§ 7.11 Maxwell's Equations (chapter 7)

❖1. the Maxwell's equations

Four Maxwell's equations in the point (differential) and integral forms are

$$(1)\quad \nabla\times\mathbf{H}=\mathbf{J}_v+\frac{\partial\vec{\mathbf{D}}}{\partial t} \longrightarrow \oint_l \vec{H}\bullet d\vec{l}=\int_s\left(\vec{J}_v+\frac{\partial\vec{D}}{\partial t}\right)\bullet d\vec{\mathbf{s}}$$

$$(2)\quad \nabla\times\vec{\mathbf{E}}=-\frac{\partial\vec{\mathbf{B}}}{\partial t} \longrightarrow \oint_l \vec{\boldsymbol{E}}\bullet d\vec{l}=-\frac{\partial}{\partial t}\int_s \vec{\boldsymbol{B}}\bullet d\vec{s}$$

$$(3)\quad \nabla\bullet\vec{\mathbf{D}}=\rho_v \longrightarrow \oint_s \vec{\boldsymbol{D}}\bullet d\vec{\mathbf{s}}=\int_v \rho_v dv$$

$$(4)\quad \nabla\bullet\vec{\mathbf{B}}=0 \longrightarrow \oint_s \vec{\mathbf{B}}\bullet d\vec{\mathbf{s}}=0$$

$$(5)\quad \nabla\bullet\vec{\mathbf{J}}_v=-\frac{\partial\rho_v}{\partial t} \Leftrightarrow \oint_s \vec{\mathbf{J}}_v\bullet d\vec{s}=-\int_v\frac{\partial\rho_v}{\partial t}dv=-\frac{\partial}{\partial t}\int_v\rho_v dv$$

•Equation(1) states that

a time-varying magnetic field can be produced not only by conduction current but also by displacement current

(represents a time-varying electric field)

$$(1)\quad \nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}_v + \frac{\partial \vec{\boldsymbol{D}}}{\partial t}$$

Equation (2) states that
a time-varying magnetic field
produces a time-varying electric field
(a rotational field)

$$(2) \quad \nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Equation (1) and (2) can state that

- A time-varying magnetic field produces a time-varying electric field, which in turn produces a time-varying magnetic field. (2) $\nabla \times \vec{\mathbf{E}} = -\dfrac{\partial \vec{\mathbf{B}}}{\partial t}$
- That is, the energy from the magnetic field can be transferred to the electric field, which then transfers it back to the magnetic field. (1) $\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}_v + \dfrac{\partial \vec{\boldsymbol{D}}}{\partial t}$

The knowledge enabled Maxwell to predict the propagation of electromagnetic energy in any medium.

Equation(3) states that

ρ_v is the source of the flux of the electric field.

Equation(4) states that

the source of the flux of the magnetic field does not exist.

The magnetic flux lines are always continuous.

- 2. the equation of continuity of current

$$(5)\quad \nabla \bullet \vec{\mathbf{J}}_v = -\frac{\partial \rho_v}{\partial t} \Leftrightarrow \oint_s \vec{\mathbf{J}}_v \bullet d\vec{s} = -\int_v \frac{\partial \rho_v}{\partial t} dv = -\frac{\partial}{\partial t}\int_v \rho_v dv$$

- **3.relationships among these equations**
  - **(1) equation (4) and (5) can be derived from equation (1), (2) and (3)**

- **taking the divergence of both besides of equation(2), we have**

$$\nabla \bullet (\nabla \times \vec{\mathbf{E}}) = -\frac{\partial}{\partial t}\nabla \bullet \vec{\mathrm{B}} = 0$$

then

$$\nabla \bullet \vec{\mathbf{B}} = \text{constant}$$

When the constant is zero, equation (4) is proved.

- taking the divergence of both besides of equation(1), we have

$$\nabla \bullet (\nabla \times \vec{\mathbf{H}}) = \nabla \bullet \vec{\mathbf{J}} + \frac{\partial}{\partial t} \nabla \bullet \vec{\mathbf{D}} = 0$$

and from equation(3) $\nabla \bullet \vec{D} = \rho_v$

the above equation can be rewritten as

$$\nabla \bullet \vec{\mathbf{J}} + \frac{\partial}{\partial t} \rho_v = 0 \text{ or } \nabla \bullet \vec{\mathbf{J}} = -\frac{\partial \rho_v}{\partial t}$$

equation(5) is proved.

⇒equation(1), (2) and (3) are independent of each other, equation(4) and (5) can be derived from them, and dependent of them.

❖(2) equation(3) and (4) can be derived from equation(1), (2) and (5)

it states that equation(1), (2) and (5) are independent of each other, equation(3) and (4) are dependent of them. Exercise

❖4.solutions of these equations

Variables: $\vec{E}, \vec{D}, \vec{B}, \vec{H}, \vec{J}$ and ρ_v

There are sixteen scalar variables .

those independent equations: (1), (2) and (5)

or (1), (2) and (3)

represent seven scalar equations

⇒thus solutions of these sixteen scalar quantities can not be defined.

❖**5.the constitutive equations** **page 311**

the constitutive equations, which can define the relationships between the field quantities in a linear, homogeneous and isotropic medium are

$$(6)\quad \vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}}$$

$$(7)\quad \vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$$

$$(8)\quad \vec{B} = \mu\vec{H}$$

the constitutive equations represent nine scalar equations.

Thus, equation(1), (2), (5) and these three constitutive equations together represent sixteen scalar equations which can define sixteen scalar quantities.

Example:

Show that the volume charge density ρ_v in an ideal conductor is zero.

Solution:

Since equation(5) and (7) are

$$(5) \quad \nabla \bullet \vec{\mathbf{J}}_v = -\frac{\partial \rho_v}{\partial t} \quad \textbf{and} \quad (7) \quad \vec{\boldsymbol{J}}_v = \sigma \vec{\mathbf{E}}$$

we have

$$\nabla \bullet \sigma \vec{\mathbf{E}} = -\frac{\partial \rho_v}{\partial t} \qquad \textbf{(7.3.1)}$$

From equation(3) $\nabla \bullet \vec{D} = \rho_v$ and (6) $\vec{D} = \varepsilon\vec{E}$

we have $\nabla \bullet \vec{\mathbf{E}} = \dfrac{\rho_v}{\varepsilon}$ (7.3.2)

from equation(7.3.1) and (7.3.2), we can obtain

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

the solution of the equation is

$$\rho_v = \rho_0 e^{-(\sigma/\varepsilon)t} \qquad \textbf{(7.3.3)}$$

since the ideal conductor $\sigma \to \infty$, from (7.3.3) we can obtain

$$\rho_v \to 0 \text{ when } \sigma \to \infty$$

§ 7.12 Poynting's theorem (page314)

the electromagnetic field is an entity, it has its energy which is distributed in a certain. The energy can be transferred from one place to another. This case can be specified by Poynting's theorem.

❖1. Poynting's theorem

in terms of equation

$$\nabla \bullet (\vec{\mathbf{E}} \times \vec{\mathbf{H}}) = \vec{\mathbf{H}} \bullet (\nabla \times \vec{\mathbf{E}}) - \vec{\mathbf{E}} \bullet (\nabla \times \vec{\mathbf{H}})$$

and Maxwell's equations, we have

$$
\begin{aligned}
\nabla \bullet (\vec{\mathbf{E}} \times \vec{\mathbf{H}}) &= \vec{\mathbf{H}} \bullet (\nabla \times \vec{\mathbf{E}}) - \vec{\mathbf{E}} \bullet (\nabla \times \vec{\mathbf{H}}) \\
&= \vec{\boldsymbol{H}} \bullet \left(-\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \right) - \vec{\boldsymbol{E}} \bullet \left(\vec{\boldsymbol{J}}_v + \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \right) \\
&= \vec{\boldsymbol{H}} \bullet \left(-\frac{\partial (\mu \vec{\boldsymbol{H}})}{\partial t} \right) - \vec{\boldsymbol{E}} \bullet \left(\vec{\boldsymbol{J}}_v + \frac{\partial (\varepsilon \vec{\boldsymbol{E}})}{\partial t} \right) \\
&= -\frac{\partial}{\partial t} \left(\frac{1}{2} \mu H^2 \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon E^2 \right) - \vec{\boldsymbol{J}}_v \bullet \vec{\boldsymbol{E}}
\end{aligned}
$$

we can rewrite the above differential equation in integral form as

$$-\oint_{s}(\vec{\mathbf{E}}\times\vec{\mathbf{H}})\bullet d\vec{\mathbf{s}}=\frac{d}{dt}\int_{v}\left(\frac{1}{2}\mu H^{2}\right)dv+\frac{d}{dt}\int_{v}\left(\frac{1}{2}\varepsilon E^{2}\right)dv+\int_{v}\vec{\boldsymbol{J}}_{v}\bullet\vec{\boldsymbol{E}}dv \quad (7.101)$$

where the volume *v* is bounded by a surface *s*.

- **(1)the first term on right-hand side of equation (7.101) represents the rate of change of stored magnetic energy.**

When this integral is positive, an external source is supplying energy to the magnetic field, resulting in an increase in the magnetic field.

$$-\oint_s (\vec{\mathbf{E}} \times \vec{\mathbf{H}}) \bullet d\vec{\mathbf{s}} = \frac{d}{dt}\int_v \left(\frac{1}{2}\mu H^2\right) dv + \frac{d}{dt}\int_v \left(\frac{1}{2}\varepsilon E^2\right) dv + \int_v \vec{J}_v \bullet \vec{E} dv \tag{7.101}$$

(2)The second term on the right-hand side of equation(7.101) represents the rate of change of stored energy in the electric field. When this integral is positive, an external source is supplying energy to the electric field, resulting in an increase in the electric field.

(3)The final term represents the power supplied to those charged particles by the field. When this integral is positive, the field is doing work on those charged particles.

$$-\oint_s (\vec{\mathbf{E}}\times\vec{\mathbf{H}})\bullet d\vec{\mathbf{s}} = \frac{d}{dt}\int_v\left(\frac{1}{2}\mu H^2\right)dv + \frac{d}{dt}\int_v\left(\frac{1}{2}\varepsilon E^2\right)dv + \int_v \vec{J}_v \bullet \vec{E}dv$$

$$\int_v \vec{\mathbf{J}} \bullet \vec{\mathbf{E}}dv = \int_v \rho_v \vec{\mathbf{\upsilon}} \bullet \vec{\mathbf{E}}dv = q\vec{\mathbf{\upsilon}} \bullet \vec{\mathbf{E}} \qquad \textbf{(7.101)}$$

$\dfrac{q\vec{E}\bullet\Delta\vec{l}}{\Delta t} = \dfrac{\vec{\mathbf{F}}\bullet\Delta\vec{l}}{\Delta t}$ **is work per unit time.**

- **the power supplied to ohmic power loss.**

In a conductive medium $\vec{J}_v = \sigma\vec{E}$**, this term represents power dissipation or ohmic power loss.**

$$\int_v \vec{\mathbf{J}} \bullet \vec{\mathbf{E}}dv = \int_v \sigma\vec{\mathbf{E}} \bullet \vec{\mathbf{E}}dv = \int_v \sigma E^2 dv$$

- **(4) The left-hand side of equation(7.101) represents**

$$-\oint_s (\vec{\mathbf{E}} \times \vec{\mathbf{H}}) \bullet d\vec{\mathbf{s}} = \frac{d}{dt}\int_v \left(\frac{1}{2}\mu H^2\right) dv + \frac{d}{dt}\int_v \left(\frac{1}{2}\varepsilon E^2\right) dv + \int_v \vec{J}_v \bullet \vec{E} dv$$

(7.101)

When the integral is positive, the term with the negative sign represents the net inward flux of the vector field. According to the law of conservation of energy, the inward flux must be the power which flowing into volume *v* bounded by surface *s*.

2. the Poynting's vector

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}}$$

represent the power density, watts per unit square meter(W/m ^{2}). It is called the Poynting's vector.

Its direction is normal to the plane containing $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$, it represents the direction of power flow.

Thus equation(2.4.1) can be rewritten in integral form as

$$-\oint_s \vec{\mathbf{S}} \bullet d\vec{\mathbf{s}} = \frac{d}{dt}\int_v \left(\frac{1}{2}\mu H^2\right)dv + \frac{d}{dt}\int_v \left(\frac{1}{2}\varepsilon E^2\right)dv + \int_v \vec{\boldsymbol{J}}_v \bullet \vec{\boldsymbol{E}}dv \quad \textbf{(7-2 }\boldsymbol{a}\textbf{)}$$

or it can also be rewritten in differential (point) form as

$$-\nabla \bullet \vec{\mathbf{S}} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) + \vec{\boldsymbol{J}}_v \bullet \vec{\boldsymbol{E}} \qquad (7\text{-}2b)$$

the negative sign on the left-hand side of the above equation indicates that the net power must flow into volume *v* (or a point, sink) in order to account for the power dissipation in the region(point) as heat and the increase in the energy stored in both the electric and magnetic fields.

* for static fields, (2.4.2*a*) becomes

$$-\oint_s \vec{\mathbf{S}} \bullet d\vec{\mathbf{s}} = \int_v \vec{J}_v \bullet \vec{E} dv$$

which simply states that the net power flowing through surface *s* into volume *v* is equal to the power dissipation in that volume.

Example:

A coaxial cable with two concentric, cylindrical conductors of radii *a* and *b*, carries a current *I* . *U* is the potential difference between the two conductors. The conductivity is σ, a material with its permittivity being ε fills the space between the two conductors. ❑Find

(1) Poynting vector $\vec{S}$ and the transmitted power *P* by the coaxial cable when $\sigma=\infty$

(2)the power dissipation within the inner conductor if the current *I* is uniformly distributed within the inner conductor when σ is finite.

- Solution:

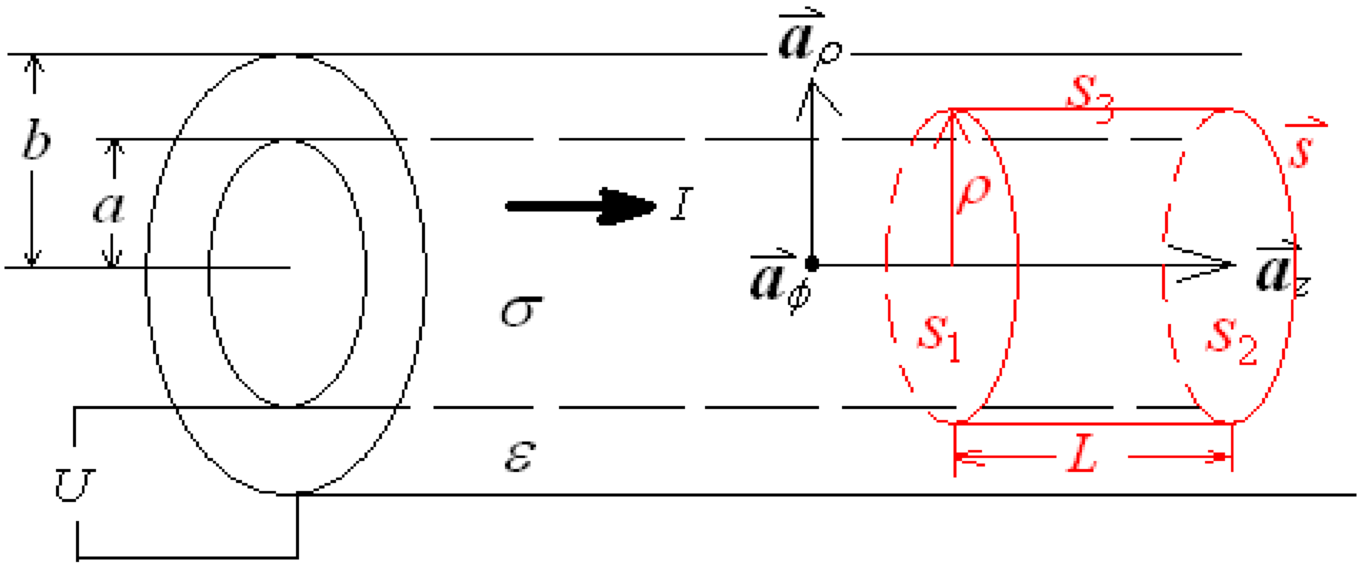
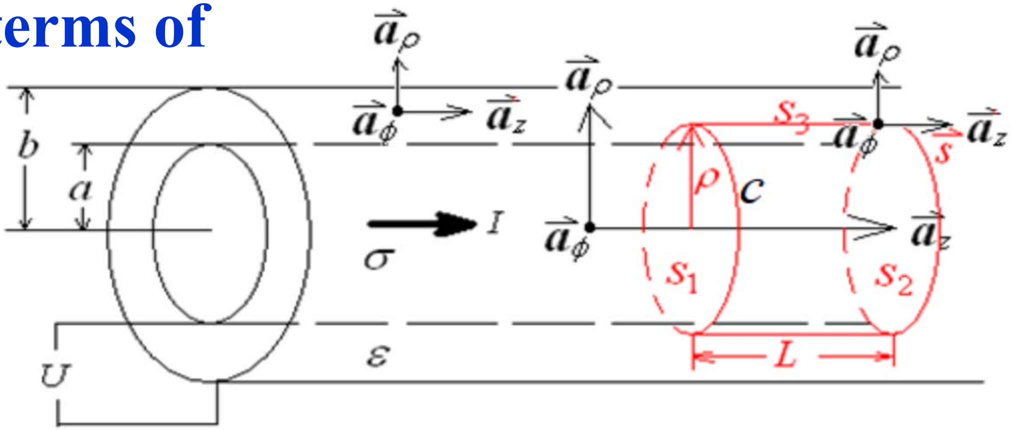


fig.1 a coaxial cable

(1)

let a closed surface *s* close the inner conductor, as shown in fig.1 where $a \le \rho \le b$, the closed surface s includes section s1, section s2 and cylinder s3. In terms of

$$\oint_s \vec{D} \bullet d\vec{s} = q = \int_v \rho_v dv$$



where q is closed by the surface s, we have

$$\oint_s \varepsilon \vec{\mathbf{E}} \bullet d\vec{\mathbf{s}} = q, \quad \oint_s \vec{\mathbf{E}} \bullet d\vec{\mathbf{s}} = q / \varepsilon$$

$$\int_{s1} \vec{\mathbf{E}} \bullet d\vec{\mathbf{s}} + \int_{s2} \vec{\mathbf{E}} \bullet d\vec{\mathbf{s}} + \int_{s3} \vec{\mathbf{E}} \bullet d\vec{\mathbf{s}} = q / \varepsilon$$

since the direction of $\vec{\mathbf{E}}$ is normal to the directions of the vector surface s1 and s2, we can obtain

$$\int_{s3} \vec{\mathbf{E}} \bullet d\vec{\mathbf{s}} = q / \varepsilon$$

the magnitude of $\vec{\mathbf{E}}$ is the same at any point on the cylinder s3, the direction of $\vec{\mathbf{E}}$ is in accord with the direction of $d\vec{s}$, thus the above equation can be rewritten as

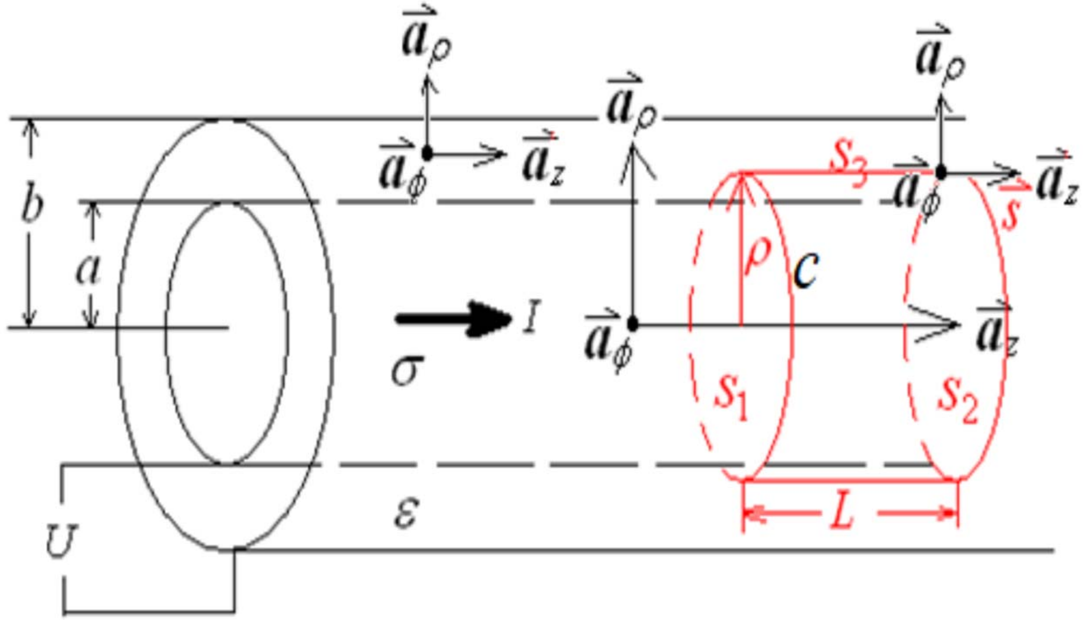
$$\int_{s3} E\vec{\mathbf{a}}_{\boldsymbol{\rho}} \bullet \vec{\mathbf{a}}_{\boldsymbol{\rho}} ds = q/\varepsilon, \qquad \int_{s3} Eds = q/\varepsilon$$

$$2\pi\rho LE = \frac{q}{\varepsilon}$$

$$E = \frac{q}{2\pi\rho L\varepsilon}$$

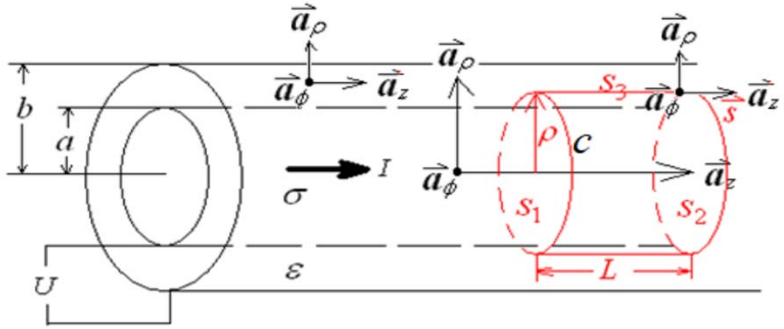
or

$$\vec{\mathbf{E}} = \frac{q}{2\pi\rho L\varepsilon}\vec{\mathbf{a}}_{\boldsymbol{\rho}}$$



since the potential difference *U* can be given by

$$U = \int_c \vec{E} \bullet d\vec{l}$$



we have

$$E = \frac{q}{2\pi\rho L\varepsilon}$$

$$U = \int_a^b \vec{\mathbf{E}} \bullet \vec{\mathbf{a}}_\rho d\rho = \int_a^b E\vec{\mathbf{a}}_\rho \bullet \vec{\mathbf{a}}_\rho d\rho = \int_a^b E d\rho = \int_a^b \frac{q}{2\pi\rho L\varepsilon} d\rho$$

$$= \frac{q}{2\pi\varepsilon L} \ln\frac{b}{a}$$

Hence,

$$\vec{\mathbf{E}} = \frac{U}{\rho \ln\frac{b}{a}} \vec{\mathbf{a}}_\rho$$

where $a \le \rho \le b$ **from Ampere's law for static field**

$$\oint_c \vec{H} \bullet d\vec{l} = I$$

where I is the net current intercepted by the area enclosed by the path c.

if the area is section s1, the path *c* is a circle of its radius *ρ*. Therefore,

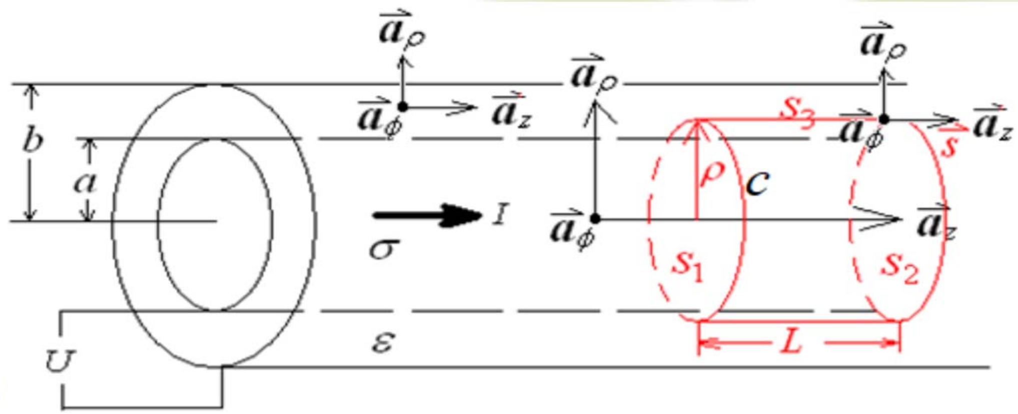
$$\oint_c H\vec{\mathbf{a}}_l \bullet \vec{\mathbf{a}}_l dl = I, \quad \oint_c Hdl = I, \qquad 2\pi\rho H = I$$

$$\therefore \qquad \vec{H} = \frac{I}{2\pi\rho}\vec{\mathbf{a}}_\varphi \qquad\qquad \vec{\mathbf{E}} = \frac{U}{\rho \ln \frac{b}{a}}\vec{\mathbf{a}}_\rho$$

from the definition of Poynting's vector, we can obtain

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}} = \frac{U}{\rho \ln \frac{b}{a}}\vec{\mathbf{a}}_\rho \times \frac{I}{2\pi\rho}\vec{\mathbf{a}}_\varphi$$

$$= \vec{\mathbf{a}}_z \frac{UI}{2\pi\rho^2 \ln \frac{b}{a}}$$



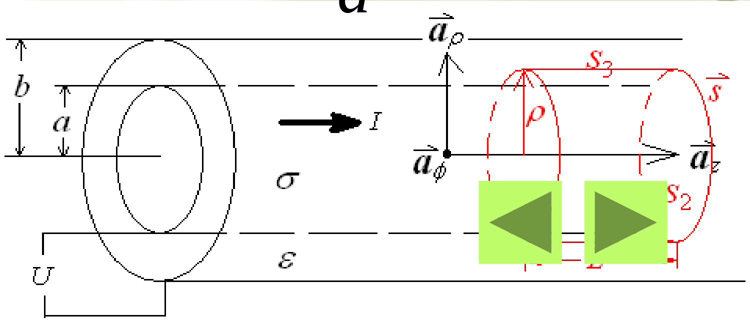
obviously, the Poynting's vector field exists in the medium with permittivity ε. Thus, we can obtain the transmitted power *P* by the coaxial cable. that is,

$$P = \int_S \vec{\mathbf{S}} \bullet d\vec{\mathbf{s}}$$

the integral is over the surface which is a annular disc of inner radius *a* and outer radius *b*. the power is the flux of Poynting's vector passing through the disc. That is,

$$P = \int_S \vec{\mathbf{S}} \bullet d\vec{\mathbf{s}} = \int_s \frac{UI}{2\pi\rho^2 \ln\frac{b}{a}} \vec{\mathbf{a}}_z \bullet \vec{\mathbf{a}}_z ds = \int_a^b \frac{UI}{2\pi\rho^2 \ln\frac{b}{a}} 2\pi\rho d\rho$$

$$= UI$$



(2) since I is the uniform current distribution within the inner conductor, the uniform volume current density is

$$\vec{\mathbf{J}}_v = \frac{I}{\pi a^2}\vec{\mathbf{a}}_z$$

the current $\vec{J}_v$ is produced by a certain electric field $\vec{E}'$ in terms of Ohmic's law, we have

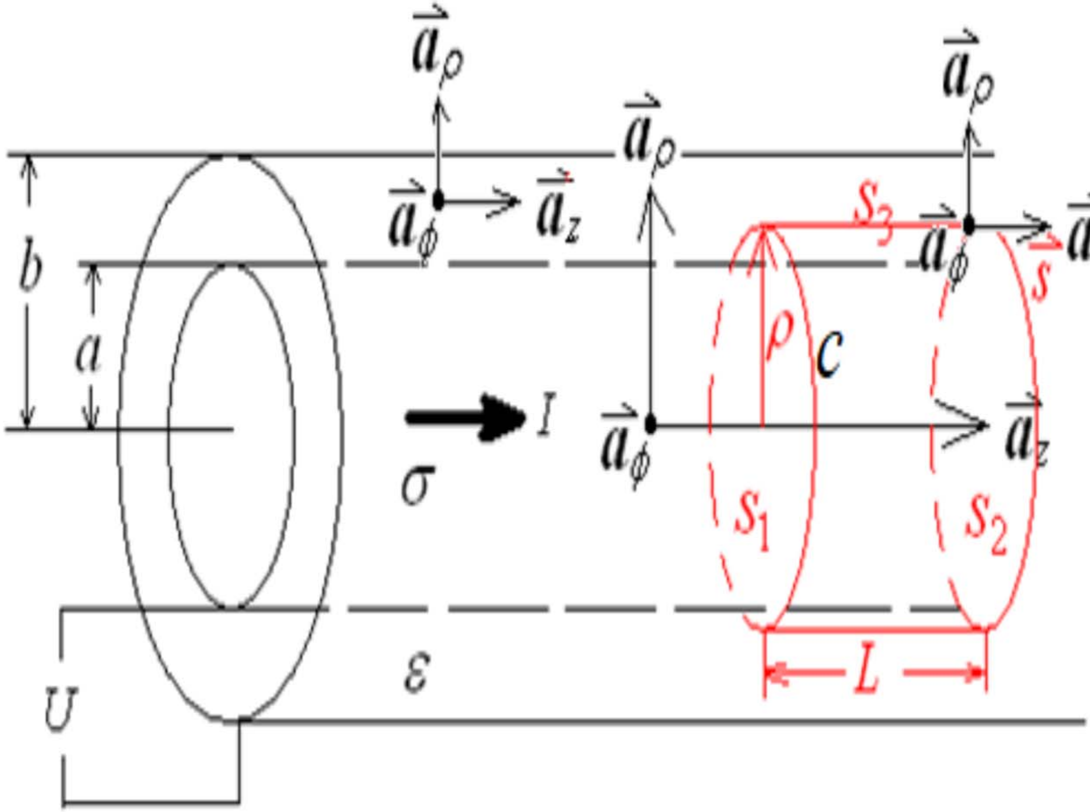
$$\vec{\mathbf{E}}' = \vec{J}_v / \sigma = \frac{I}{\pi a^2 \sigma}\vec{\mathbf{a}}_z$$

Note that the electric field is constant within the inner conductor, the Poynting's vector at any point on the cylinder with its radius being *a*(or the surface enclosing the inner conductor) is

$$\vec{\mathbf{S}}' = \vec{\mathbf{E}}' \times \vec{\mathbf{H}}\Big|_{\rho=a}$$

$$= \frac{I}{\pi a^2 \sigma}\vec{\mathbf{a}}_z \times \frac{I}{2\pi\rho}\vec{\mathbf{a}}_\varphi\Big|_{\rho=a}$$

$$= -\vec{\mathbf{a}}_\rho \frac{I^2}{2\pi^2 a^3 \sigma}$$



for Poynting's vector field $\vec{S}'$ the flux from the closed surface bounding the inner conductor is the power dissipation within the inner conductor. That is,

$$P_L = \oint_s \vec{\mathbf{S}}' \bullet d\vec{\mathbf{s}} = \int_{s1} \vec{\mathbf{S}}' \bullet d\vec{\mathbf{s}} + \int_{s2} \vec{\mathbf{S}}' \bullet d\vec{\mathbf{s}} + \int_{s3} \vec{\mathbf{S}}' \bullet d\vec{\mathbf{s}}$$

$$= \int_{s3} \vec{\mathbf{S}}' \bullet d\vec{\mathbf{s}} = -\int_{s3} \vec{\mathbf{a}}_\rho \frac{I^2}{2\pi^2 a^3 \sigma} \bullet \vec{\mathbf{a}}_\rho ds$$

$$= -\frac{I^2}{2\pi^2 a^3 \sigma} 2\pi a L$$

$$= -\frac{I^2}{\pi a^2 \sigma} L$$

$$= -I^2 \frac{L}{\pi a^2 \sigma} = -I^2 R$$