Chapter 5 magnetostatics

Introduction

A magnetic field is associated with each magnet in the same way as an electric field is associated with a charge. Magnetic lines of force (outside the magnet) are said to emanate from the north pole and terminate at the south pole.

In the chapter,we begin our discussion with the Biot-savart law and use it as a basic tool to calculate the magnetic field set up by any given distribution of currents.

5.2 The Biot-Savart Law

It has been found experimentally that the magnetic flux density produced at a point P from an element of length $d\vec{l}$ of a filamentary wire carrying a steady current *I*,

$$d\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\mu_0}{4\pi}\frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{a}}_R}{R^2} \qquad \textbf{(5.1)}$$

as shown in fig.1, is

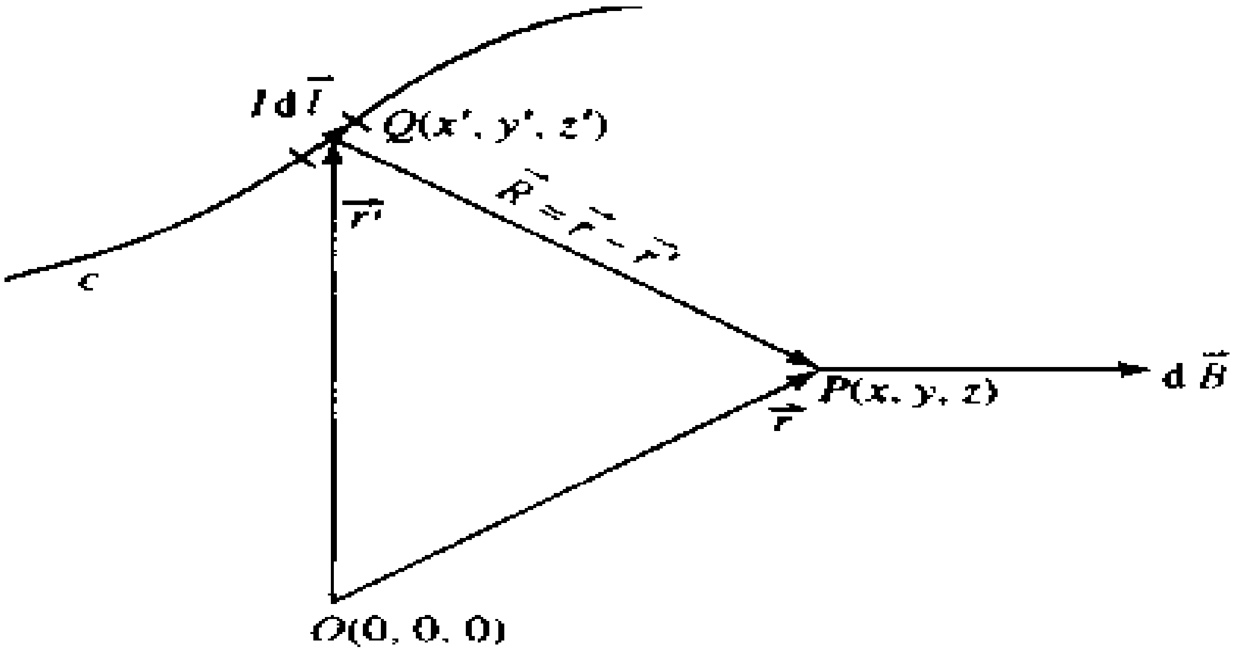


fig.1 Magnetic flux density at a point P (x, y, z) produced by current element at Q (x', y', z')

$$d\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\mu_0}{4\pi} \frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{a}}_R}{R^2} \qquad \textbf{(5.1)}$$

$$d\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\mu_0}{4\pi}\frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{a}}_R}{R^2} \qquad \textbf{(5.1)}$$

$d\vec{\boldsymbol{B}}$ **is the elemental magnetic flux density in teslas (T),**
where one tesla is equal to one weber per square meter (wb/m^2)

- $d\vec{\boldsymbol{l}}$ **is an element of length in the direction of the current.**
- $\vec{\boldsymbol{a}}_R$ **is the unit vector pointing from** $d\vec{l}$ **to P.**
- **The point *P* is at a distance *R* from the current element** $d\vec{l}$ **.**
- $\boldsymbol{\mu_0 = 4\pi\times10^{-7}}$ **is the free space permeability**

integrating (5.1), we obtain

$$\vec{B} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{\boldsymbol{l}} \times \vec{\boldsymbol{a}}_R}{R^2} \quad \textbf{(5.2)}$$

is a vector, is the magnetic flux density at point *P*(*x*, *y*, *z*) due to a filamentary wire carrying steady current *I*.

The direction of $\vec{B}$ **is perpendicular to the plane containing** $d\vec{l}$ **and** $\vec{R} = \vec{r} - \vec{r}'$

The integrand in (5.2) involves six variables: *x, y, z, x', y'* and *z'*. the unprimed variables *x, y,* and *z* are the coordinates of point *P* and are not involved in the integration process. However, the primed variables (also known as the dummy variables) *x', y'* and *z'* are the coordinates of Point Q and are involved in the integration process.

The integration process eliminates the primed variables(*x', y', z'*). Thus,

$\vec{B}$ **is a function of unprimed variables(*x, y, z*) only.**

We can express the current element $I d\vec{\boldsymbol{l}}$ **in terms of the volume current density** $\vec{\boldsymbol{J}}_v$ **as**

$$I d\vec{\boldsymbol{l}} = \vec{\boldsymbol{J}}_v dv$$

we can rewrite (5.2)

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{a}}_R}{R^2}$$

as

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{\boldsymbol{J}}_v \times \vec{\boldsymbol{R}}}{R^3} dv = \frac{\mu_0}{4\pi}\int_v \frac{\vec{\boldsymbol{J}}_v \times \vec{\boldsymbol{a}}_R}{R^2} dv \qquad \textbf{(5.2)}$$

We can also express the current element $I d\vec{\boldsymbol{l}}$ in terms of the surface current density $\vec{\boldsymbol{J}}_s$ as

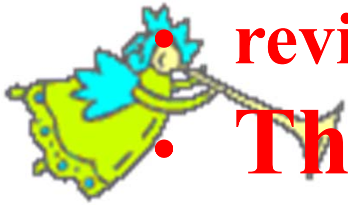
$$I d\vec{\boldsymbol{l}} = \vec{\boldsymbol{J}}_s ds$$

we can also rewrite (5.2)

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{R}}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{a}}_R}{R^2}$$

as

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\int_s \frac{\vec{\boldsymbol{J}}_s \times \vec{\boldsymbol{R}}}{R^3} ds = \frac{\mu_0}{4\pi}\int_s \frac{\vec{\boldsymbol{J}}_s \times \vec{\boldsymbol{a}}_R}{R^2} ds \qquad \textbf{(5.2)}$$



- **review:**
- **The magnetic fields (magnetostatics) are produced by steady currents.**

$$\vec{B} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{l} \times \vec{R}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{l} \times \vec{a}_R}{R^2} \qquad (5.2)$$

or

$$\vec{B} = \frac{\mu_0}{4\pi}\int_s \frac{\vec{J}_s \times \vec{R}}{R^3} ds = \frac{\mu_0}{4\pi}\int_s \frac{\vec{J}_s \times \vec{a}_R}{R^2} ds$$

For the surface current density.

$$\vec{B} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}_v \times \vec{R}}{R^3} dv = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}_v \times \vec{a}_R}{R^2} dv$$

For the volume current density.

Example 5.1 read by yourselves.

5.3 Ampère's Force Law

Most of the experiments conducted by Ampère were related to determination of the force that one current-carrying conductor experiences in the presence of another current-carrying conductor. From his experiments, Ampère was able to demonstrate that when two current-carrying elements $I_1 d\vec{l}_1$ and $I_2 d\vec{l}_2$ interact, the elemental magnetic force exerted by element 1 upon element 2 is

$$d\vec{\boldsymbol{F}}_2 = I_2 d\vec{\boldsymbol{l}}_2 \times \frac{\mu_0}{4\pi} \frac{I_1 d\vec{\boldsymbol{l}}_1 \times \vec{\boldsymbol{R}}_{21}}{R_{21}^3}$$

where $\vec{R}_{21}$ **is the distance vector from elements** $I_1 d\vec{l}_1$ **to** $I_2 d\vec{l}_2$**. If each current-carrying element is a part of a current-carrying conductor, the magnetic force exerted by current-carrying conductor 1 upon current-carrying conductor 2 is**

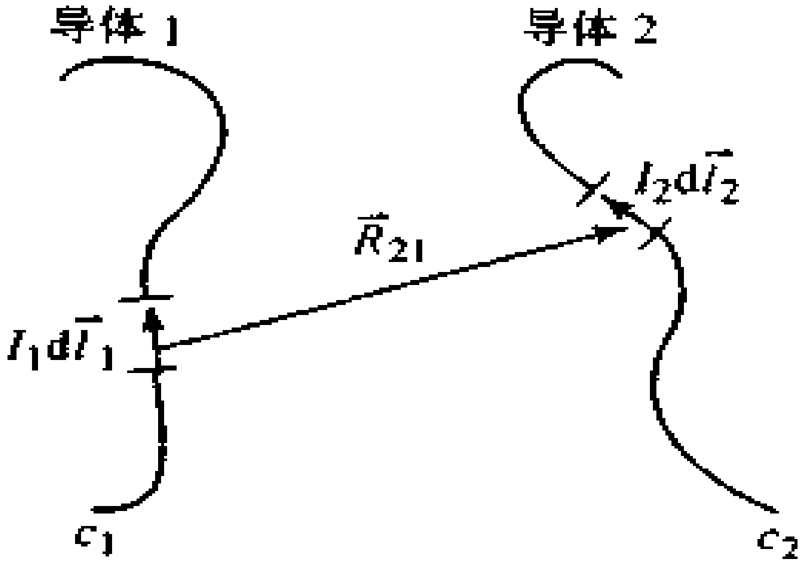


fig. magnetic force on conductor 2 exerted by conductor 1

$$\vec{F}_2 = \frac{\mu_0}{4\pi} \oint_{c2} I_2 d\vec{l}_2 \times \oint_{c1} \frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3} \qquad \textbf{(5.11a)}$$

this equation is referred to as Ampère’s Force Law

Ampère’s Force Law:

(1) Magnetic fields are produced by currents.

$$\vec{F}_2 = \oint_{c2} I_2 d\vec{l}_2 \times \frac{\mu_0}{4\pi} \oint_{c1} \frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3}$$

$$= \oint_{c2} I_2 d\vec{l}_2 \times \vec{B}_1 \qquad \textbf{(5.11b)}$$

where $\vec{B}_1$, the magnetic flux density produced by a current-carrying conductor 1 at the location of current-carrying element $I_2 d\vec{l}_2$, is given as

$$\vec{B}_1 = \frac{\mu_0}{4\pi} \oint_{c1} \frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3} \qquad \textbf{(5.11c)}$$

(2) Magnetic fields exert magnetic field forces upon currents.

•when a current-carrying conductor is placed in an external magnetic field $\vec{\mathrm{B}}$, the magnetic force $\vec{F}$ experienced by the conductor is

$$\vec{\boldsymbol{F}} = \oint_c I d\,\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}} \qquad \textbf{(5.12a)}$$

the equation can specifies the fact that the magnetic field force was exerted by the magnetic field upon a current-carrying conductor which is placed in the magnetic field.

We can express the current element $Id\vec{\boldsymbol{l}}$ **in terms of the volume current density** $\vec{\boldsymbol{J}}_v$ **as**

$$Id\,\vec{\boldsymbol{l}} = \vec{\boldsymbol{J}}_v\, dv$$

the equation (5.12a) can be rewritten as

$$\vec{\boldsymbol{F}} = \int_v \vec{J}_v \times \vec{\boldsymbol{B}}\, dv \qquad \textbf{(5.12b)}$$

By replacing $\vec{J}_v dv$ with $\vec{J}_s ds$ we can obtain an expression for the magnetic force experienced by a surface current distribution in an external magnetic field. If ρ_v is the volume charge density

is the average velocity of the charge, the magnetic force experienced by the charge q

$$\vec{F} = \int_v \vec{J}_v \times \vec{B}\, dv = \int_v \rho_v \vec{v} \times \vec{B}\, dv = q\vec{v} \times \vec{B}$$

if an electric field also exists, we have

$$\vec{F} = q\vec{E} + q\bar{v} \times \vec{B}$$

Review:

Magnetic fields exert magnetic field forces upon currents/moving charges.

$$\vec{\boldsymbol{F}} = \oint_c I d\,\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}}$$

$$\vec{\boldsymbol{F}} = \int_v \vec{J}_v \times \vec{\boldsymbol{B}}\, dv$$

$$\vec{F} = \int_v \vec{J}_v \times \vec{B}\, dv = \int_v \rho_v \vec{v} \times \vec{B}\, dv = q\,\vec{v} \times \vec{B}$$

$$\vec{\boldsymbol{F}} = q\,\vec{\boldsymbol{E}} + q\,\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$$

5.5 magnetic flux and Gauss' law

1. Magnetic flux

Since $\vec{B}$ is called as the magnetic field flux density, and the magnetic flux density $\vec{B}$ may or may not be uniform over the entire surface. The magnetic flux passing through an open(or enclosed) surface s is given by

$$\phi = \sum_{i}^{n} \vec{B}_i \bullet \Delta \vec{s}_i \qquad or \qquad \phi = \int_s \vec{B} \bullet d\vec{s}$$

8字形线圈

1 物理模型

在 xy 平面内有一通有恒定电流 I(电流方向如图所示)的“8”形线圈,电流方向如图所示。

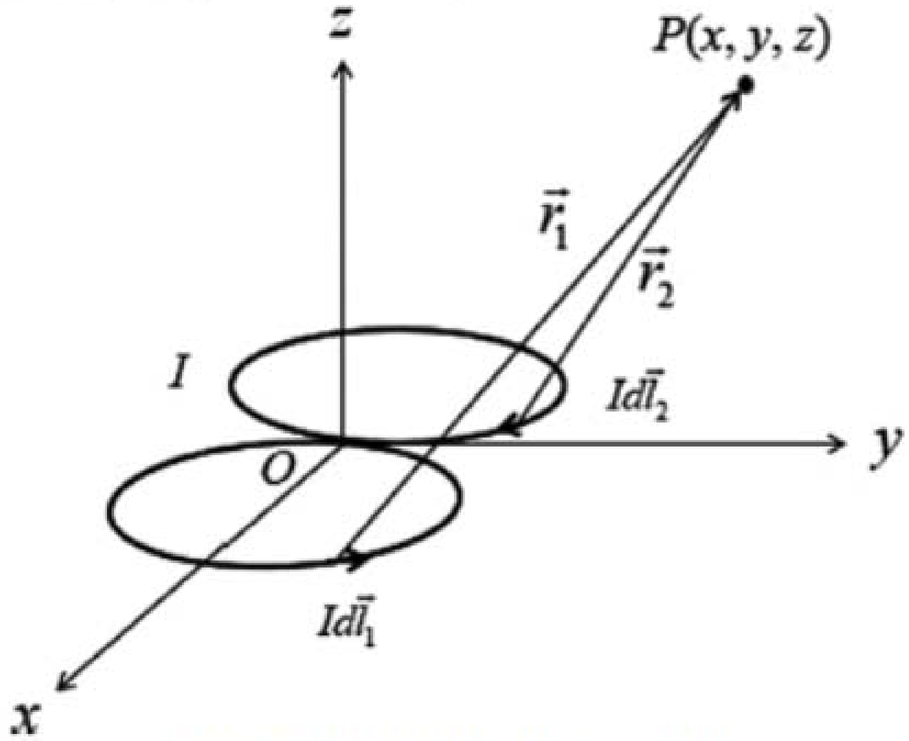


图 1 载流线圈示意图

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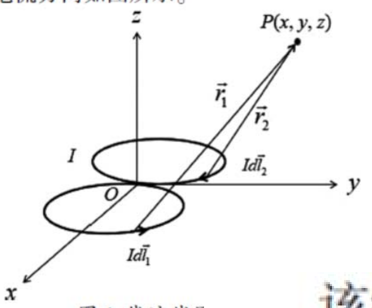

图 1 载流线

2. 磁场计算

该线圈形状由两个半径均为 R 的圆环相切而成,两圆环交点位于坐标原点 O。在两个圆环上与 x 轴正向夹角 α 处分别取一电流元 $I\vec{dl}$,即:

$$I\vec{dl}_1 = IRd\alpha(-\sin\alpha\hat{i}+\cos\alpha\hat{j}) \tag{1}$$

$$I\vec{dl}_2 = IRd\alpha(\sin\alpha\hat{i}-\cos\alpha\hat{j}) \tag{2}$$

两个电流元的位置坐标分别为:

$$\begin{cases} x'_1 = R+R\cos\alpha \\ y'_1 = R\sin\alpha \\ z'_1 = 0 \end{cases} \tag{3}$$

$$\begin{cases} x'_2 = -R+R\cos\alpha \\ y'_2 = R\sin\alpha \\ z'_2 = 0 \end{cases} \tag{4}$$

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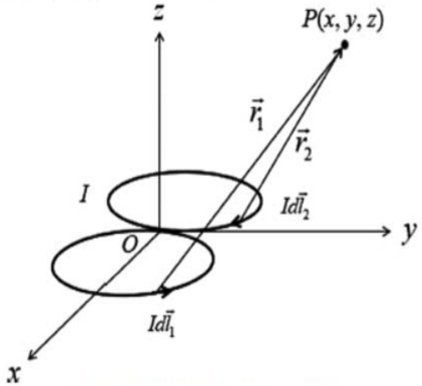

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假设在三维空间中任取一点 P(x,y,z),因此两个线圈上电流元 $I\vec{dl}$ 指向点 P 的位置矢量分别表示为:

$$\vec{r}_1 = (x-R-R\cos\alpha)\hat{i}+(y-R\sin\alpha)\hat{j}+z\hat{k} \tag{5}$$

$$\vec{r}_2 = (x+R-R\cos\alpha)\hat{i}+(y-R\sin\alpha)\hat{j}+z\hat{k} \tag{6}$$

根据毕奥-萨伐尔定律,该“8”形载流线圈在 P 点的磁感应强度等于两个相切圆环激发磁场的矢量和:

$$\vec{B} = \frac{\mu_0}{4\pi}\oint_{L_1}\frac{I\vec{dl}_1\times\vec{r}_1}{r_1^3}+\frac{\mu_0}{4\pi}\oint_{L_2}\frac{I\vec{dl}_2\times\vec{r}_2}{r_2^3} \tag{7}$$

因此,场分布在三维直角坐标系中的分量可表示为:

$$\begin{aligned} B_x = &\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{IRz\cos\alpha}{[(x-R-R\cos\alpha)^2+(y-R\sin\alpha)^2+z^2]^{3/2}}d\alpha \\ &+\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{-IRz\cos\alpha}{[(x+R-R\cos\alpha)^2+(y-R\sin\alpha)^2+z^2]^{3/2}}d\alpha \end{aligned} \tag{8}$$

$$\begin{aligned} B_y = &\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{IRz\sin\alpha}{[(x-R-R\cos\alpha)^2+(y-R\sin\alpha)^2+z^2]^{3/2}}d\alpha \\ &+\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{-IRz\sin\alpha}{[(x+R-R\cos\alpha)^2+(y-R\sin\alpha)^2+z^2]^{3/2}}d\alpha \end{aligned} \tag{9}$$

$$\begin{aligned} B_z = &\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{-IR\sin\alpha(y-R\sin\alpha)-IR\cos\alpha(x-R-R\cos\alpha)}{[(x-R-R\cos\alpha)^2+(y-R\sin\alpha)^2+z^2]^{3/2}}d\alpha \\ &+\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{IR\sin\alpha(y-R\sin\alpha)+IR\cos\alpha(x+R-R\cos\alpha)}{[(x+R-R\cos\alpha)^2+(y-R\sin\alpha)^2+z^2]^{3/2}}d\alpha \end{aligned} \tag{10}$$

我们借助 MATLAB 软件对(8)、(9)和(10)三式进行数值积分即可得到该“8”形圆环周围空间的磁场分布。

【科技前沿】浅谈经颅磁刺激技术

经颅磁刺激技术与脑磁图、正电子发射断层成像、功能核磁共振成像，被誉为二十世纪的四大脑科学技术。

工作原理

通过输入数千安培的脉冲电流，在线圈内外产生脉冲磁场，进而在大脑皮层产生反向的感应电流；感应电流影响神经元的膜电位：当膜电位超过阈值时，就会引起去极化（兴奋）或超极化（抑制），进而产生一系列生理生化反应[1]。

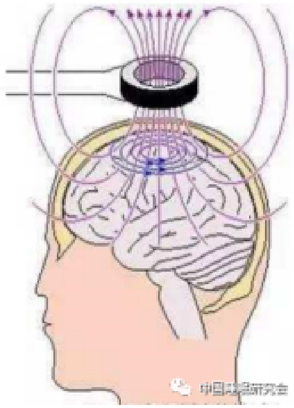


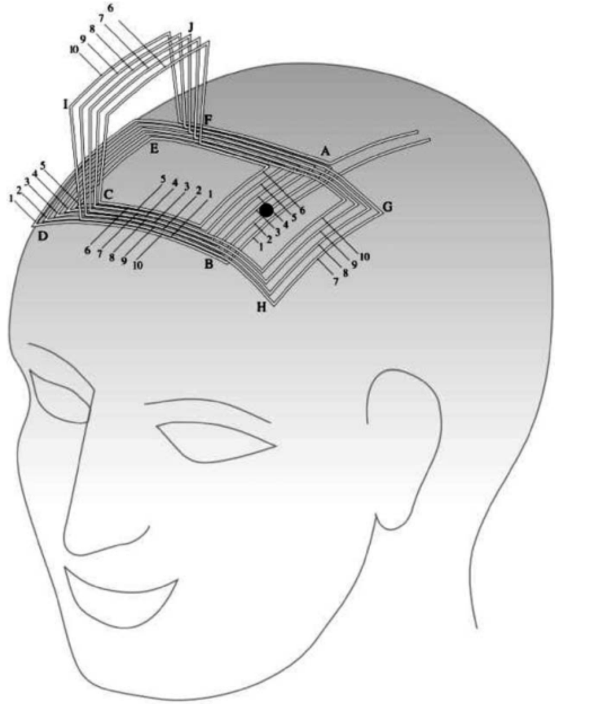
图1 TMS脉冲磁场示意图

刺激线圈作为初级回路，当输入脉冲电流时产生时变磁场。该磁场可以使附近的次级回路（人体或大脑皮层）产生感应电场。据Faraday电磁感应定律，时变磁场 *B(r,t)* 在组织内矢径 *(r)*的任一点处的感应电场 *E(r,t)* 可由下式给出：

$$\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}$$

磁场衰减速率 H1 vs 8字形

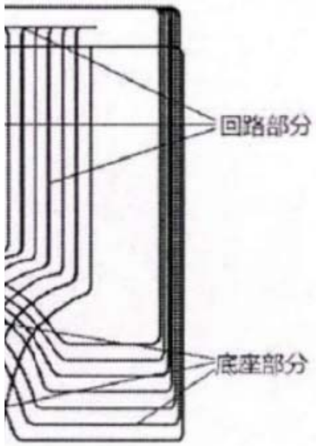
H1线圈（英智科技&Brainsway）如右图所示	条数	方向	平均长度/cm
	10	沿+z方向传输电流（A-B/G-H）	11
	5	径向（C-I/J-F）	8
	5	放置再对侧半球的头部（D-E）	11
	5	连接条带和返回路径之间的导线（B-C/F-A）	9



图中所示的线圈方向用于对右侧APB(用黑点表示)进行最佳刺激。

5. Zangen A, et al. Transcranial magnetic stimulation of deep brain regions: evidence for efficacy of the H-coil. Clin Neurophysiol. 116:775-9 (2005).

边缘系统组织，影响腹侧被盖区-伏隔核-大
用有复杂的结构设计，底座部分是刺激主要
亍；回路部分远离刺激目标区域，避免组织
用，同时减弱对目标区域的影响[18]。常见



2. Gauss's Law

Because the lines of magnetic flux form concentric circles around an infinitely long current-carrying conductor. and the lines of magnetic flux are always continuous. In other words, the flux penetrating a closed surface is equal to the flux leaving the closed surface. Therefore, for a closed surface,

$$\oint_S \vec{B} \bullet d\vec{s} = 0 \qquad \textbf{(5.23a)}$$

$$\oint_s \vec{F} \bullet d\vec{s} = \int_V \nabla \bullet \vec{F} dV$$

Equation (5.23a) is known as the integral form of Gauss's Law for magnetic fields.

The closed surface integral can, however, be converted into a volume integral by the direct application of the divergence theorem. That is,

$$\nabla \bullet \vec{B} = 0 \qquad \text{(5.23b)/homework}$$

Equation (5.23b) is known as the point form or differential form of Gauss's law for magnetic fields. Since the divergence of B is always zero, the magnetic flux density is solenoid.

$$\oint_S \vec{F} \bullet d\vec{s} = \int_V \nabla \bullet \vec{F} dV$$

3. the magnetic field intensity

(a) the magnetic moment

an orbiting electron produces a ring current of magnitude

$$I = \frac{e\upsilon}{2\pi\rho}$$

Where *e* is the magnitude of the charge on the electron, υ is its speed, and ρ is the radius. The orbiting electron gives rise to an orbital magnetic moment

$$\vec{\boldsymbol{m}} = \frac{e\upsilon\rho}{2}\vec{\boldsymbol{a}}_n$$

The electron spinning motion involves circulating charge and it gives an electron a spin magnetic moment

$$\vec{m}_s = \frac{he}{8\pi m_e}\vec{a}_n$$

The net magnetic moment $\vec{m}_i$ of the atom is obtained by combining both the orbital and spin moments of the electron.

$$\vec{m}_i = \frac{e\upsilon\rho}{2}\vec{a}_n + \frac{he}{8\pi m_e}\vec{a}'_n$$

The net magnetic moment produces a far field similar to that produced by a current loop (magnetic dipole).

•If there are *n* atoms in a material and $\vec{m}_i$ is the Magnetic moment of the *i*th atom, the magnetic moment is define a

$$\vec{p}_m = \sum_i^n \vec{m}_i = \sum_i^n I_i \vec{S}_i$$

•a magnetic dipole: a current loop

•a magnetic dipole moment can be given as

$$\vec{p}_m = I\vec{S}$$

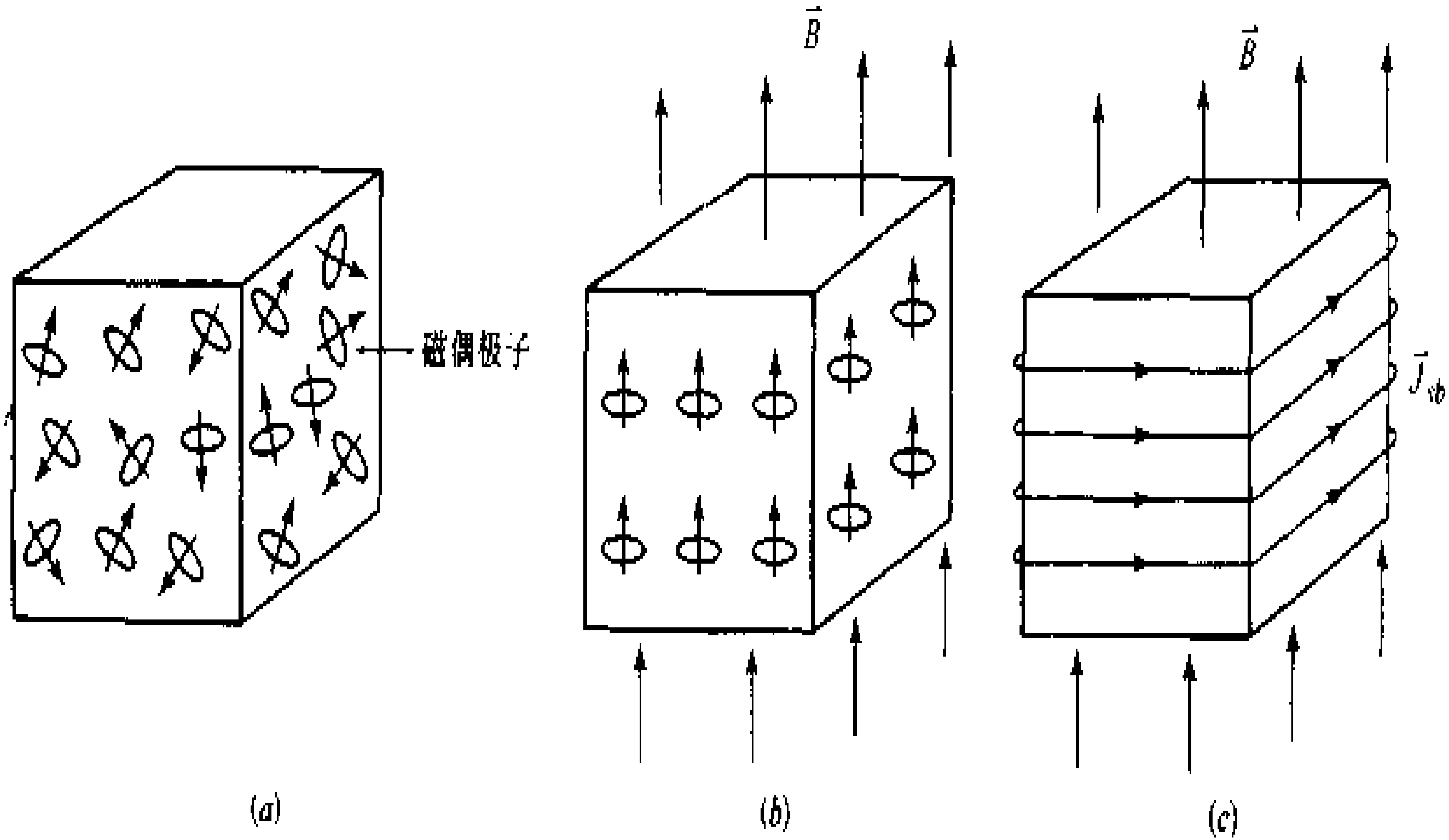
If there are *n* atoms in a material and $\vec{m}_i$ is the magnetic moment of the *i*th atom, the magnetic moment is define as

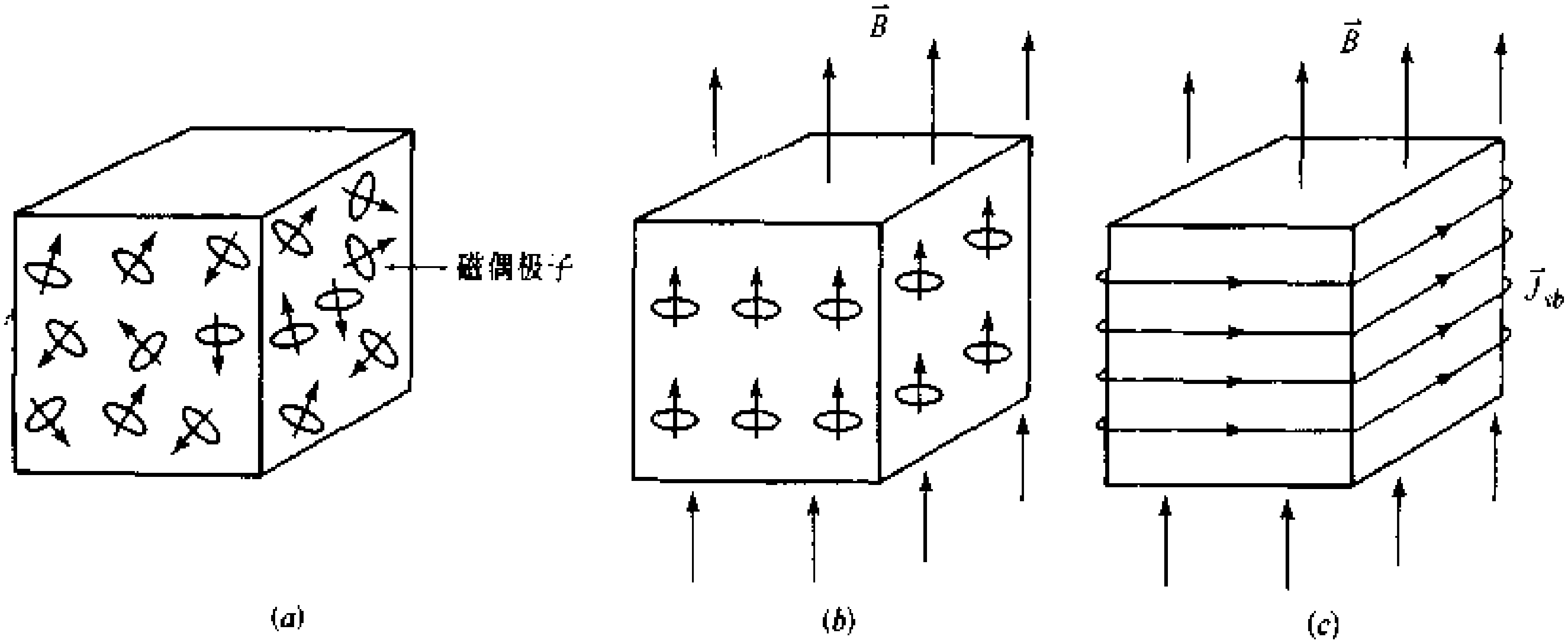
$$\vec{p}_m = \sum_i^n \vec{m}_i = \sum_i^n I_i \vec{S}_i$$

(b)The magnetic moment per unit volume is defined as

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_i^n \vec{m}_i}{\Delta V} = \lim_{\Delta v \to 0} \frac{\Delta \vec{p}_m}{\Delta V}$$

In the presence of an extern magnetic field, each magnetic dipole experiences a torque that tends to align it with the magnetic field.



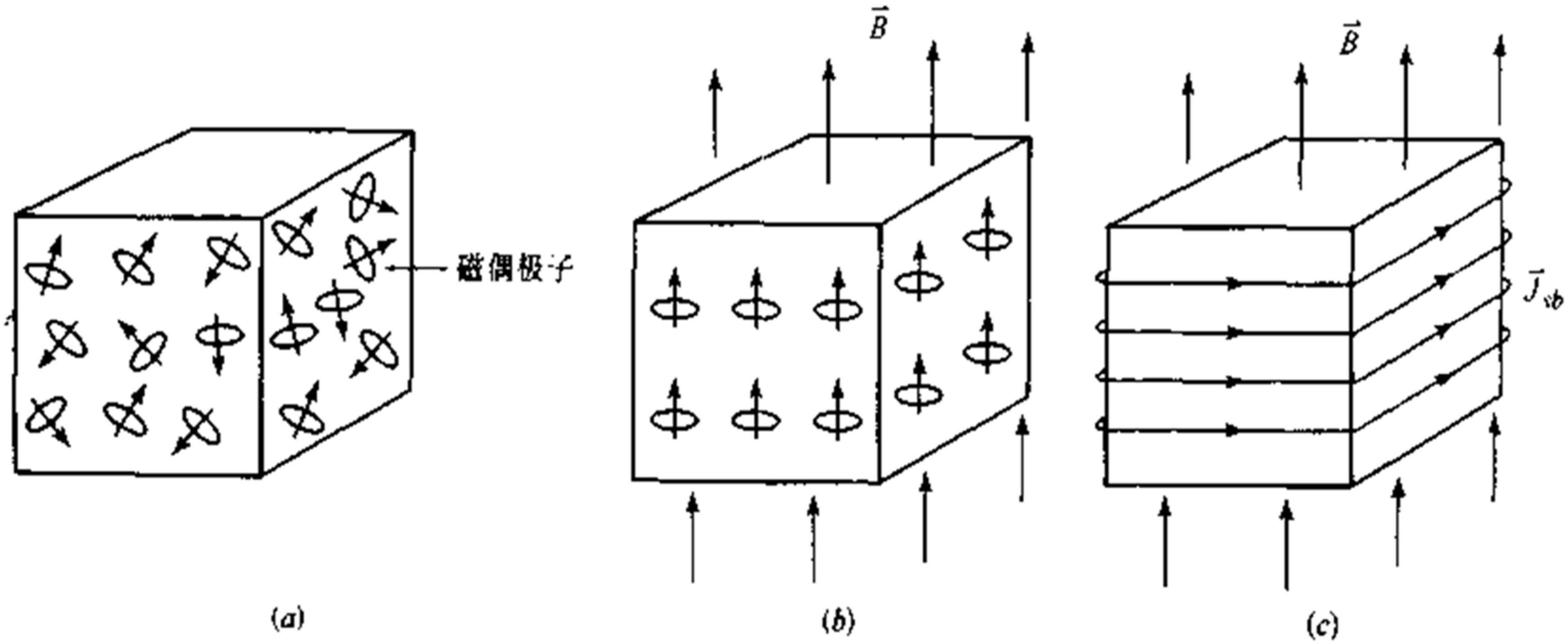


a) a piece of magnetic material with randomly oriented magnetic dipoles;

b) an extern magnetic field causes the magnetic dipoles to align with it;

b) an extern magnetic field causes the magnetic dipoles to align with it;

c) the small aligned current loops of (b) are equivalent to a current along the surface of the material.



(c) the magnetic field intensity $\vec{H}$

In the presence of an external magnetic field, each magnetic dipoles experiences a torque that tends to align it with the magnetic field. the magnetic moment will distort the external magnetic field.

The magnetic field in the material will include:

- **the magnetic field created by the magnetic moment** $\vec{M}$
- **the external magnetic field** $\vec{B}$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

The alignment of electric dipoles always decreases the original electric field, whereas the alignment of the magnetic dipole in paramagnetic and ferromagnetic materials increases the original magnetic field. That is ,

顺磁性的

铁磁性的

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

if the material is linear, isotropic and homogeneous, we have

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the magnetic susceptibility.
In terms of relative permeability， the magnetic susceptibility is $\chi_m = \mu_r - 1$

$\chi_m > 0$, $\mu_r > 1$: paramagnetic;
$\chi_m < 0$, $0 < \mu_r < 1$: diamagnet;
$\chi_m = 0$, vacuum.

$$
\begin{aligned}
\vec{B} &= \mu_0\left(\vec{H} + \vec{M}\right) \\
&= \mu_0\left(\vec{H} + \chi_m \vec{H}\right) \\
&= \mu_0\left(1 + \chi_m\right)\vec{H} \\
&= \mu_0 \mu_r \vec{H} \\
&= \mu \vec{H}
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon_0 \vec{E} + \vec{P} \\
&= \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E} \\
&= \varepsilon_0 (1 + \chi_e) \vec{E} \\
&= \varepsilon_0 \varepsilon_r \vec{E} \\
&= \varepsilon \vec{E} \\
&= \vec{D}
\end{aligned}
$$

5.7 Ampere' circuital law

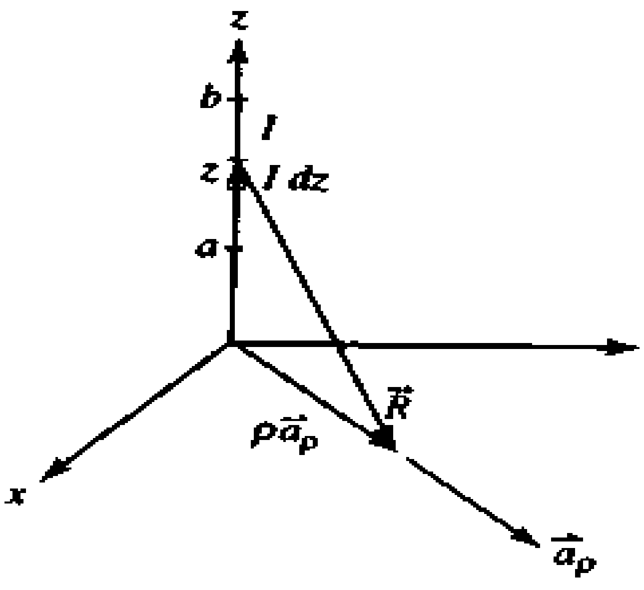
•**In the study of electrostatic fields we defined the electric flux density in terms of the electric field intensity as $\vec{D}=\varepsilon\vec{E}$ so that $\vec{D}$ was independent of the permittivity of the medium. We shall now define the magnetic field intensity $\vec{H}$ in free space as**

$$\vec{\boldsymbol{B}} = \mu_0 \vec{H}$$

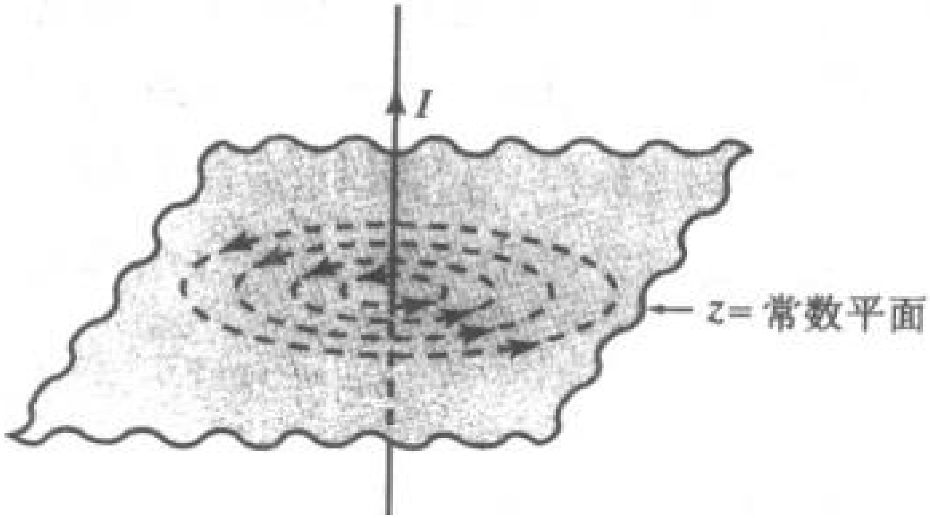
$$\vec{H} = \frac{\vec{\boldsymbol{B}}}{\mu_0}$$

Here $\vec{M}$=0.

(1)example5.1 page 58.
a filamentary wire of finite length extends from *z=a* to *z=b*, as shown in fig.5.5a, determine the magnetic flux density at a point P in the xy plane. What is the magnetic flux density at P if *a*→-∞ and *b*→∞ ?



(a)



$$\vec{B} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{l} \times \vec{R}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{l} \times \vec{a}_R}{R^2}$$

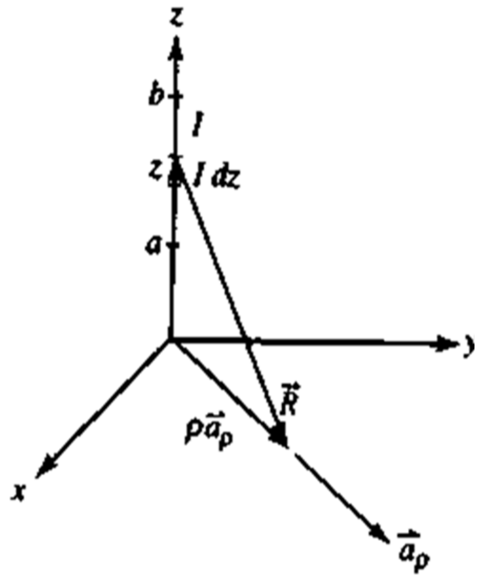
solution

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{\mathbf{l}} \times \vec{\mathbf{R}}}{R^3}$$

$$= \frac{\mu_0}{4\pi}\int_c \frac{I dz \vec{\mathbf{a}}_{\mathbf{z}} \times (\rho \vec{\mathbf{a}}_{\boldsymbol{\rho}} - z\vec{\mathbf{a}}_{\mathbf{z}})}{[\rho^2 + z^2]^{3/2}}$$

$$= \frac{\mu_0}{4\pi}\int_c \frac{I dz \vec{\mathbf{a}}_{\mathbf{z}} \times \rho \vec{\mathbf{a}}_{\boldsymbol{\rho}}}{[\rho^2 + z^2]^{3/2}}$$

$$= \frac{\mu_0}{4\pi}\int_c \frac{I \rho dz \vec{\mathbf{a}}_{\phi}}{[\rho^2 + z^2]^{3/2}}$$



(a)

$$\vec{B} = \mathbf{a}_\phi \frac{\mu_0 I \rho}{4\pi} \int_c \frac{dz}{[\rho^2 + z^2]^{3/2}}$$

$$= \mathbf{a}_\phi \frac{\mu_0 I \rho}{4\pi} \int_a^b \frac{dz}{[\rho^2 + z^2]^{3/2}}$$

$$= \mathbf{a}_\phi \frac{\mu_0 I}{4\pi\rho} \left[\frac{b}{\sqrt{\rho^2 + b^2}} - \frac{a}{\sqrt{\rho^2 + a^2}} \right]$$

By setting $a = -\infty$ and $b = \infty$ in the preceding expression, we obtain the magnetic field $\vec{B}$ produced at a point by a wire of infinite extent as

$$\vec{B} = \mathrm{a}_\phi \frac{\mu_0 I}{4\pi\rho} [1 - (-1)]$$

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi\rho}\vec{\mathbf{a}}_\phi$$

Since we can define the magnetic flux density in a medium in terms of the current(Biot-Savart law) as

$$\vec{\mathbf{B}} = \frac{\mu}{4\pi}\int_c \frac{I d\vec{l} \times \vec{\boldsymbol{R}}}{R^3}$$

we obtain the magnetic field $\vec{B}$ produced at a point in a medium μ by a wire of infinite extent as

$$\vec{\mathbf{B}} = \frac{\mu I}{2\pi\rho}\vec{\mathbf{a}}_\phi$$

(2) Ampère's Circuital Law page304

it states that the line integral of the magnetic field intensity around a closed path equals t-he current enclosed. That is

$$\oint_c \vec{H} \bullet d\vec{l} = \oint_c \frac{\vec{\mathbf{B}}}{\mu} \bullet d\vec{l} = \oint_c \frac{\vec{B}}{\mu} \bullet \vec{a}_l \, dl$$

$$= \int_0^{2\pi} \frac{\vec{\mathbf{B}}}{\mu} \bullet \vec{\mathbf{a}}_\phi \, \rho \, d\phi$$

$$= I$$

where *I* is the uniform current enclosed by contour c.

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi\rho} \vec{\mathbf{a}}_\phi$$

•**Ampere' circuital law**

•**Ampere' circuital law states that the line integral of the magnetic field intensity around a closed path equals the current enclosed. That is**

$$\oint_c \vec{H} \bullet d\vec{l} = I$$

Where *I* is the net current intercepted by the area enclosed by the path. We will refer to it as the integral form of Ampere' circuital law.

Since the current can be expressed in terms of volume current density as $I = \int_s \vec{J}_v \bullet d\vec{s}$

Thus, applying Stoke's theorem we have

$$\oint_c \vec{H} \bullet d\vec{l} = I$$

$$\int_s (\nabla \times \vec{H}) \bullet d\vec{s} = \int_s \vec{J}_v \bullet d\vec{s}$$

$$\nabla \times \vec{H} = \vec{J}_v$$

$$\int_s \left(\nabla \times \vec{F}\right) \bullet d\vec{s} = \oint_c \vec{F} \bullet d\vec{l}$$

- Review:

- Concept: $\vec{H},\vec{B}$ ------ $\vec{E}, \vec{D}$
- Characteristics :

  divergence of $\vec{B}$ -----flux of $\vec{B}$

$$\nabla \bullet \vec{B} = 0 \qquad\qquad \oint_S \vec{B} \bullet d\vec{s} = 0$$

  curl of $\vec{H}$------------circulation of $\vec{H}$

$$\nabla \times \vec{H} = \vec{J}_v \qquad\qquad \oint_c \vec{H} \bullet d\vec{l} = I$$

- Relationships:

$$\begin{aligned}\vec{B} &= \mu_0 \left(\vec{H} + \vec{M}\right) \\ &= \mu_0 \left(\vec{H} + \chi_m \vec{H}\right) \\ &= \mu_0 \left(1 + \chi_m\right)\vec{H} \\ &= \mu_0 \mu_r \vec{H} \\ &= \mu \vec{H}\end{aligned}$$

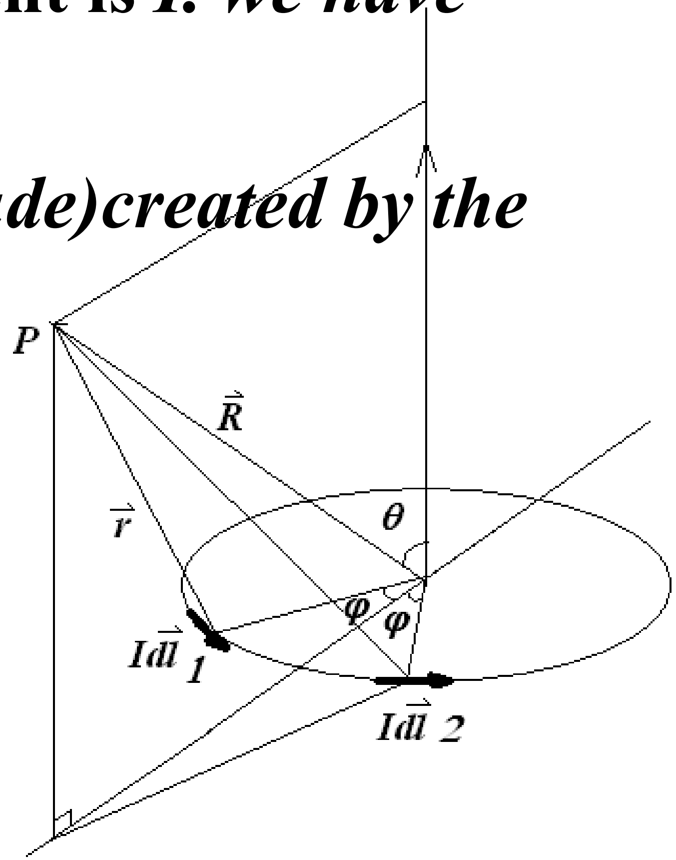
5.8 Magnetic materials

1.The magnetic potential produced by a magnetic moment

The radius of a current –carrying loop is *a*, namely, the area is *S*=π*a*².**the current is *I. we have***

The magnetic moment $\vec{p}_m = I\vec{S}$.

The magnetic potential $d\vec{A}$ ***(magnitude)created by the current element*** $Id\vec{l}$ ***1 and 2 is given***



$$2dA\cos\varphi = \frac{\mu_0}{4\pi}\frac{Idl}{r}2\cos\varphi$$

$$\vec{A} = \frac{\mu}{4\pi}\int_c \frac{Id\vec{l}}{R}$$

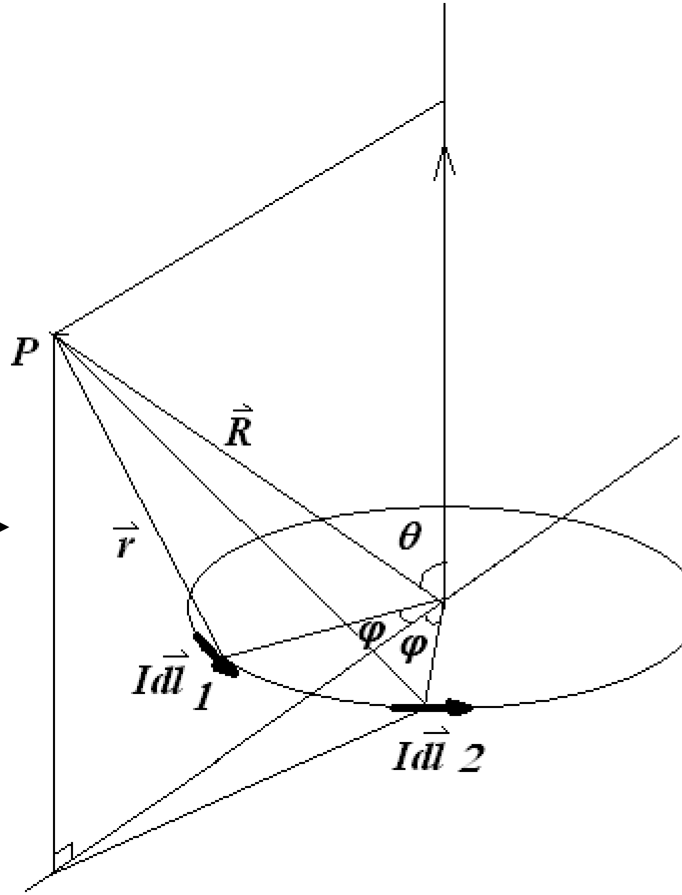
5.8 Magnetic materials

The magnetic moment p_m=IS.

The magnetic potential $d\vec{A}$ (magnitude)created by the current element $Id\vec{l}$ 1 and 2 is given

$$2dA\cos\varphi = \frac{\mu_0}{4\pi}\frac{Idl}{r}2\cos\varphi$$

The direction is along the unit vector $\vec{a}_\varphi$, The magnetic potential $\vec{A}$ created by the loop is given by



$$\vec{A} = \vec{a}_\varphi \int_0^\pi \frac{\mu_0}{4\pi} 2\cos\varphi \frac{Iad\varphi}{\sqrt{(R\sin\theta - a\cos\varphi)^2 + (a\sin\varphi)^2 + (R\cos\theta)^2}}$$

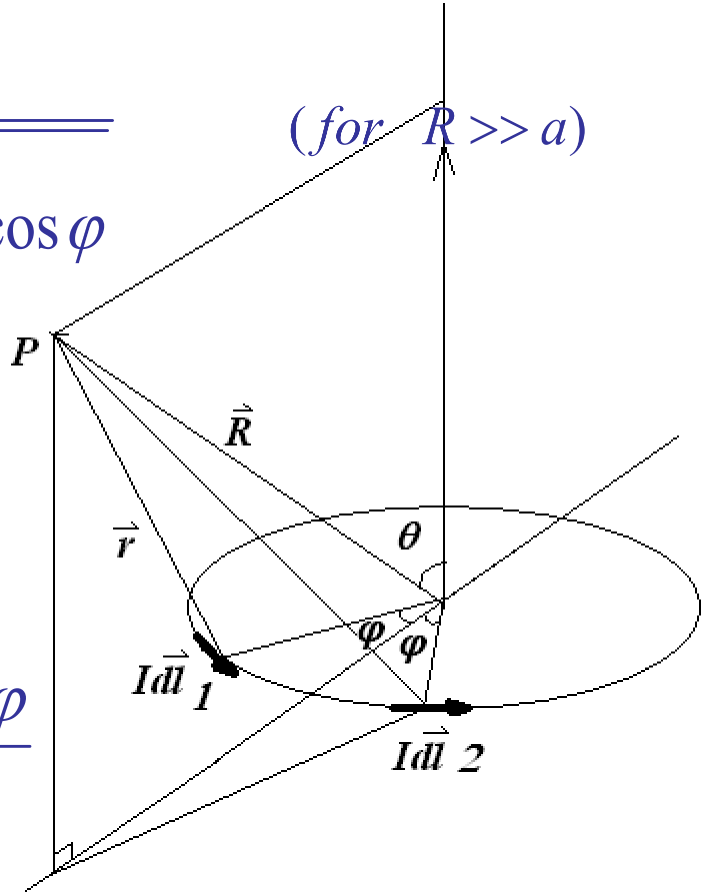
$$= \vec{a}_\varphi \int_0^\pi \frac{\mu_0}{4\pi} 2\cos\varphi \frac{Iad\varphi}{\sqrt{R^2 + a^2 - 2Ra\cos\varphi\sin\theta}}$$

$$= \vec{a}_\varphi \int_0^\pi \frac{\mu_0}{4\pi} 2\cos\varphi \frac{Iad\varphi}{R\sqrt{1 + \frac{a^2}{R^2} - 2\frac{a}{R}\sin\theta\cos\varphi}}$$

$$(for\ R >> a)$$

$$\approx \vec{a}_\varphi \int_0^\pi \frac{\mu_0}{4\pi} 2\cos\varphi \frac{Iad\varphi}{R\sqrt{1 - 2\frac{a}{R}\sin\theta\cos\varphi}}$$

$$\approx \vec{a}_\varphi \int_0^\pi \frac{\mu_0 I a}{2\pi}\left(1 + \frac{a}{R}\sin\theta\cos\varphi\right)\frac{\cos\varphi d\varphi}{R}$$

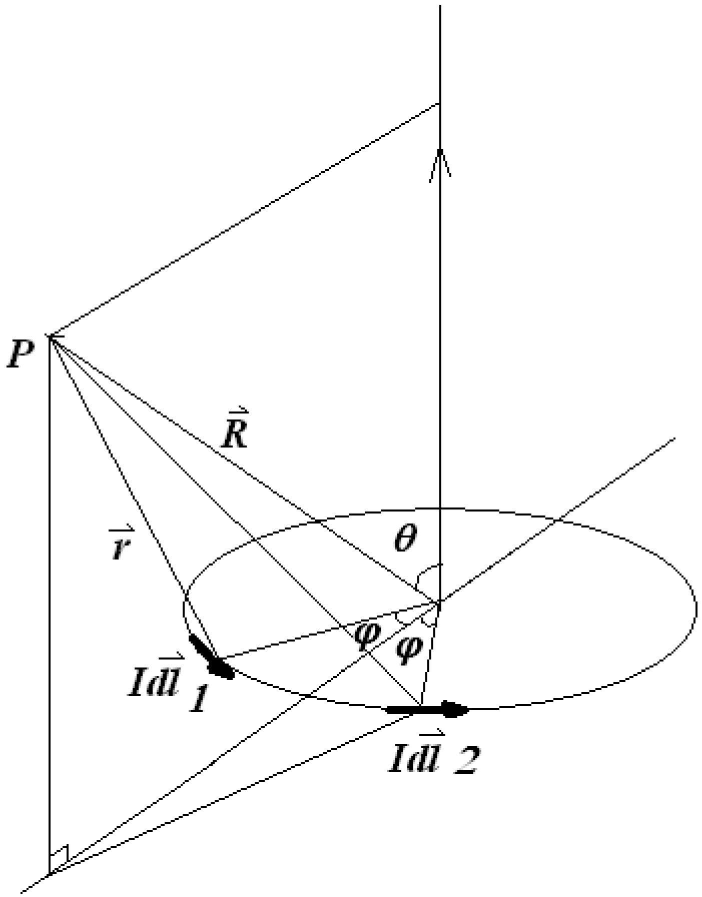


$$\vec{A}=\vec{a}_\varphi\int_0^\pi \frac{\mu_0 Ia}{2\pi}\left(1+\frac{a}{R}\sin\theta\cos\varphi\right)\frac{\cos\varphi d\varphi}{R}$$

$$=\vec{a}_\varphi\frac{\mu_0 Ia}{2\pi}\left(0+\frac{a}{R^2}\sin\theta\frac{1}{2}\pi\right)$$

$$=\vec{a}_\varphi\frac{\mu_0 I\pi a^2}{4\pi R^2}\sin\theta$$

$$=\frac{\mu_0\vec{p}_m\times\vec{a}_R}{4\pi R^2}=\frac{\mu_0\vec{p}_m\times\vec{R}}{4\pi R^3}$$



5.8 Magnetic materials

A material is said to be magnetized if M $\neq$ **0, the magnetic dipole moment** $d\vec{p}_m$ **for an elemental volume** dv' **is** $d\vec{p}_m = \vec{M}\,dv'$**.**

The magnetic vector potential set up by $d\vec{p}_m$ **is**

$$d\vec{A} = \frac{\mu_0 d\vec{\boldsymbol{p}}_{\boldsymbol{m}} \times \vec{\boldsymbol{a}}_R}{4\pi R^2} = \frac{\mu_0 \vec{M} \times \vec{\boldsymbol{R}}}{4\pi R^3} dv'$$

If v ' is the volume of the magnetized material, the magnetic vector potential that it produces is

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{M} \times \vec{\boldsymbol{R}}}{R^3} dv' = \frac{\mu_0}{4\pi}\int_{v'} \vec{M} \times \frac{\vec{\boldsymbol{R}}}{R^3} dv' = \frac{\mu_0}{4\pi}\int_{v'} \vec{M} \times \nabla'(\frac{1}{R}) dv'$$

5.8 Magnetic materials

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{v'}\vec{M}\times\nabla'\left(\frac{1}{R}\right)dv'=-\frac{\mu_0}{4\pi}\int_{v'}\left[\nabla'\times\left(\frac{\vec{M}}{R}\right)-\frac{\nabla'\times\vec{M}}{R}\right]dv'$$

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{v'}\frac{\nabla'\times\vec{M}}{R}dv'-\frac{\mu_0}{4\pi}\int_{v'}\left[\nabla'\times\left(\frac{\vec{M}}{R}\right)\right]dv'$$

$$=\frac{\mu_0}{4\pi}\int_{v'}\frac{\nabla'\times\vec{\boldsymbol{M}}}{R}dv'+\frac{\mu_0}{4\pi}\oint_{s'}\left(\frac{\vec{\boldsymbol{M}}}{R}\right)\times d\vec{\boldsymbol{s}}'$$

$$=\frac{\mu_0}{4\pi}\int_{v'}\frac{\nabla'\times\vec{\boldsymbol{M}}}{R}dv'+\frac{\mu_0}{4\pi}\oint_{s'}\frac{\vec{\boldsymbol{M}}\times\vec{\boldsymbol{a}}_n}{R}ds'$$

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{J}_{v'}}{R}dv' \qquad and \qquad \vec{A}=\frac{\mu_0}{4\pi}\oint_{s'}\frac{\vec{J}_{s'}}{R}ds'$$

In terms of equations

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_{v'}}{R} dv' \qquad and \qquad \vec{A} = \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{J}_{s'}}{R} ds'$$

We can obtain

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_{vb}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{J}_{sb}}{R} ds'$$

Which is created by the volume v' of the magnetized material.

Where $\vec{J}_{vb} = \nabla' \times \vec{M} \quad and \quad \vec{J}_{sb} = \vec{\boldsymbol{M}} \times \vec{\boldsymbol{a}}_n$

are the bound volume current density and the bound surface current density, respectively.

Review:

1.Faraday and Ampère's Force Law states:

1)The current produces the magnetic field

$$\vec{B} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{l} \times \vec{R}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{l} \times \vec{a}_R}{R^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}_v \times \vec{R}}{R^3}dv = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}_v \times \vec{a}_R}{R^2}dv$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_s \frac{\vec{J}_s \times \vec{R}}{R^3}ds = \frac{\mu_0}{4\pi}\int_s \frac{\vec{J}_s \times \vec{a}_R}{R^2}ds$$

2)Magnetic fields exert force upon currents.

$$\vec{F} = \oint_c Id\vec{l} \times \vec{B} \qquad \textbf{or} \qquad \vec{F} = \int_v \vec{J}_v \times \vec{B}dv$$

2. Gauss's Law

$$\oint_S \vec{B} \bullet d\vec{s} = 0 \qquad \nabla \bullet \vec{B} = 0$$

3. a magnetic dipole: a current loop $\vec{p}_m = I\vec{S}$

4.The magnetic moment per unit volume is $\vec{M} = \lim_{\Delta v \to 0} \frac{\Delta \vec{p}_m}{\Delta V}$

5.The magnetic field intensity

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$\vec{M} = \chi_m \vec{H}$$

$$\begin{aligned} \vec{B} &= \mu_0 \left(\vec{H} + \vec{M} \right) \\ &= \mu_0 \left(\vec{H} + \chi_m \vec{H} \right) \\ &= \mu_0 (1 + \chi_m) \vec{H} \\ &= \mu_0 \mu_r \vec{H} \\ &= \mu \vec{H} \end{aligned}$$

6.The magnetic vector potential $\vec{B} = \nabla \times \vec{A}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{l'} \frac{I d\vec{l}\,'}{R} dl' \qquad \vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_{v'}}{R} dv' \qquad \vec{A} = \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{J}_{s'}}{R} ds'$$

$$\vec{A} = \frac{\mu_0 \vec{p}_m \times \vec{a}_R}{4\pi R^2} = \frac{\mu_0 \vec{p}_m \times \vec{R}}{4\pi R^3}$$

7. Magnetic materials are magnetized if $\overrightarrow{M} \neq 0$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\nabla' \times \vec{M}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{M} \times \vec{a}_n}{R} ds'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_{vb}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{J}_{sb}}{R} ds'$$

$\vec{J}_{vb} = \nabla' \times \vec{M}$ **the bound volume current density and**

$\vec{J}_{sb} = \vec{M} \times \vec{a}_n$ **the bound surface current density**

8.Ampere’ circuital law states that the line integral of the magnetic field intensity around a closed path equals the enclosed current

$$\oint_c \vec{H} \bullet d\vec{l} = I \qquad\qquad \nabla \times \vec{H} = \vec{J}_v$$

9.Homework，作业

自己阅读：

5.8 电感

5.9 磁场能量

5.10 磁场力

中文教材：**p254-256**，**T5.2**；**T5.3**；**T5.4**；**T5.5**

T5.19；**T5.20**；**T5.21**；**T5.22**