3.2.6 隐函数存在定理及

隐函数的微分法

一、一个方程所确定的隐函数
及其导数

二、方程组所确定的隐函数组
及其导数

本节讨论：

1) 方程 $F(x, y) = 0$ 在什么条件下才能确定隐函数？

例如, 方程 $x^2 + \sqrt{y} + C = 0$

当 $C < 0$ 时, 能确定隐函数;

当 $C > 0$ 时, 不能确定隐函数;

2) 在方程能确定隐函数时, 研究其连续性、可微性

及求导方法问题 .

一、一个方程所确定的隐函数及其导数

定理1. 设函数 $F(x,y)$ 在点 $P(x_0,y_0)$ 的某一邻域内满足

① 具有连续的偏导数;

② $F(x_0\,,\,y_0)=0$;

③ $F_y(x_0\,,\,y_0)\neq 0$

则方程 $F(x,y)=0$ 在点 x_0 的某邻域内可唯一确定一个单值连续函数 $y=f(x)$，满足条件 $y_0=f(x_0)$，并有连续导数

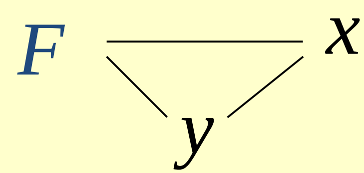
$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

(隐函数求导公式)

定理证明从略，仅就求导公式推导如下：

设 $y=f(x)$ 为方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,f(x))\equiv 0$$



$\Big\downarrow$ 两边对 x 求导

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\equiv 0$$

$\Big\downarrow$ 在 $(x_0\,,\,y_0)$ 的某邻域内 $F_y\neq 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

若$F(x, y)$的二阶偏导数也都连续，则还有

二阶导数：

$$\begin{tikzcd}
-\frac{F_x}{F_y} \arrow[rr, dash] \arrow[dr, dash] & & x \\
& y \arrow[ur, dash] &
\end{tikzcd}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{d}{d\,x}\left(-\frac{F_x}{F_y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right) + \frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= -\frac{F_{xx}F_y - F_{yx}F_x}{F_y^2} - \frac{F_{xy}F_y - F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$= -\frac{F_{xx}F_y{}^2 - 2F_{xy}F_xF_y + F_{yy}F_x{}^2}{F_y^3}$$

例1. 验证方程 $\sin y + e^x - xy - 1 = 0$ 在点(0,0)某邻域可确定一个单值可导隐函数 $y = f(x)$，并求

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}, \quad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}$$

解: 令 $F(x, y) = \sin y + e^x - xy - 1$，则

① $F_x = e^x - y,\ F_y = \cos y - x$ 连续，

② $F(0,0) = 0$，

③ $F_y(0,0) = 1 \neq 0$

由 定理1 可知，在 $x = 0$ 的某邻域内方程存在单值可导的隐函数 $y = f(x)$，且

$$\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x=0} = -\frac{F_x}{F_y}\bigg|_{x=0} = -\frac{e^x - y}{\cos y - x}\bigg|_{x=0,\, y=0} = -1$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\bigg|_{x=0}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x - y}{\cos y - x}\right)\bigg|_{x=0,\ y=0,\ y'=-1}$$

$$= -\frac{(e^x - y')(\cos y - x) - (e^x - y)(-\sin y \cdot y' - 1)}{(\cos y - x)^2}\Bigg|_{\substack{x=0\\ y=0\\ y'=-1}}$$

$$= -3$$

导数的另一求法 — 利用隐函数求导

$$\sin y + e^x - xy - 1 = 0, \quad y = y(x)$$

两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

$$y'\Big|_{x=0} = -\left.\frac{e^x - y}{\cos y - x}\right|_{(0,0)} = -1$$

两边再对 x 求导

$$-\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - x y'' = 0$$

令 $x = 0$, 注意此时 $y = 0$, $y' = -1$

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0} = -3$$

定理2. 若函数 $F(x,y,z)$ 满足:

① 在点 $P(x_0,y_0,z_0)$ 的某邻域内具有**连续偏导数**,

② $F(x_0,y_0,z_0)=0$

③ $F_z(x_0,y_0,z_0)\neq 0$

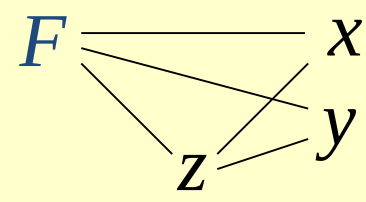
则方程 $F(x,y,z)=0$ 在点 (x_0,y_0) 某一邻域内可唯一确定一个单值连续函数 $z=f(x,y)$, 满足 $z_0=f(x_0,y_0)$, 并有连续偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

定理证明从略, 仅就求导公式推导如下:

设 $z=f(x,y)$ 是方程 $F(x,y,z)=0$ 所确定的隐函数，则

$$F(x,y,f(x,y))\equiv 0$$



$\Big\downarrow$ 两边对 x 求偏导

$$F_x+F_z\frac{\partial z}{\partial x}\equiv 0$$

$\Big\downarrow$ 在 (x_0,y_0,z_0) 的某邻域内 $F_z\neq 0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

同样可得 $$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

例2. 设 $x^2+y^2+z^2-4z=0$, 求 $\dfrac{\partial^2 z}{\partial x^2}$.

解法1 利用隐函数求导

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

解法2 利用公式

设 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $F_x=2x,\ F_z=2z-4$

$$\therefore \quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x}{z-2}=\frac{x}{2-z}$$

两边对 x 求偏导

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

例3. 设$F(x, y)$具有连续偏导数, 已知方程 $F(\frac{x}{z}, \frac{y}{z}) = 0$, 求 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$.

解: **利用偏导数公式.** 设 $z = f(x, y)$ 是由方程 $F(\frac{x}{z}, \frac{y}{z}) = 0$ 确定的隐函数, 则

$$\frac{\partial z}{\partial x} = -\frac{F_1' \cdot \frac{1}{z}}{F_1' \cdot (-\frac{x}{z^2}) + F_2' \cdot (-\frac{y}{z^2})} = \frac{z F_1'}{x F_1' + y F_2'}$$

$$\frac{\partial z}{\partial y} = -\frac{F_2' \cdot \frac{1}{z}}{F_1' \cdot (-\frac{x}{z^2}) + F_2' \cdot (-\frac{y}{z^2})} = \frac{z F_2'}{x F_1' + y F_2'}$$

解法2 **微分法.** 利用一阶微分形式不变性，求微分:

$$dF(\frac{x}{z},\frac{y}{z})=0 \qquad F_1'\cdot \mathrm{d}(\frac{x}{z})+F_2'\cdot \mathrm{d}(\frac{y}{z})=0$$

$$F_1'\cdot(\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2})+F_2'\cdot(\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2})=0$$

$$\frac{xF_1'+yF_2'}{z^2}\mathrm{d}z=\frac{F_1'\mathrm{d}x+F_2'\mathrm{d}y}{z}$$

$$\mathrm{d}z=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

例4. 设 $z = f(x+y+z, xyz)$，求 $\frac{\partial z}{\partial x}, \frac{\partial x}{\partial z}, \frac{\partial x}{\partial y}$.

解：利用全微分形式不变性同时求出各偏导数.

$$dz = df(x+y+z, xyz)$$

$$\mathrm{d}z = f_1' \cdot (\mathrm{d}x + \mathrm{d}y + \mathrm{d}z) + f_2' \cdot (yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z)$$

解出 $\mathrm{d}x$:

$$\mathrm{d}x = \frac{-(f_1' + xz f_2')\mathrm{d}y + (1 - f_1' - xy f_2')\mathrm{d}z}{f_1' + yz f_2'}$$

由 $\mathrm{d}y, \mathrm{d}z$ 的系数即可得

$$\frac{\partial x}{\partial y} = -\frac{f_1' + xz f_2'}{f_1' + yz f_2'} \qquad \frac{\partial x}{\partial z} = \frac{1 - f_1' - xy f_2'}{f_1' + yz f_2'}$$

二、方程组所确定的隐函数组及其导数

隐函数存在定理还可以推广到方程组的情形.

以两个方程确定两个隐函数的情况为例，即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

由 F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F、G 的**雅可比(Jacobi)**行列式.

定理3. 设函数 $F(x,y,u,v), G(x,y,u,v)$ 满足:

① 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0, y_0,u_0,v_0)=0,\ G(x_0, y_0,u_0,v_0)=0$;

③ $J\Big|_P=\left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_P\neq 0$

则方程组 $F(x,y,u,v)=0,\ G(x,y,u,v)=0$ 在点(x_0,y_0)的某一邻域内可**唯一**确定一组满足条件 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$的**单值连续函数** $u=u(x,y), v=v(x,y)$, 且有偏导数公式 :

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{x},v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{y},v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{x})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}$$

定理证明略.仅推导偏导数公式如下:

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{y})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}$$

设方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 有隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$，则

$$\begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0 \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$$

两边对 x 求导得 $\begin{cases} F_x+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$

这是关于 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ 的线性方程组，在点P 的某邻域内

系数行列式 $J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0$，故得

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}$$

同样可得

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$

例4. 设 $xu - yv = 0,\ yu + xv = 1$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解: 方程组两边对 x 求导，并移项得

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}$$

由题设 $J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$

故有
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{1}{J}\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix} = -\dfrac{xu + yv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial x} = \dfrac{1}{J}\begin{vmatrix} x & -u \\ y & -v \end{vmatrix} = -\dfrac{xv - yu}{x^2 + y^2} \end{cases}$$

练习: 求 $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$

答案:

$$\begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{yu - xv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial y} = -\dfrac{xu + yv}{x^2 + y^2} \end{cases}$$

例5. 设函数 $x = x(u,v),\ y = y(u,v)$ 在点(u,v) 的某一邻域内有连续的偏导数, 且 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$

1) 证明函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 (u, v) 对应的点 (x, y) 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u = u(x,y),\ v = v(x, y)$.

2) 求 $u = u(x,y),\ v = v(x, y)$ 对 x , y 的偏导数.

解: 1) 令 $F(x,y,u,v) \equiv x - x(u,v) = 0$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

则有 $$J=\frac{\partial(F,G)}{\partial(u,v)}=\frac{\partial(x,y)}{\partial(u,v)}\neq 0,$$

由定理 3 可知结论 1) 成立.

2) 求反函数的偏导数.

$$\begin{cases} x \equiv x(u(x,y),v(x,y)) \\ y \equiv y(u(x,y),v(x,y)) \end{cases} \qquad ①$$

①式两边对 x 求导, 得

$$\begin{cases} 1=\dfrac{\partial x}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial x}{\partial v}\cdot\dfrac{\partial v}{\partial x} \\ 0=\dfrac{\partial y}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial y}{\partial v}\cdot\dfrac{\partial v}{\partial x} \end{cases} \qquad ②$$

注意$J\neq 0$, 从方程组②解得

$$\frac{\partial u}{\partial x}=\frac{1}{J}\begin{vmatrix}1 & \frac{\partial x}{\partial v}\\ 0 & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v},\quad \frac{\partial v}{\partial x}=\frac{1}{J}\begin{vmatrix}\frac{\partial x}{\partial u} & 1\\ \frac{\partial y}{\partial u} & 0\end{vmatrix}=-\frac{1}{J}\frac{\partial y}{\partial u}$$

同理, ①式两边对y求导, 可得

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial x}{\partial v},\qquad \frac{\partial v}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

注意$J \neq 0$，从方程组②解得

$$\frac{\partial u}{\partial x} = \frac{1}{J}\begin{vmatrix} 1 & \frac{\partial x}{\partial v} \\ 0 & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{J}\frac{\partial y}{\partial v},$$

$$\frac{\partial v}{\partial x} = \frac{1}{J}\begin{vmatrix} \frac{\partial x}{\partial u} & 1 \\ \frac{\partial y}{\partial u} & 0 \end{vmatrix} = -\frac{1}{J}\frac{\partial y}{\partial u}$$

同理, ①式两边对 y 求导, 可得

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v}, \qquad \frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u}$$

例5的应用: 计算极坐标变换 $x = r\cos\theta\,,\ y = r\sin\theta$ 的反变换的导数.

$$\boxed{\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{J}\frac{\partial y}{\partial \theta}\\ \frac{\partial \theta}{\partial x} &= -\frac{1}{J}\frac{\partial y}{\partial r}\end{aligned}}$$

由于 $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

所以 $\dfrac{\partial r}{\partial x} = \dfrac{1}{J}\dfrac{\partial y}{\partial \theta} = \dfrac{1}{r} r\cos\theta = \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r} = -\frac{1}{r}\sin\theta = -\frac{y}{x^2+y^2}$$

同样有 $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$ $\qquad \dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$

内容小结

1. 隐函数(组) 存在定理

2. 隐函数 (组) 求导方法

方法1. 利用复合函数求导法则直接计算

方法2. 代公式

方法3. 利用微分形式不变性

思考与练习

设 $z=f(x+y+z,xyz)$,求 $\dfrac{\partial z}{\partial x},\ \dfrac{\partial x}{\partial z},\ \dfrac{\partial x}{\partial y}$.

提示: $z = f(x+y+z, xyz)$

- $$\frac{\partial z}{\partial x} = f_1' \cdot (1 + \frac{\partial z}{\partial x}) + f_2' \cdot (yz + xy\frac{\partial z}{\partial x})$$

$$\Longrightarrow \frac{\partial z}{\partial x} = \frac{f_1' + yz f_2'}{1 - f_1' - xy f_2'}$$

- $$1 = f_1' \cdot (\frac{\partial x}{\partial z} + 1) + f_2' \cdot (yz\frac{\partial x}{\partial z} + xy)$$

$$\Longrightarrow \frac{\partial x}{\partial z} = \frac{1 - f_1' - xy f_2'}{f_1' + yz f_2'}$$

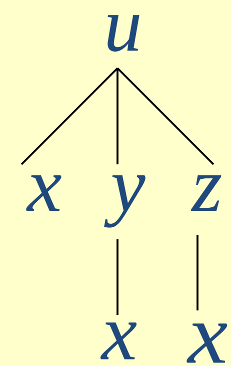
- $$0 = f_1' \cdot (\frac{\partial x}{\partial y} + 1) + f_2' \cdot (yz\frac{\partial x}{\partial y} + xz)$$

$$\Longrightarrow \frac{\partial x}{\partial y} = -\frac{f_1' + xz f_2'}{f_1' + yz f_2'}$$

练习题 **1.** 设 $u=f(x,y,z)$ 有连续的一阶偏导数，又函数 $y=y(x)$ 及 $z=z(x)$ 分别由下列两式确定：

$$\underline{e^{xy}-xy=2}\,,\ \underline{e^{x}=\int_0^{x-z}\frac{\sin t}{t}\,\mathrm{d}t},\ 求\ \frac{\mathrm{d}u}{\mathrm{d}x}.$$ **(2001考研)**

解: 两个隐函数方程两边对 x 求导，得



$$\begin{cases} e^{xy}(y+xy')-(y+xy')=0 \\ e^{x}=\dfrac{\sin(x-z)}{x-z}(1-z') \end{cases}$$

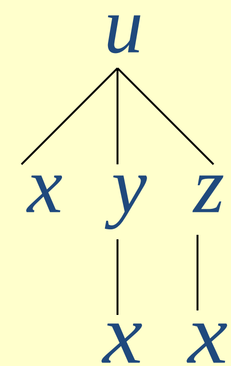
解得 $\quad y'=-\dfrac{y}{x},\quad z'=1-\dfrac{e^{x}(x-z)}{\sin(x-z)}$

因此 $\quad \dfrac{\mathrm{d}u}{\mathrm{d}x}=f_1'-\dfrac{y}{x}f_2'+\left[1-\dfrac{e^{x}(x-z)}{\sin(x-z)}\right]f_3'$

2. 设 $u=f(x,y,z)$ 有连续的一阶偏导数，又函数 $y=y(x)$ 及 $z=z(x)$ 分别由下列两式确定：

$$e^{xy}-xy=2\,,\quad e^{x}=\int_0^{x-z}\frac{\sin t}{t}\,\mathrm{d}t,\ \text{求}\ \frac{\mathrm{d}u}{\mathrm{d}x}.$$

解: 两个隐函数方程两边对 x 求导，得



$$\begin{cases} e^{xy}(y+xy')-(y+xy')=0 \\ e^{x}=\dfrac{\sin(x-z)}{x-z}(1-z') \end{cases}$$

解得 $\quad y'=-\dfrac{y}{x},\quad z'=1-\dfrac{e^{x}(x-z)}{\sin(x-z)}$

因此 $\quad \dfrac{\mathrm{d}u}{\mathrm{d}x}=f_1'-\dfrac{y}{x}f_2'+\left[1-\dfrac{e^{x}(x-z)}{\sin(x-z)}\right]f_3'$

3. 设 $y=y(x), z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解法1 分别在各方程两端对 x 求导, 得

$$\begin{cases} z' = f + x\cdot f'\cdot(1+y') \\ F_x + F_y\cdot y' + F_z\cdot z' = 0 \end{cases} \Longrightarrow \begin{cases} -xf'\cdot y' + z' = f + xf' \\ F_y\cdot y' + F_z\cdot z' = -F_x \end{cases}$$

$$\therefore\quad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\begin{vmatrix} -xf' & f+xf' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -xf' & 1 \\ F_y & F_z \end{vmatrix}} = \frac{(f+xf')F_y - xf'\cdot F_x}{F_y + xf'\cdot F_z}$$

$$(F_y + xf'\cdot F_z \neq 0)$$

解法2 微分法.

$$z = xf(x+y),\ \ F(x,y,z)=0$$

对各方程两边分别求微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + xf'\cdot(\mathrm{d}x+\mathrm{d}y) \\ F_1'\mathrm{d}x + F_2'\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

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$$\begin{cases} (f+xf')\,\mathrm{d}x + x\,f'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1'\,\mathrm{d}x + F_2'\,\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

消去$\mathrm{d}\,y$可得 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

4. 设 $z=f(u)$，方程 $u=\varphi(u)+\int_y^x p(t)\,dt$ 确定 u 是 x,y 的函数，其中 $f(u),\varphi(u)$ 可微，$p(t),\varphi'(u)$ 连续，且 $\varphi'(u)\neq 1$，求 $p(y)\dfrac{\partial z}{\partial x}+p(x)\dfrac{\partial z}{\partial y}$.

解: $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$，$\dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\left\{\begin{aligned}\frac{\partial u}{\partial x}&=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}&=\frac{-p(y)}{1-\varphi'(u)}\end{aligned}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\right]=0$$

思考题

已知$\dfrac{x}{z}=\varphi(\dfrac{y}{z})$，其中$\varphi$为可微函数，

求$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=?$

思考题解答

记$F(x,y,z)=\dfrac{x}{z}-\varphi(\dfrac{y}{z})$，　则$F_x=\dfrac{1}{z}$，

$$F_y=-\varphi'(\frac{y}{z})\cdot\frac{1}{z},\quad F_z=\frac{-x}{z^2}-\varphi'(\frac{y}{z})\cdot\frac{(-y)}{z^2},$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{z}{x-y\varphi'(\frac{y}{z})},\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{-z\varphi'(\frac{y}{z})}{x-y\varphi'(\frac{y}{z})},$$

于是$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z.$

练 习 题

一、填空题:

1、设$\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$, 则

$\frac{dy}{dx}=$________________________________.

2、设$z^x=y^z$, 则

$\frac{\partial z}{\partial x}=$________________________________,

$\frac{\partial z}{\partial y}=$________________________________.

二、设$2\sin(x+2y-3z)=x+2y-3z$,

证明: $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$.

三、如果函数 $f(x,y,z)$ 对任何 t 恒满足关系式 $f(tx,ty,tz)=t^k f(x,y,z)$, 则称函数 $f(x,y,z)$为 k次齐次函数, 试证: k次齐次函数满足方程

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z).$$

四、设$z^3-3xyz=a^3$,求$\dfrac{\partial^2 z}{\partial x\partial y}$.

五、求由下列方程组所确定的函数的导数或偏导数:

1、 设$\begin{cases} z=x^2+y^2 \\ x^2+2y^2+3z^2=20 \end{cases}$, 求$\dfrac{dy}{dx},\dfrac{dz}{dx}$.

2、 设$\begin{cases} u=f(ux,v+y) \\ v=g(u-x,v^2y) \end{cases}$, 求$\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$.

（其中 f,g具有一阶连续偏导数）

六、设函数$u(x)$由方程组$\begin{cases} u = f(x,y) \\ g(x,y,z) = 0 \\ h(x,z) = 0 \end{cases}$所确定,

且$\dfrac{\partial g}{\partial y} \neq 0, \dfrac{\partial h}{\partial z} \neq 0$,求$\dfrac{du}{dx}$.（$f,g,h$均可微）

七、设$y = f(x,t)$,而t是由方程$F(x,y,t) = 0$所确定的x,y的函数,求$\dfrac{dy}{dx}$.

八、设$z = z(x,y)$由方程$F(x+\dfrac{x}{y}, y+\dfrac{z}{x})$=0 所确定,

证明:$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z - xy$.

练习题答案

一、1、$\dfrac{x+y}{x-y}$；　　　2、$\dfrac{-z^{x}\ln z}{xz^{x-1}-y^{z}\ln y}$；

3、$\dfrac{zy^{z-1}}{xz^{x-1}-y^{z}\ln y}$.

四、$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{z(z^4-2xyz^2-x^2y^2)}{(z^2-xy)^3}$.

五、1、$\dfrac{dy}{dx}=\dfrac{-x(6z+1)}{2y(3z+1)},\dfrac{dz}{dx}=\dfrac{x}{3z+1}$；

2、$\dfrac{\partial u}{\partial x}=\dfrac{-uf_1'(2yvg_2'-1)-f_2'\cdot g_1'}{(xf_1'-1)(2yvg_2'-1)-f_2'\cdot g_1'}$,

$\dfrac{\partial v}{\partial x}=\dfrac{g_1'(xf_1'+uf_1'-1)}{(xf_1'-1)(2yvg_2'-1)-f_2'\cdot g_1'}$.

六、$$\frac{du}{dx}=f'_x-\frac{f'_x\cdot g'_x}{g'_y}+\frac{f'_y\cdot g'_z\cdot h'_x}{g'_y\cdot h'_z}$$

$$=\frac{f'_xg'_yh'_z-f'_xg'_xh'_z+f'_yg'_zh'_x}{g'_yh'_z}.$$

七、$$\frac{dy}{dx}=\frac{F'_t\cdot f'_x-F'_x\cdot f'_t}{F'_t+F'_y\cdot f'_t}.$$