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}\int_0^1 x^{n+2} f''(x)dx$$

$$\int_0^1 x^n f(x)dx - [\frac{f(1)}{n} - \frac{f(1)+f'(1)}{n^2}] = [\frac{1}{n^2} - \frac{1}{n(n+1)}]f(1)$$

$$+ [\frac{1}{n^2} - \frac{1}{(n+1)(n+2)}]f'(1) + \frac{1}{(n+1)(n+2)}\int_0^1 x^{n+2} f''(x)dx \qquad (1)$$

因$f''(x)$在$[0,1]$上连续，所以$\exists M>0, \forall x\in[0,1], |f''(x)|\le M.$

$$\Rightarrow |\int_0^1 x^{n+2} f''(x)dx| \le M\int_0^1 x^{n+2} = \frac{M}{(n+3)} \to 0 (n\to\infty).$$

由(1)可得

$$\lim_{n\to\infty} n^2\{\int_0^1 x^n f(x)dx - [\frac{f(1)}{n} - \frac{f(1)+f'(1)}{n^2}]\} = 0$$

$$\Rightarrow \quad \int_0^1 x^n f(x)dx = \frac{f(1)}{n} - \frac{f(1)+f'(1)}{n^2} + o(\frac{1}{n^2})\ (n\to\infty).$$

7. (原题有误，将f单调增改为单调减)

令$F(x)=2\int_a^x tf(t)dt - x\int_0^x f(t)dt + a\int_0^a f(x)dx, x\in[a,b]$, 则$F(a)=0$, 因$f(x)$单调减，所以有

$$F'(x)=xf(x)-\int_0^x f(t)dt=\int_0^x [f(x)-f(t)]dt\le 0,$$

故$F(x)$在$[a,b]$上单调减，$\Rightarrow F(b)\le F(a)=0$,

$$\Rightarrow 2\int_a^b xf(x)dx \le b\int_0^b f(x)dx - a\int_0^a f(x)dx.$$

8. 由定积分第一中值定理知存在$\xi\in(0,a)$, 满足$|f(\xi)|=\frac{1}{a}\int_0^a |f(x)|dx$. 于是由于$f(x)$在$[0,2\pi]$上连续可导, 可得

$$|f(0)|-\frac{1}{a}\int_0^a |f(x)|dx = |f(0)|-|f(\xi)| \le |f(0)-f(\xi)| = \Big|\int_0^\xi f'(x)dx\Big|$$

$$\le \int_0^\xi |f'(x)|dx \le \int_0^a |f'(x)|dx.$$

因此

$$|f(0)|\le \frac{1}{a}\int_0^a |f(x)|dx + \int_0^a |f'(x)|dx.$$

9. 因$f(x)$在$[0,1]$上连续, 则可设$f(\eta)=\max\limits_{0\le x\le 1}|f(x)|, \eta\in[0,1]$. 由积分中值定理可知, 存在$\xi\in[0,1]$, 使得$|\int_0^1 f(x)dx|=|f(\xi)|$.
若$\xi=\eta$, 则不等式显然成立. 下设$\xi\neq\eta$, 则

$$|f(\eta)-f(\xi)|=\left|\int_\eta^\xi f'(x)dx\right|\le\int_0^1|f'(x)|dx,$$

即

$$|f(\xi)|\ge|f(\eta)|-\int_0^1|f'(x)|dx.$$

因此

$$\left|\int_0^1 f(x)dx\right|\ge|f(\eta)|-\int_0^1|f'(x)|dx.$$

即

$$\max_{0\le x\le 1}|f(x)|\le\left|\int_0^1 f(x)dx\right|+\int_0^1|f'(x)|dx.$$

因为对$\forall x\in[0,1]$, $|f(x)|\le\max\limits_{0\le x\le 1}|f(x)|$, 所以

$$|f(x)|\le\left|\int_0^1 f(x)dx\right|+\int_0^1|f'(x)|dx\le\int_0^1[|f(x)|+|f'(x)|]dx.$$

12. 证明：若函数$f(x)$在$(-\infty,+\infty)$的任意有界闭区间$[\alpha,\beta]$上可积，且对$\forall x,y\in[\alpha,\beta]$, 有$f(x+y)=f(x)+f(y)$, 则$f(x)=cx, c=f(1)$.

证：$\forall x\in\mathbb{R}, x\neq 0, f(t+y)=f(t)+f(y)$, 两边对$t$从0到$x$积分，得

$$\int_0^x f(t+y)dt=\int_0^x f(t)dt+\int_0^x f(y)dt=\int_0^x f(t)dt+xf(y),$$

或

$$xf(y)=\int_0^x f(t+y)dt-\int_0^x f(t)dt.$$

令$t+y=u$, 有

$$\int_0^x f(t+y)dt=\int_y^{x+y} f(u)du=\int_0^{x+y} f(u)du-\int_0^y f(u)du,$$

$$\Rightarrow xf(y)=\int_0^{x+y} f(u)du-\int_0^y f(u)du-\int_0^x f(u)du,$$

交换x与y的位置，右端积分的代数和不变，即

$$xf(y)=yf(x)\quad \text{或}\quad \frac{f(x)}{x}=\frac{f(y)}{y}.$$

于是$\dfrac{f(x)}{x}=c$, 即$f(x)=cx$. 当$x=y=0$时，$f(0)=2f(0),\Rightarrow f(0)=0$, 上式也成立. 令$x=1,\Rightarrow c=f(1)$.

13. 作变换$t = nx$, 由定积分第一中值定理知存在$\varepsilon_k \in (2(k-1)\pi, 2k\pi)$, 使

$$\begin{aligned}\int_0^{2\pi} f(x)|\sin nx|dx &= \frac{1}{n}\int_0^{2n\pi} f(\frac{x}{n})|\sin x|dx = \frac{1}{n}\sum_{k=1}^{n}\int_{2(k-1)\pi}^{2k\pi} f(\frac{x}{n})|\sin x|dx \\ &= \frac{1}{n}\sum_{k=1}^{n} f(\frac{\xi_k}{n})\int_{2(k-1)\pi}^{2k\pi}|\sin x|dx = \frac{4}{n}\sum_{k=1}^{n} f(\frac{\xi_k}{n}) \\ &= \frac{2}{\pi}\sum_{k=1}^{n} f(\frac{\xi_k}{n})\frac{2\pi}{n},\end{aligned}$$

而$\sum_{k=1}^{n} f(\frac{\xi_k}{n})\frac{2\pi}{n}$是$f(x)$将$[0, 2\pi]$区间$n$等分的积分和, 由于$f(x)$在$[0, 2\pi]$上连续, 故

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{\xi_k}{n})\frac{2\pi}{n} = \int_0^{2\pi} f(x)dx,$$

从而

$$\lim_{n\to\infty}\int_0^{2\pi} f(x)|\sin nx|dx = \frac{2}{\pi}\int_0^{2\pi} f(x)dx.$$

14. 证：不妨设$0<h<1$($-1<h<0$时，同法可证). 因

$$\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx = \int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx+\int_{-\sqrt{h}}^{\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx + \int_{\sqrt{h}}^{1}\frac{h}{h^2+x^2}f(x)dx \qquad (1)$$

对（1）式右端第一个积分，由于$h\to 0^+$时，有

$$\left|\int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx\right| \le M\int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}dx = M(-\arctan\frac{1}{\sqrt{h}}+\arctan\frac{1}{h})\to 0,$$

故$\lim\limits_{h\to 0^+}\int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx=0$. 同理可得$\lim\limits_{h\to 0^+}\int_{\sqrt{h}}^{1}\frac{h}{h^2+x^2}f(x)dx=0$. 对(1)式右端第二个积分，由积分中值定理，$\exists\xi_h\in(-\sqrt{h},\sqrt{h})$, 使得

$$\begin{aligned}\int_{-\sqrt{h}}^{\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx &= f(\xi_h)\int_{-\sqrt{h}}^{\sqrt{h}}\frac{h}{h^2+x^2}dx = f(\xi_h)\arctan\frac{x}{h}\Big|_{-\sqrt{h}}^{\sqrt{h}} \\ &= f(\xi_h)\cdot 2\arctan\frac{1}{\sqrt{h}}\to\pi f(0)\quad(h\to 0^+),\end{aligned}$$

所以$\lim\limits_{h\to 0+}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx=\pi f(0)$.

类似可证，$\lim\limits_{h\to 0-}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx=\pi f(0)$. 故，原式成立.

15. 证：

$$\int_{\pi}^{n\pi}\frac{|\sin(x)|}{x}dx=\sum_{k=1}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(x)|}{x}dx,$$

$$\int_{k\pi}^{(k+1)\pi}\frac{|\sin(x)|}{x}dx\overset{x=k\pi+t}{=}\int_{0}^{\pi}\frac{|\sin(t)|}{k\pi+t}dt>\int_{0}^{\pi}\frac{\sin(t)}{(k+1)\pi}dt=\frac{2}{(k+1)\pi}.$$

又$\int_{n}^{n+1}\frac{dx}{x}<\int_{n}^{n+1}\frac{dx}{n}=\frac{1}{n}$, 于是

$$\begin{aligned}\int_{\pi}^{n\pi}\frac{|\sin(x)|}{x}dx&=\sum_{k=1}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(x)|}{x}dx>\sum_{k=1}^{n-1}\frac{2}{(k+1)\pi}=\frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{k+1}\\&>\frac{2}{\pi}\sum_{k=1}^{n-1}\int_{k+1}^{k+2}\frac{1}{x}dx=\frac{2}{\pi}\int_{2}^{n+1}\frac{1}{x}dx=\frac{2}{\pi}\ln\frac{n+1}{2}.\end{aligned}$$

16.提示：当$n \neq m$时，不防设$n < m$, 并记$a_n = \frac{1}{2^n n!}$. 连续应用m次分部积分公式：

$$\int_{-1}^{1} P_m(x)P_n(x)dx = a_n \int_{-1}^{1} P_m(x)d\Big(\frac{d^n-1}{dx^{n-1}}(x^2-1)^n\Big) = \cdots$$

注意，当$k \leq m$时，有$P_m^{(k)}(x)\dfrac{d^{k-1}}{dx^{k-1}}(x^2-1)^n\Big|_{-1}^{1} = 0$.

当$n = m$时，连续应用n次分部积分：

$$\int_{-1}^{1} P_n(x)P_n(x)dx = \frac{2(2n)!}{2^{2n}(n!)^2}\int_0^1 (1-x^2)^n dx,$$

令$x = \cos(t)$, 有

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2(2n)!}{2^{2n}(n!)^2}\int_0^{\pi/2} \sin^{2n+1}(t)dt,$$

再应用Page135，例5.2.11.