启明学院　2017 - 2018 学年第二学期

《微积分（一）》（下）课程考试试卷(A 卷) (闭卷)

参考答案与评分标准

一、填空题（每小题 4 分，共 28 分）

1. $[0,2)$;　　2. $e^{x^2y}(2xydx+x^2dy)$;　　3. -5；

4. $\frac{2}{3}$;

5. $\int_0^1 dx\int_0^{x^2} f(x,y)dy+\int_1^{\sqrt{2}} dx\int_0^{\sqrt{2-x^2}} f(x,y)dy$;　　6. $11x+3y-4z-11=0$;

7. $a=2, b=2$.

二、判断题(每小题 2 分，共 8 分). 请在正确说法相应的括号中画“ √ ”，在错误说法的括号中画“×”.

8. √；　9. √；　10. ×；　11. ×.

三、解答题（每小题 6 分, 共 12 分）

12. 解法 1：由于区域Ω关于yoz, xoz平面对称，从而

$$\iiint_\Omega xdxdydz=\iiint_\Omega ydxdydz=0. \qquad (2 分)$$

因此 $I=\iiint_\Omega zdxdydz=\int_0^1 dz\iint_{\Omega(z)} zdxdy+\int_1^{\sqrt{3}} dz\iint_{\Omega(z)} zdxdy$

$$=\int_0^1 2\pi z^2dz+\int_1^{\sqrt{3}}\pi z(3-z^2)dz$$

$$=\frac{5}{3}\pi. \qquad (6 分)$$

解法 2：由于区域Ω关于yoz, xoz平面对称，从而

$$\iiint_\Omega xdxdydz=\iiint_\Omega ydxdydz=0.$$

又因为Ω在xoy平面上的投影区域$D=\{(x,y)|x^2+y^2\le 2\}$　　(2 分)

因此 $$I=\iiint_\Omega zdxdydz=\int_0^{2\pi} d\theta\int_0^{\sqrt{2}} rdr\int_{\frac{r^2}{2}}^{\sqrt{3-r^2}} zdz$$

$$=2\pi\int_0^{\sqrt{2}} r(\frac{3-r^2}{2}-\frac{r^4}{8})\,dr$$

$$=2\pi\left(\frac{3}{4}r^2-\frac{1}{8}r^4-\frac{1}{48}r^6\right)\Big|_0^{\sqrt{2}}$$

$$=\frac{5}{3}\pi. \qquad (6 分)$$

13. 解：由于$\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{y^2-x^2}{x^2+y^2}$.　　(2 分)

从而有

$$\int_L\frac{xdy-ydx}{x^2+y^2}=\oint_{L+\overrightarrow{CA}}\frac{xdy-ydx}{x^2+y^2}-\int_{\overrightarrow{CA}}\frac{xdy-ydx}{x^2+y^2} \qquad (4 分)$$

$$= \oint_{x^2+y^2=\epsilon^2} \frac{xdy-ydx}{x^2+y^2} - \int_0^2 \frac{dt}{1+t^2}$$

$$= 2\pi - \arctan\ 2.$$ (6 分)

四、计算题（每小题 7 分，共 28 分）

14. 解： 设$f(x)$的 Fourier 级数展开式为

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx),$$

其中

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^2;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = (-1)^n \frac{4}{n^2};$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0.$$

因此$f(x)$的 Fourier 级数展开式为

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} \cos nx\,.$$

(4 分)

由 Dirichlet 收敛定理可知

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} \cos nx\,, \quad x \in [-\pi, \pi].$$

令$\mathrm{x} = \pi$得

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

(5 分)

又由 Parseval 等式得

$$\frac{1}{2}\left(\frac{2}{3}\pi^2\right)^2 + 16\sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2}{5}\pi^4.$$

于是有

$$\sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{1}{16}\left(\frac{2}{5}\pi^4 - \frac{2}{9}\pi^4\right) = \frac{\pi^4}{90}.$$

(7 分)

15.解：设$S_1 = \{(x,y,0) \in \mathbb{R}^3 | x^2 + y^2 \le 1\}$且取下侧为正定向，$S + S_1$所围成的区域为 $\Omega = \{(x,y,z) \in \mathbb{R}^3 | 0 \le z \le 1, x^2 + y^2 \le 1 - z\}$. (2 分)

则由 Gauss 公式可得

$$I = \iint_S (x^2 - z)dxdy + (z^2 - y)dzdx$$

$$= \iint_{S+S_1} (x^2 - z)dxdy + (z^2 - y)dzdx - \iint_{S_1} (x^2 - z)dxdy + (z^2 - y)dzdx$$

$= \iiint_{\Omega} -2dxdydz - \iint_{s_1}(x^2-z)dxdy + (z^2-y)dzdx$ (5 分)

$= -2\int_0^1 \pi(1-z)dz + \iint_{x^2+y^2\le 1} x^2dxdy$

$= -2\pi\int_0^1 (1-z)dz + \frac{1}{2}\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy$

$= -\pi + \frac{\pi}{4}$

$= -\frac{3}{4}\pi.$

(7 分)

16.解：设Σ为平面$x+y+z=0$与球体$x^2+y^2+z^2\le a^2$交得得圆盘，取法向量$\vec{n}=\frac{1}{\sqrt{3}}(1,1,1)$为正定向. (2 分)

则由 Stokes 公式可知，

$$I = \oint_C ydx + zdy + xdz = -\iint_{\Sigma} dydz + dzdx + dxdy \quad (5\text{ 分})$$

$$= -\iint_{\Sigma}\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)d\sigma$$

$$= -\sqrt{3}\iint_{\Sigma} d\sigma$$

$$= -\sqrt{3}\pi a^2. \quad (7\text{ 分})$$

17. 解：令$u=\sqrt{a}t$, 则$du=\sqrt{a}dt$. (2 分)

从而

$$\int_0^{+\infty} t^6 e^{-at^2}dt$$

$$= \int_0^{+\infty}\frac{u^6}{a^3}e^{-u^2}\frac{du}{\sqrt{a}}$$

$$= \frac{1}{2a^{\frac{7}{2}}}2\int_0^{+\infty}u^{2\cdot\frac{7}{2}-1}e^{-u^2}du$$

$$= \frac{\Gamma(\frac{7}{2})}{2a^{\frac{7}{2}}} \quad (5\text{ 分})$$

$$= \frac{15\sqrt{\pi}}{16a^{\frac{7}{2}}}. \quad (7\text{ 分})$$

五、证明题（每小题 6 分，共 24 分）

18. 证明：$\forall\epsilon>0$, 不妨取$\delta_1=1$且$(x-1)^2+(y-2)^2\le 1$, 则有$0\le x\le 2, 1\le y\le 3$.

因此当$(x-1)^2+(y-2)^2\le 1$时，

$$|x^2+y^2-5| = |x^2-1+y^2-4|$$
$$\le |x+1||x-1| + |y+2||y-2|$$
$$\le 5|x-1| + 5|y-2|$$
$$\le 5\sqrt{2}\sqrt{(x-1)^2+(y-2)^2}.$$

取$\delta=\min\{\frac{\epsilon}{5\sqrt{2}},\delta_1\}$. (4 分)

对于任意的(x,y)满足$0<\sqrt{(x-1)^2+(y-2)^2}<\delta$时，

$$|x^2+y^2-5|<\epsilon$$

成立. 因此$\lim_{(x,y)\to(1,2)}(x^2+y^2)=5$. (6 分)

19. 证明：(i). 当$x = 1$时，

$$\sum_{n=0}^{+\infty}(-1)^n x^n(1-x) = \sum_{n=0}^{+\infty} 0 = 0.$$

此时该级数在$x = 1$时绝对收敛.

当$0 \le x < 1$时，

$$\sum_{n=0}^{+\infty}|(-1)^n x^n(1-x)| = \sum_{n=0}^{+\infty} x^n(1-x) = (1-x)\sum_{n=0}^{+\infty} x^n = 1,$$

此时也绝对收敛.

因此该级数在[0,1]上绝对收敛. (2 分)

(ii). 令$a_n(x) = x^n(1-x), b_n(x) = (-1)^n$.

因为$a_n(x) \le n^n(1-x)\left(\frac{x}{n}\right)^n \le n^n\left(\frac{1}{n+1}\right)^n = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \le \frac{1}{n+1}$, 因此函数列$a_n(x)$在[0,1]上一致趋于零. 又因为对于固定的$x \in [0,1]$, $a_n(x)$单调. 对于任意的n,

$$\left|\sum_{k=0}^{n} b_k\right| \le 1.$$

由 Dirichlet 收敛定理可知，

$$\sum_{n=0}^{+\infty}(-1)^n x^n(1-x) = \sum_{n=0}^{+\infty} a_n(x) b_n(x)$$

在[0,1]上一致收敛. (4 分)

(iii). 因为当$n \to +\infty$时

$$S_n(x) = \sum_{k=0}^{n-1} x^n(1-x) \to S(x) = \begin{cases} 1, 0 \le x < 1 \\ 0, \quad\quad x = 1 \end{cases}.$$

由于和函数不连续，因此该级数在[0,1]上不一致收敛. (6 分)

20. 证明：由链式法则可知

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}, \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v},$$

所以

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{\partial^2 w}{\partial x \partial y}\left(\frac{\partial y}{\partial u}\frac{\partial x}{\partial u}\right) + \frac{\partial^2 w}{\partial y \partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial u^2},$$

$$\frac{\partial^2 w}{\partial v^2} = \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2 + \frac{\partial^2 w}{\partial x \partial y}\left(\frac{\partial y}{\partial v}\frac{\partial x}{\partial v}\right) + \frac{\partial^2 w}{\partial y \partial x}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial v^2} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial v^2}.$$

因此有

$$\begin{aligned}\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} &= \left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2\right] + 2\frac{\partial^2 w}{\partial x \partial y}\left[\frac{\partial y}{\partial u}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\frac{\partial x}{\partial v}\right] \\ &+ \left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2\right] + \frac{\partial w}{\partial x}\left[\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right] + \frac{\partial w}{\partial y}\left[\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2}\right].\end{aligned}$$

(3 分)

又因为

$$\frac{\partial x}{\partial u}=\frac{\partial y}{\partial v},\frac{\partial x}{\partial v}=-\frac{\partial y}{\partial u}.$$

因此有

$$\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2=\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]\left(\frac{\partial x}{\partial u}\right)^2=0,$$

$$\frac{\partial y}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial y}{\partial v}\frac{\partial x}{\partial v}=\frac{\partial y}{\partial u}\frac{\partial x}{\partial u}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial u}=0,$$

$$\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2=\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]\left(\frac{\partial x}{\partial v}\right)^2=0,$$

$$\frac{\partial^2 x}{\partial u^2}=\frac{\partial^2 y}{\partial v\partial u},\frac{\partial^2 x}{\partial v^2}=-\frac{\partial^2 y}{\partial u\partial v},\text{从而}\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}=0,$$

$$\frac{\partial^2 y}{\partial u^2}=-\frac{\partial^2 x}{\partial v\partial u},\frac{\partial^2 y}{\partial v^2}=\frac{\partial^2 x}{\partial u\partial v},\text{从而}\frac{\partial^2 y}{\partial u^2}+\frac{\partial^2 y}{\partial v^2}=0.$$

因此$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=0.$ (6 分)

21. 证明：反证法，设存在点(x_0,y_0)使得$u(x_0,y_0)$<0. 因为$u(x,y)$在$x^2+y^2\le 1$上连续，从而存在最小值$u(\xi,\eta)$，其中$\xi^2+\eta^2\le 1$. 从而有$u(\xi,\eta)\le u(x_0,y_0)<0$.

又因为在圆周$x^2+y^2=1$上有$u(x,y)\ge 0$，所以(ξ,η)在单位圆盘的内部，从而(ξ,η)为$u(x,y)$在单位圆盘内的一个极小值点. (3 分)

因此$u(x,y)$在(ξ,η)处的 Hessian 矩阵为非负定矩阵，即

$$\begin{pmatrix}\frac{\partial^2 u}{\partial x^2}(\xi,\eta) & \frac{\partial^2 u}{\partial x\partial y}(\xi,\eta)\\ \frac{\partial^2 u}{\partial x\partial y}(\xi,\eta) & \frac{\partial^2 u}{\partial y^2}(\xi,\eta)\end{pmatrix}\ge 0.$$

从而$u(\xi,\eta)=\frac{\partial^2 u}{\partial x^2}(\xi,\eta)+\frac{\partial^2 u}{\partial y^2}(\xi,\eta)\ge 0$，与$u(\xi,\eta)<0$矛盾． 因此假设不成立，即在$x^2+y^2\le 1$上$u(x,y)\ge 0$. (6 分)