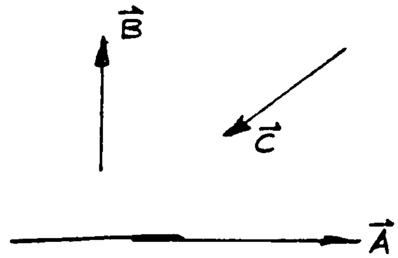
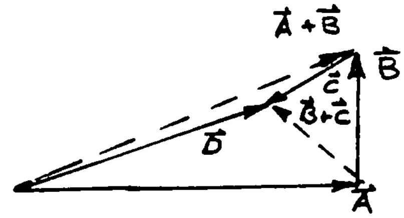
CHAPTER - 2 "VECTOR ANALYSIS"

Exercise 2.1





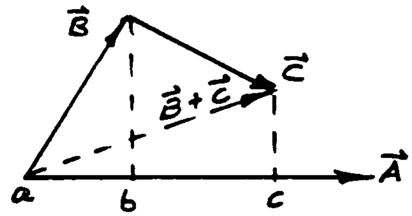
$\vec{D} = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

Exercise 2.2

$\vec{A} \cdot \vec{B} = AB\cos\theta$ if $\vec{A} \neq 0$ and $\vec{B} \neq 0$, then

for $\vec{A} \cdot \vec{B} = 0 \Rightarrow \cos\theta = 0$ or $\theta = \pm \frac{\pi}{2}$.

Exercise 2.3



since $\vec{A} \cdot \vec{B} = AB\cos\theta$, where $B\cos\theta$ is the projection of B onto A.
Thus, $\vec{A} \cdot (\vec{B} + \vec{C})$ is the product of $\vec{A}$ and projection of $\vec{B} + \vec{C}$ on to A, which is ac.
However, projection of $\vec{B}$ is ab and that of $\vec{C}$ is bc. Thus,

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

Exercise 2.4

$$|\vec{A} + \vec{B}|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$
$$= A^2 + B^2 + 2AB\cos\theta$$

If $\vec{A} \perp \vec{B}$, then $\cos\theta = 0$.

Hence,

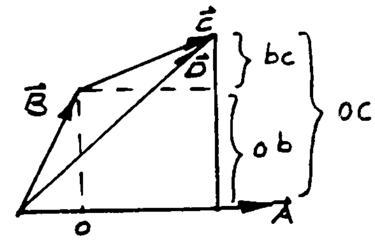
$$|\vec{A} + \vec{B}|^2 = A^2 + B^2 \quad \text{if } \vec{A} \perp \vec{B}$$

Exercise 2.5

$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

$\vec{B} + \vec{C} = \vec{D}$ $\qquad \vec{A} \times \vec{B} = A\ ob\ \vec{a}_n$

$\vec{A} \times \vec{C} = A\ bc\ \vec{a}_n$



$$\vec{A} \times \vec{B} + \vec{A} \times \vec{C} = A[ob + bc]\,\vec{a}_n$$
$$= A\ oc\ \vec{a}_n$$
$$= \vec{A} \times \vec{D} = \vec{A} \times (\vec{B} + \vec{C})$$

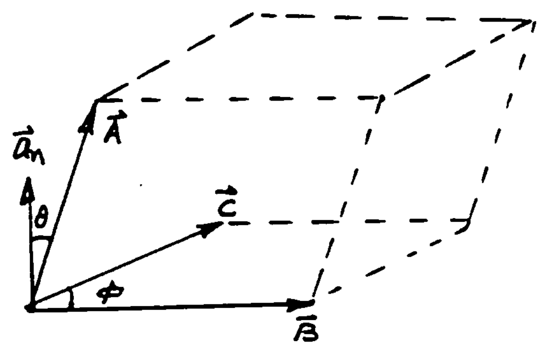
$\vec{a}_n$ is unit normal perpendicular to the plane of paper containing $\vec{A}$, $\vec{B}$, and $\vec{C}$.

Exercise 2.6

Since $|\vec{A} \times \vec{B}| = |AB \sin\theta|$

If $\vec{A} \neq 0$, $\vec{B} \neq 0$ but $|\vec{A} \times \vec{B}| = 0$, then $\sin\theta = 0 \Rightarrow \theta = 0 \Rightarrow \vec{A} \parallel \vec{B}$

Exercise 2.7



$\vec{A} \cdot (\vec{B} \times \vec{C}) = ABC \sin\phi \cos\theta$ is a scalar tripple product. The scalar tripple product can be permuted cyclically without changing its value. This can be visualized by constructing a parallelopiped as shown.

Exercise 2.8 $\quad (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$

Let $\vec{C} \times \vec{D} = \vec{E}$, then $(\vec{A} \times \vec{B}) \cdot \vec{E} = \vec{A} \cdot (\vec{B} \times \vec{E}) = \vec{A} \cdot [\vec{B} \times (\vec{C} \times \vec{D})]$

$$= \vec{A} \cdot [(\vec{B} \cdot \vec{D})\vec{C} - (\vec{B} \cdot \vec{C})\vec{D}]$$

$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

Exercise 2.9

$\vec{A} = 2\vec{a}_x + 0.3\vec{a}_y - 1.5\vec{a}_z$ and $\vec{B} = 10\vec{a}_x + 1.5\vec{a}_y - 7.5\vec{a}_z$

Since $\vec{B} = 5\vec{A}$, $\vec{A}$ and $\vec{B}$ are dependent vectors.

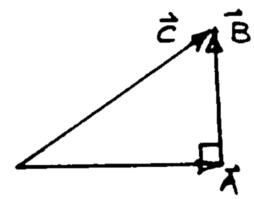
Exercise 2.10 Position vectors for $P(0,-2,1)$ and $Q(-2,0,3)$ are

$$\vec{r}_1 = \overrightarrow{OP} = -2\vec{a}_y + \vec{a}_z \quad \text{and} \quad \vec{r}_2 = \overrightarrow{OQ} = -2\vec{a}_x + 3\vec{a}_z$$

Hence, distance vector from P to Q is

$$\vec{R} = \vec{r}_2 - \vec{r}_1 = -2\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z$$

Exercise 2.11



Note that $\vec{A} + \vec{B} = \vec{C}$. and

$\vec{A}\cdot\vec{B} = 0$. Thus $\vec{A} \perp \vec{B}$ and

$\vec{A}$, $\vec{B}$, and $\vec{C}$ form a right angle triangle.

Exercise 2.12

$$\vec{S} + \vec{G} = 3\vec{a}_x + 4\vec{a}_y + 12\vec{a}_z$$

$$|\vec{S} + \vec{G}| = \sqrt{3^2 + 4^2 + 12^2} = 13$$

$$\vec{a}_{|\vec{S}+\vec{G}|} = \frac{3}{13}\vec{a}_x + \frac{4}{13}\vec{a}_y + \frac{12}{13}\vec{a}_z$$

$$\vec{a}_{|\vec{S}+\vec{G}|}\cdot\vec{a}_x = \cos\theta = \frac{3}{13} \Rightarrow \theta = 76.66^\circ$$

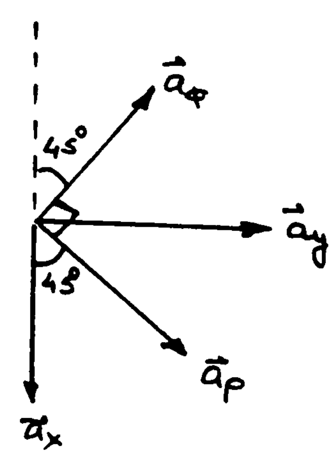
Exercise 2.13 $A_x = A_\rho\cos\phi - A_\phi\sin\phi$, $A_y = A_\rho\sin\phi + A_\phi\cos\phi$

Thus,

$$A_\rho = \frac{\begin{vmatrix} A_x & -\sin\phi \\ A_y & \cos\phi \end{vmatrix}}{\begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix}} = A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = \frac{\begin{vmatrix} \cos\phi & A_x \\ \sin\phi & A_y \end{vmatrix}}{\begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix}} = -A_x\sin\phi + A_y\cos\phi$$

Exercise 2.14



$$\vec{C} = -2\vec{a}_x + 3\vec{a}_y + 4\vec{a}_z$$

$$C_\rho = \vec{C}\cdot\vec{a}_\rho = -2\vec{a}_x\cdot\vec{a}_\rho + 3\vec{a}_y\cdot\vec{a}_\rho + 4\vec{a}_z\cdot\vec{a}_\rho$$

$$= -2\cos45^\circ + 3\cos45^\circ = 0.707$$

$$C_\phi = \vec{C}\cdot\vec{a}_\phi = -2\vec{a}_x\cdot\vec{a}_\phi + 3\vec{a}_y\cdot\vec{a}_\phi + 4\vec{a}_z\cdot\vec{a}_\phi$$

$$= 2\cos45^\circ + 3\cos45^\circ = 3.535$$

$$C_z = \vec{C}\cdot\vec{a}_z = 4$$

$$\vec{C} = 0.707\,\vec{a}_\rho + 3.535\,\vec{a}_\phi + 4\,\vec{a}_z$$

Exercise 2.15 $\quad x = \rho\cos\phi \quad y = \rho\sin\phi \quad \rho = \sqrt{x^2+y^2}$

a. $\vec{F} = \rho\sin\phi\ \vec{a}_\rho - \rho\cos\phi\ \vec{a}_\phi$

Using (2.38),

$$A_x = 2\rho\cos\phi\sin\phi = \frac{2xy}{\sqrt{x^2+y^2}}$$

$$A_y = \rho(\sin^2\phi - \cos^2\phi) = \frac{y^2-x^2}{\sqrt{x^2+y^2}}$$

$$A_z = 0$$

b. $\vec{H} = \frac{1}{\rho}\ \vec{a}_\rho$

$$H_x = \frac{1}{\rho}\cos\phi = \frac{x}{x^2+y^2}$$

$$H_y = \frac{1}{\rho}\sin\phi = \frac{y}{x^2+y^2}$$

$$H_z = 0$$

Exercise 2.16 $\quad$ Transform: $P(1,\pi,0) \Rightarrow P(-1,0,0)$

$Q(0,-\frac{\pi}{2},2) \Rightarrow Q(0,0,2)$

$$\vec{PQ} = \vec{R} = \vec{a}_x + 2\vec{a}_z \Rightarrow R = \sqrt{5} \qquad \vec{QP} = -\vec{PQ} = -\vec{a}_x - 2\vec{a}_z$$

Exercise 2.17 $\quad x = R\sin\theta\cos\phi,\ y = R\sin\theta\sin\phi,\ z = R\cos\theta$

$$A_r = x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta$$
$$= R\sin^2\theta\cos^2\phi + R\sin^2\theta\sin^2\phi + R\cos^2\theta = R$$

$$A_\theta = x\cos\theta\cos\phi + y\cos\theta\sin\phi - z\sin\theta$$
$$= R\sin\theta\cos\theta\cos^2\phi + R\sin\theta\cos\theta\sin^2\phi - R\sin\theta\cos\theta = 0$$

$$A_\phi = -x\sin\phi + y\cos\phi$$
$$= -R\sin\theta\sin\phi\cos\phi + R\sin\theta\sin\phi\cos\phi = 0$$

Hence $\quad \vec{r} = R\,\vec{a}_R$

Exercise 2.18 $\quad \vec{F} = r\,\vec{a}_r + r\tan\theta\,\vec{a}_\theta + r\sin\theta\cos\phi\,\vec{a}_\phi$

$$F_x = r\sin\theta\cos\phi + r\frac{\sin\theta}{\cos\theta}\cos\theta\cos\phi - r\sin\theta\sin\phi\cos\phi$$
$$= 2r\sin\theta\cos\phi - r\sin\theta\cos\phi\sin\phi = 2x - \frac{xy}{\sqrt{x^2+y^2}}$$

$$F_y = r\sin\theta\sin\phi + r\frac{\sin\theta}{\cos\theta}\cos\theta\sin\phi + r\sin\theta\cos^2\phi$$
$$= 2y + \frac{x^2}{\sqrt{x^2+y^2}}$$

$$F_z = r\cos\theta - r\frac{\sin^2\theta}{\cos\theta} = z - \frac{x^2+y^2}{z} = \frac{z^2-x^2-y^2}{z} = 2z - \frac{x^2+y^2+z^2}{z}$$

$$\vec{F} = \left[2x - \frac{xy}{\sqrt{x^2+y^2}}\right]\vec{a}_x + \left[2y + \frac{x^2}{\sqrt{x^2+y^2}}\right]\vec{a}_y + \left[2z - \frac{x^2+y^2+z^2}{z}\right]\vec{a}_z$$

Exercise 2.19 $P(2, \pi/2, 3\pi/4) \Rightarrow P(-1.414, 1.414, 0)$

$Q(10, \pi/4, \pi/2) \Rightarrow Q(0, 7.07, 7.07)$

$$\vec{R} = \vec{PQ} = 1.414\vec{a}_x + 5.656\vec{a}_y + 7.07\vec{a}_z \qquad |\vec{PQ}| = 9.164$$

Exercise 2.20

$\vec{S} = 12\vec{a}_r + 5\vec{a}_\theta + \pi\vec{a}_\phi$ @ $(2, \pi, \pi/2) \Rightarrow \vec{S} = -\pi\vec{a}_x - 5\vec{a}_y - 12\vec{a}_z$

$\vec{T} = 2\vec{a}_r + 0.5\pi\vec{a}_\theta$ @ $(5, \pi/2, \pi/2) \Rightarrow \vec{T} = 2\vec{a}_y - 0.5\pi\vec{a}_z$

$\vec{S} + \vec{T} = -\pi\vec{a}_x - 3\vec{a}_y - 13.57\vec{a}_z$, $\quad \vec{S}\cdot\vec{T} = 0 - 10 + 6\pi = 8.85$

$\vec{S}\times\vec{T} = 31.85\vec{a}_x - 4.93\vec{a}_y - 6.28\vec{a}_z \Rightarrow \vec{a}_{\vec{S}\times\vec{T}} = \dfrac{\vec{S}\times\vec{T}}{|\vec{S}\times\vec{T}|}$

$= 0.97\vec{a}_x - 0.15\vec{a}_y - 0.19\vec{a}_z$

Exercise 2.21

since $g = g(u(t), v(t), s(t))$,

$$\frac{dg}{dt} = \frac{\partial g}{\partial u}\frac{du}{dt} + \frac{\partial g}{\partial v}\frac{dv}{dt} + \frac{\partial g}{\partial s}\frac{ds}{dt}$$

Exercise 2.22 For $G(x, y, z, t)$

$$\frac{dG}{dt} = \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} + \frac{\partial G}{\partial z}\frac{dz}{dt} + \frac{\partial G}{\partial t}$$

Exercise 2.23

$$\frac{\partial \vec{F}}{\partial y} = \lim_{\Delta y \to 0} \frac{\vec{F}(x, y+\Delta y, z) - \vec{F}(x, y, z)}{\Delta y}$$

$$\frac{\partial \vec{F}}{\partial z} = \lim_{\Delta z \to 0} \frac{\vec{F}(x, y, z+\Delta z) - \vec{F}(x, y, z)}{\Delta z}$$

Exercise 2.24 $\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z \Rightarrow \vec{dr} = dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z$

Exercise 2.25 $\vec{l} = \rho\vec{a}_\rho + z\vec{a}_z = \rho\cos\phi\,\vec{a}_x + \rho\sin\phi\,\vec{a}_y + z\vec{a}_z$

$\vec{dl} = d\rho\cos\phi\,\vec{a}_x - \rho\sin\phi\, d\phi\,\vec{a}_x + d\rho\sin\phi\,\vec{a}_y + \rho\cos\phi\, d\phi\,\vec{a}_y + dz\,\vec{a}_z$

$= (\cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y)\,d\rho + (\cos\phi\,\vec{a}_y - \sin\phi\,\vec{a}_x)\,\rho\, d\phi + dz\,\vec{a}_z$

$= d\rho\,\vec{a}_\rho + \rho\, d\phi\,\vec{a}_\phi + dz\,\vec{a}_z$

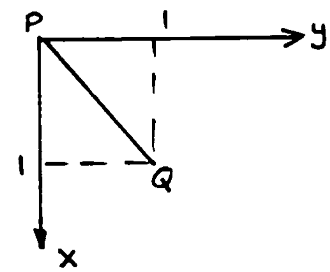
Exercise 2.26

$$\vec{l} = r\vec{a}_r = r\sin\theta\cos\phi\,\vec{a}_x + r\sin\theta\sin\phi\,\vec{a}_y + r\cos\theta\,\vec{a}_z$$

$$d\vec{l} = \vec{a}_x[\sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi]$$
$$+ \vec{a}_y[\sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi]$$
$$+ \vec{a}_z[\cos\theta\,dr - r\sin\theta\,d\theta]$$
$$= [\sin\theta\cos\phi\,\vec{a}_x + \sin\theta\sin\phi\,\vec{a}_y + \cos\theta\,\vec{a}_z]dr$$
$$+ [\cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z]r\,d\theta$$
$$+ [-\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y]r\sin\theta\,d\phi$$
$$= dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi$$

Exercise 2.27

$g = 20xy \quad d\vec{l} = dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z$



$z=0$
$dz=0$
$y=x$
$dy=dx$

a) $$\int_P^Q g\,d\vec{l} = \int_P^Q 20xy\,(dx\,\vec{a}_x + dy\,\vec{a}_y)$$

$$= \int_{x=0}^{1} 20x^2(\vec{a}_x+\vec{a}_y)\,dx = \frac{20}{3}[\vec{a}_x+\vec{a}_y]$$

b) $$\int_{x=0}^{1} 20x(4x^2)[dx\,\vec{a}_x + 8x\,dx\,\vec{a}_y] \qquad y = 4x^2 \quad dy = 8x\,dx$$

$$= 20\,\vec{a}_x + 128\,\vec{a}_y$$

Exercise 2.28 $\quad \vec{P}\cdot d\vec{l} = (b\,\vec{a}_\rho)\cdot(b\,d\phi\,\vec{a}_\phi) = 0 \quad \because \vec{a}_\rho\cdot\vec{a}_\phi = 0$

Hence $\oint_c \vec{P}\cdot d\vec{l} = 0$

Exercise 2.29 $\quad \vec{r}\cdot d\vec{s} = (b\,\vec{a}_r)\cdot(\vec{a}_r\,b^2\sin\theta\,d\theta\,d\phi)$

$$= b^3\sin\theta\,d\theta\,d\phi$$

$$\oint \vec{r}\cdot d\vec{s} = b^3\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= 4\pi b^3$$

Exercise 2.26

$$\vec{l} = r\vec{a}_r = r\sin\theta\cos\phi\,\vec{a}_x + r\sin\theta\sin\phi\,\vec{a}_y + r\cos\theta\,\vec{a}_z$$

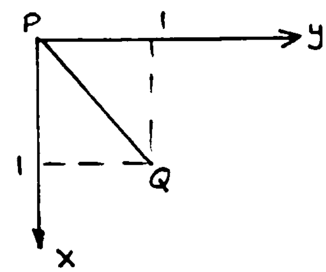
$$d\vec{l} = \vec{a}_x[\sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi]$$
$$+ \vec{a}_y[\sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi]$$
$$+ \vec{a}_z[\cos\theta\,dr - r\sin\theta\,d\theta]$$
$$= [\sin\theta\cos\phi\,\vec{a}_x + \sin\theta\sin\phi\,\vec{a}_y + \cos\theta\,\vec{a}_z]\,dr$$
$$+ [\cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z]\,r\,d\theta$$
$$+ [-\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y]\,r\sin\theta\,d\phi$$
$$= dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi$$

Exercise 2.27

$g = 20xy \qquad d\vec{l} = dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z$

a) $$\int_P^Q g\,d\vec{l} = \int_P^Q 20xy\,(dx\,\vec{a}_x + dy\,\vec{a}_y)$$

$z = 0$
$dz = 0$
$y = x$
$dy = dx$



$$= \int_{x=0}^{1} 20x^2\,(\vec{a}_x + \vec{a}_y)\,dx = \frac{20}{3}[\vec{a}_x + \vec{a}_y]$$

b) $$\int_{x=0}^{1} 20x\,(4x^2)\,[dx\,\vec{a}_x + 8x\,dx\,\vec{a}_y] \qquad y = 4x^2 \quad dy = 8x\,dx$$

$$= 20\,\vec{a}_x + 128\,\vec{a}_y$$

Exercise 2.28

$\vec{P}\cdot d\vec{l} = (b\,\vec{a}_\rho)\cdot(b\,d\phi\,\vec{a}_\phi) = 0 \quad \because \vec{a}_\rho\cdot\vec{a}_\phi = 0$

Hence $\oint_c \vec{P}\cdot d\vec{l} = 0$

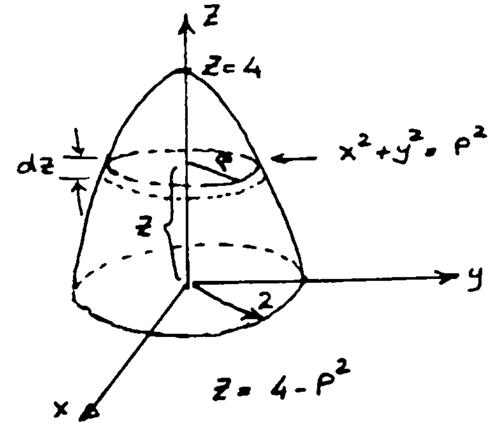
Exercise 2.29

$$\vec{r}\cdot d\vec{s} = (b\,\vec{a}_r)\cdot(\vec{a}_r\,b^2\sin\theta\,d\theta\,d\phi)$$
$$= b^3\sin\theta\,d\theta\,d\phi$$

$$\oint \vec{r}\cdot d\vec{s} = b^3\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_0^{2\pi} d\phi$$

$$= 4\pi b^3$$

Exercise 2.30



$z = 0 \Rightarrow x^2 + y^2 = 4$

$x = 0,\ y = 0 \Rightarrow z = 4$

$$V = \int_\rho \int_\phi \int_z \rho\, d\rho\, d\phi\, dz \qquad \int_{\phi=0}^{2\pi} d\phi = 2\pi$$

$$= 2\pi \int_0^2 \rho\, d\rho \int_0^{4-\rho^2} dz$$

$$= 8\pi$$

Exercise 2.31

$df = \nabla f \cdot \vec{dl}$

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \quad ①$$

$$\left.\begin{aligned} \vec{a}_x &= \cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi \\ a_y &= \sin\phi\, \vec{a}_\rho + \cos\phi\, \vec{a}_\phi \end{aligned}\right\} ②$$

$x = \rho\cos\phi \Rightarrow dx = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$

$y = \rho\sin\phi \Rightarrow dy = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$

Thus, $d\rho = \cos\phi\, dx + \sin\phi\, dy$ and $d\phi = -\frac{1}{\rho}\sin\phi\, dx + \frac{1}{\rho}\cos\phi\, dy$ ③

But $d\rho = \frac{\partial\rho}{\partial x} dx + \frac{\partial\rho}{\partial y} dy$ and $d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy$ ④

From ③ and ④

$\frac{\partial\rho}{\partial x} = \cos\phi$, $\frac{\partial\rho}{\partial y} \sin\phi$, $\frac{\partial\phi}{\partial x} = -\frac{1}{\rho}\sin\phi$, and $\frac{\partial\phi}{\partial y} = \frac{1}{\rho}\cos\phi$ ⑤

However,

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial x} = \cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial y} = \sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi} \end{aligned}\right\} ⑥$$

substitute ② and ⑥ in ①

$$\nabla = (\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi)\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$+ (\sin\phi\, \vec{a}_\rho + \cos\phi\, \vec{a}_\phi)\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}\right) + \vec{a}_z\frac{\partial}{\partial z}$$

$$= \vec{a}_\rho\frac{\partial}{\partial\rho} + \vec{a}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \vec{a}_z\frac{\partial}{\partial z}$$

Exercise 2.32 $\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z = \rho\vec{a}_\rho + z\vec{a}_z = r\vec{a}_r$

$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2} = r$

$\nabla r = \frac{\partial r}{\partial x}\vec{a}_x + \frac{\partial r}{\partial y}\vec{a}_y + \frac{\partial r}{\partial z}\vec{a}_z$ However, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$= \frac{x}{r}\vec{a}_x + \frac{y}{r}\vec{a}_y + \frac{z}{r}\vec{a}_z = \frac{\vec{r}}{r} = \vec{a}_r$

$\nabla r = \frac{\partial r}{\partial r}\vec{a}_r = \vec{a}_r$

Exercise 2.33 $f = 12x^2 + yz^2 \Rightarrow \nabla f = 24x\vec{a}_x + z^2\vec{a}_y + 2yz\vec{a}_z$

@ $P(-1,0,1)$, $\nabla f = -24\vec{a}_x + \vec{a}_y$ $\quad Q = (1,1,1)$

$(\nabla f)_x = \nabla f \cdot \vec{a}_x = 24x = -24$ @ $P(-1,0,1)$ $\quad \vec{PQ} = 2\vec{a}_x + \vec{a}_y$

$(\nabla f)_y = \nabla f \cdot \vec{a}_y = z^2 = 1$ @ $P(-1,0,1)$ $\quad \vec{a}_{PQ} = \frac{2}{\sqrt{5}}\vec{a}_x + \frac{1}{\sqrt{5}}\vec{a}_y$

$(\nabla f)_z = \nabla f \cdot \vec{a}_z = 2yz = 0$ @ $P(-1,0,1)$

$(\nabla f)_{PQ} = \nabla f \cdot \vec{a}_{PQ} = \frac{48}{\sqrt{5}}x + \frac{1}{\sqrt{5}}z^2 = -21.02$ at $P(-1,0,1)$

Exercise 2.34 $r = \rho\vec{a}_\rho + z\vec{a}_z = r\vec{a}_r$

$\nabla\cdot\vec{r} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2) + \frac{\partial}{\partial z}z = 3$ $\qquad \nabla\cdot\vec{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3) = 3$

Exercise 2.35 $\vec{F} = -xy\,\vec{a}_x + 3x^2yz\,\vec{a}_y + xz^3\,\vec{a}_z$

$\nabla\cdot\vec{F} = -y + 3x^2z + 3z^2x = 19$ at $P(1,-1,2)$

Exercise 2.36 $\nabla\cdot\vec{r} = 3$ $\qquad \vec{r} = r\,\vec{a}_r$

$$\nabla\cdot(r^2\vec{a}_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) = 4r = (2+2)\,r^{2-1}$$

$$\nabla\cdot(r^3\vec{a}_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^5) = 5r^2 = (3+2)\,r^{3-1}$$

$$\nabla\cdot(r^4\vec{a}_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^6) = 6r^3 = (4+2)\,r^{4-1}$$

$$\vdots$$

$$\nabla\cdot(r^n\vec{a}_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2}) = (n+2)\,r^{n-1}$$

Exercise 2.37 $\quad \vec F = x\,\vec a_x + xy\,\vec a_y + xyz\,\vec a_z \;\Rightarrow\; \nabla\cdot\vec F = 1 + x + xy$

$$\int_v \nabla\cdot\vec F\,dv = \int_v (1+x+xy)\,dv = \int_v (1 + r\sin\theta\cos\phi + r^2\sin^2\theta\,\sin\phi\cos\phi)\,dv$$

$$= \int_0^2 r^2dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\phi + \int_0^2 r^3dr\int_0^\pi \sin^2\theta\,d\theta\,\overbrace{\int_0^{2\pi}\cancel{\cos\phi\,d\phi}}^{0}$$

$$+\int_0^2 r^4dr\int_0^\pi \sin^3\theta\,d\theta\,\overbrace{\int_0^{2\pi}\cancel{\sin\phi\cos\phi\,d\phi}}^{0} = \frac{32}{3}\pi$$

$\vec F\cdot\vec{ds} = F_r\, r^2\sin\theta\,d\theta\,d\phi \quad$ at $r = 2$

$x = 2\sin\theta\cos\phi$

$y = 2\sin\theta\sin\phi$

$z = 2\cos\theta$

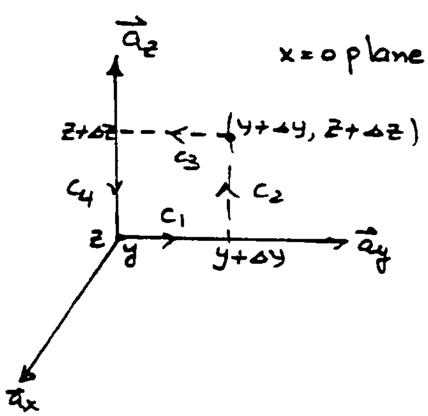
$$F_r = x\sin\theta\cos\phi + xy\sin\theta\sin\phi + xyz\cos\theta$$
$$= 2\sin^2\theta\cos^2\phi + 4\sin^3\theta\sin^2\phi\cos\phi$$
$$+ 8\sin^2\theta\cos^2\theta\sin\phi\cos\phi$$

$$\oint_s \vec F\cdot\vec{ds} = 8\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi}\cos^2\phi\,d\phi + 16\int_0^\pi \sin^4\theta\,d\theta\,\overbrace{\int_0^{2\pi}\cancel{\sin^2\phi\cos\phi\,d\phi}}^{0}$$

$$+ 32\int_0^\pi \sin^3\theta\cos^2\theta\,d\theta\,\overbrace{\int_0^{2\pi}\cancel{\sin\phi\cos\phi\,d\phi}}^{0}$$

$$= 8\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta\int_0^{2\pi}\frac{1}{2}(1+\cos 2\phi)\,d\phi = \frac{32}{3}\pi$$

Exercise 2.38 $\qquad \vec F = F_x\,\vec a_x + F_y\,\vec a_y + F_z\,\vec a_z$



$$\int_{c_1}\vec F\cdot\vec{dl} = F_y\,\Delta y\Big|_{\text{at } z}$$

$$\int_{c_2}\vec F\cdot\vec{dl} = F_z\,\Delta z\Big|_{\text{at } y+\Delta y}$$

$$\int_{c_3}\vec F\cdot\vec{dl} = -F_y\,\Delta y\Big|_{\text{at } z+\Delta z}$$

$$\int_{c_4}\vec F\cdot\vec{dl} = -F_z\,\Delta z\Big|_{\text{at } y}$$

$$F_z\,\Delta z\Big|_{y+\Delta y} - F_z\,\Delta z\Big|_{y} = \frac{\partial F_z}{\partial y}\Delta y\,\Delta z$$

$$F_y\,\Delta y\Big|_{z+\Delta z} - F_y\,\Delta y\Big|_{z} = \frac{\partial F_y}{\partial z}\Delta y\,\Delta z$$

$$\oint_c \vec F\cdot\vec{dl} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta y\,\Delta z$$

Thus, $(\nabla\times\vec F)\cdot\vec a_x = \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}$

Similarly, we can prove that

$(\nabla\times\vec F)\cdot\vec a_y = \dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}$

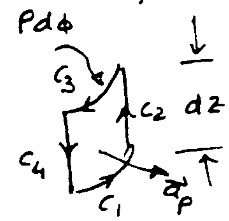
and

$(\nabla\times\vec F)\cdot\vec a_z = \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}$

Exercise 2.39 Cylindrical coordinate system

$\vec{a}_\rho$- component

$$\int_{C_1} \vec{F}\cdot d\vec{l} = F_\phi \, \rho d\phi \Big|_z \qquad d\vec{s} = \rho d\phi dz \, \vec{a}_\rho$$



$$\int_{C_3} \vec{F}\cdot d\vec{l} = -F_\phi \, \rho d\phi \Big|_{z+dz} = -\left[F_\phi + \frac{\partial F_\phi}{\partial z} dz\right] \rho d\phi$$

$$\int_{C_1} \vec{F}\cdot d\vec{l} + \int_{C_3} \vec{F}\cdot d\vec{l} = -\frac{\partial F_\phi}{\partial z} \rho d\phi \, dz$$

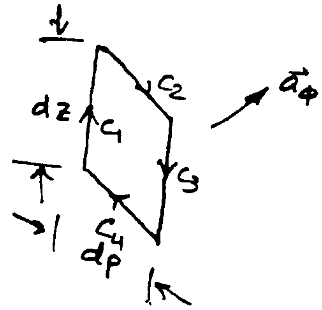
$$\int_{C_4} \vec{F}\cdot d\vec{l} = -F_z \, dz \Big|_\phi$$

$$\int_{C_2} \vec{F}\cdot d\vec{l} = F_z \, dz \Big|_{\phi+d\phi} = \left(F_z + \frac{\partial F_z}{\partial \phi} d\phi\right) dz$$

$$\int_{C_2} \vec{F}\cdot d\vec{l} + \int_{C_4} \vec{F}\cdot d\vec{l} = \frac{\partial F_z}{\partial \phi} d\phi \, dz \Rightarrow \oint_C \vec{F}\cdot d\vec{l} = \frac{\partial F_z}{\partial \phi} d\phi dz - \frac{\partial F_\phi}{\partial z} \rho d\phi dz$$

Thus, $$(\nabla \times \vec{F})\cdot \vec{a}_\rho = \lim_{ds \to 0} \frac{\oint_C \vec{F}\cdot d\vec{l}}{ds} = \frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}$$

$\vec{a}_\phi$ - component:



$$\int_{C_1} \vec{F}\cdot d\vec{l} = F_z \, dz \Big|_\rho \qquad \int_{C_4} \vec{F}\cdot d\vec{l} = -F_\rho \, d\rho \Big|_z$$

$$\int_{C_3} \vec{F}\cdot d\vec{l} = -F_z \, dz \Big|_{\rho+d\rho} = -\left[F_z + \frac{\partial F_z}{\partial \rho} d\rho\right] dz$$

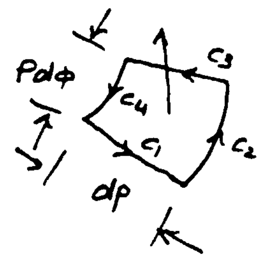
$$\int_{C_2} \vec{F}\cdot d\vec{l} = F_\rho \, d\rho \Big|_{z+dz} = \left[F_\rho + \frac{\partial F_\rho}{\partial z} dz\right] d\rho$$

$$\oint_C \vec{F}\cdot d\vec{l} = \frac{\partial F_\rho}{\partial z} dz d\rho - \frac{\partial F_z}{\partial \rho} dz d\rho \qquad d\vec{s} = dz \, d\rho \, \vec{a}_\phi$$

Thus $$(\nabla \times \vec{F})\cdot \vec{a}_\phi = \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}$$

$$d\vec{s} = \rho d\rho d\phi \, \vec{a}_z$$

$\vec{a}_z$- component:



$$\int_{C_1} \vec{F}\cdot d\vec{l} = F_\rho \, d\rho \Big|_\phi$$

$$\int_{C_3} \vec{F}\cdot d\vec{l} = -F_\rho \, d\rho \Big|_{\phi+d\phi} = -\left[F_\rho + \frac{\partial F_\rho}{\partial \phi} d\phi\right] d\rho$$

$$\int_{C_2} \vec{F}\cdot d\vec{l} = F_\phi (\rho+d\rho) d\phi = \left(F_\phi + \frac{\partial F_\phi}{\partial \rho} d\rho\right)(\rho + d\rho) d\phi$$

$$\int_{C_4} \vec{F}\cdot d\vec{l} = -F_\phi \, \rho d\phi$$

$$\oint_C \vec{F}\cdot d\vec{l} = F_\phi \, d\rho d\phi + \rho \frac{\partial F_\phi}{\partial \rho} d\rho d\phi - \frac{\partial F_\rho}{\partial \phi} d\rho d\phi$$

$$= \frac{\partial}{\partial \rho}(\rho F_\phi) d\rho d\phi - \frac{\partial F_\rho}{\partial \phi} d\rho d\phi$$

Thus $$(\nabla \times \vec{F})\cdot \vec{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\phi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}$$

Exercise 2.40 $\vec{F} = (x/r)\,\vec{a}_x \qquad r = \sqrt{x^2+y^2+z^2}$

$$\nabla\times\vec{F} = \vec{a}_y \frac{\partial F_x}{\partial z} - \vec{a}_z \frac{\partial F_x}{\partial y} = \frac{x}{r^3}\left[-z\,\vec{a}_y + y\,\vec{a}_z\right]$$

Exercise 2.41 Let $\vec{F} = F_x\vec{a}_x + F_y\vec{a}_y + F_z\vec{a}_z$, then

$$\nabla\cdot(\nabla\times\vec{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \frac{\partial^2 F_z}{\partial x\,\partial y} - \frac{\partial^2 F_y}{\partial x\,\partial z} + \frac{\partial^2 F_x}{\partial y\,\partial z} - \frac{\partial^2 F_z}{\partial y\,\partial x} + \frac{\partial^2 F_y}{\partial z\,\partial x} - \frac{\partial^2 F_x}{\partial z\,\partial y} = 0$$

Exercise 2.42 Let us use cylindrical coordinate system

$$\nabla f = \frac{\partial f}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\vec{a}_\phi + \frac{\partial f}{\partial z}\vec{a}_z$$

$$\nabla\times\nabla f = \frac{1}{\rho}\begin{vmatrix} \vec{a}_\rho & \rho\,\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial z} \end{vmatrix} = \frac{1}{\rho}\vec{a}_\rho\left(\frac{\partial^2 f}{\partial\phi\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial\phi}\right) + \vec{a}_\phi\left(\frac{\partial^2 f}{\partial z\,\partial\rho} - \frac{\partial^2 f}{\partial\rho\,\partial z}\right) + \frac{1}{\rho}\vec{a}_z\left(\frac{\partial^2 f}{\partial\rho\,\partial\phi} - \frac{\partial^2 f}{\partial\phi\,\partial\rho}\right) = 0$$

Exercise 2.43

$$\vec{F} = 10\cos\theta\,\vec{a}_r - 10\sin\theta\,\vec{a}_\theta \Rightarrow \nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 10\cos\theta & -10r\sin\theta & 0 \end{vmatrix} = 0$$

Thus $\int_S \nabla\times\vec{F}\cdot d\vec{s} = 0$

On the circular path c, $\vec{dl} = 2\,d\phi\,\vec{a}_\phi$ at $r=2$

$$\vec{F}\cdot\vec{dl} = (10\cos\theta\,\vec{a}_r - 10\sin\theta\,\vec{a}_\theta)\cdot 2\,d\phi\,\vec{a}_\phi = 0$$

Hence $\oint_C \vec{F}\cdot\vec{dl} = 0$

Exercise 2.44 $\quad g = 25x^2yz + 12xy^2$

$$\nabla g = (50xyz + 12y^2)\vec{a}_x + (25x^2z + 24xy)\vec{a}_y + 25x^2y\,\vec{a}_z$$

$$\nabla\cdot\nabla g = 50yz + 24x$$

$$\nabla^2 g = \frac{\partial^2}{\partial x^2}(25x^2yz + 12y^2) + \frac{\partial^2}{\partial y^2}(25x^2yz + 12xy^2) + \frac{\partial^2}{\partial z^2}(25x^2yz + 12xy^2)$$

$$= 50yz + 24x$$

Exercise 2.45 $\quad f = 2x^2y^3 + 3yz^3 \Rightarrow \nabla^2 f = 4y^3 + 12x^2y + 18yz$

$$\nabla f = 4xy^3\vec{a}_x + (6x^2y^2 + 3z^3)\vec{a}_y + 9yz^2\vec{a}_z$$

$$\nabla\cdot\nabla f = 4y^3 + 12x^2y + 18yz$$

Exercise 2.46 $\quad h = \rho^2\sin 2\phi + z^3\cos\phi$

$$\nabla h = 2\rho\sin 2\phi\,\vec{a}_\rho + \left(2\rho\cos 2\phi - \frac{z^3}{\rho}\sin\phi\right)\vec{a}_\phi + 3z^2\cos\phi\,\vec{a}_z$$

$$\nabla\cdot\nabla h = 4\sin 2\phi - 4\sin 2\phi - \frac{z^3}{\rho^2}\cos\phi + 6z\cos\phi = 6z\cos\phi - \frac{z^3}{\rho^2}\cos\phi$$

$$\nabla^2 h = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}(\rho^2\sin 2\phi + z^3\cos\phi)\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}(\rho^2\sin 2\phi + z^3\cos\phi) + \frac{\partial^2}{\partial z^2}(z^3\cos\phi)$$

$$= 4\sin 2\phi - 4\sin 2\phi - \frac{z^3}{\rho^2}\cos\phi + 6z\cos\phi = 6z\cos\phi - \frac{z^3}{\rho^2}\cos\phi$$

Exercise 2.47 $\quad \Phi = k[\ln b - \ln\rho]$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}(k\ln b - k\ln\rho)\right] + \cancelto{0}{\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}} + \cancelto{0}{\frac{\partial^2\Phi}{\partial z^2}}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(-\frac{k}{\rho}\right)\right] = 0$$

Exercise 2.48 $\quad \nabla\Phi = \frac{\partial\Phi}{\partial\rho}\vec{a}_\rho = -\frac{k}{\rho}\vec{a}_\rho \qquad \nabla^2\Phi = 0$

$$\cancelto{0}{\int_v \Phi\nabla^2\Phi\,dv} + \int_v |\nabla\phi|^2\,dv = \oint_s \Phi\nabla\Phi\cdot\vec{ds}$$

$$\int_v |\nabla\phi|^2\,dv = k^2\int_a^b \frac{1}{\rho^2}\rho\,d\rho\int_0^{2\pi}d\phi\int_0^1 dz = 2\pi k^2\ln\left(\frac{b}{a}\right) \quad ①$$

at surface $\rho = b \quad \Phi = k\ln(b/b) = 0 \Rightarrow$ no contribution from this surface

at surface $\rho = a \quad \Phi = k\ln(b/a)$ and $\nabla\Phi = -\frac{k}{a}\vec{a}_\rho$, $\vec{ds} = -a\,d\phi\,dz\,\vec{a}_\rho$

$$\oint_s \Phi\nabla\Phi\cdot\vec{ds} = k\ln\left(\frac{b}{a}\right)\int_0^{2\pi}\frac{k}{a}a\,d\phi\int_0^1 dz = 2\pi k^2\ln(b/a) \quad ②$$

Note: ① = ② Green's Theorem satisfied.

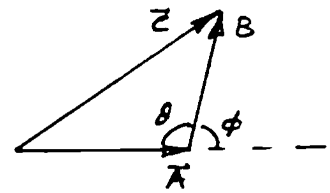
Problem 2.1

$$\vec{C} = \vec{A} + \vec{B}$$

$$C^2 = \vec{C}\cdot\vec{C} = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$$

$$= A^2 + B^2 + 2AB\cos\phi$$

$$= A^2 + B^2 - 2AB\cos\theta$$



Problem 2.2 $\vec{D} = \vec{B}\times\vec{C}$ is a vector normal to $\vec{B}$ and $\vec{C}$. Thus, $\vec{D}\perp\vec{A}$. Hence $\vec{D}\cdot\vec{A} = 0$

Problem 2.3 $\vec{R} = (x-2)\vec{a}_x + (y-3)\vec{a}_y + (z-4)\vec{a}_z$

$$\vec{R}\cdot\vec{R} = R^2 = (x-2)^2 + (y-3)^2 + (z-4)^2$$

R is the radius of the sphere.

Problem 2.4

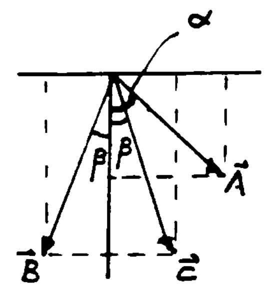
$\beta < \alpha$

$$\vec{A} = \cos\alpha\,\vec{a}_x + \sin\alpha\,\vec{a}_y$$

$$A = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$$

$$\vec{B} = \cos\beta\,\vec{a}_x - \sin\beta\,\vec{a}_y \Rightarrow B = 1$$

$$\vec{C} = \cos\beta\,\vec{a}_x + \sin\beta\,\vec{a}_y \Rightarrow C = 1$$



$$\vec{A}\times\vec{B} = -\vec{a}_z AB\sin(\alpha+\beta)$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & -\sin\beta & 0 \end{vmatrix}$$

$$= -\vec{a}_z[\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\vec{A}\times\vec{C} = -\vec{a}_z AC\sin(\alpha-\beta)$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$$= -\vec{a}_z[\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

Hence, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Problem 2.5 $\vec{A} = \vec{a}_x + \vec{a}_y + \vec{a}_z$, $\vec{B} = 4\vec{a}_x + 4\vec{a}_y + \vec{a}_z$

$$\vec{D} = \vec{B} - \vec{A} = 3\vec{a}_x + 3\vec{a}_y$$

$$D = \sqrt{3^2 + 3^2} = 4.24$$

Problem 2.6 $\vec{A} = 3\vec{a}_x + 2\vec{a}_y - \vec{a}_z$, $\vec{B} = \vec{a}_x - 2\vec{a}_y + 3\vec{a}_z$

$\vec{A}+\vec{B} = 4\vec{a}_x + 2\vec{a}_z$ $\quad \vec{A}\cdot\vec{B} = 3-4-3 = -4$

$\vec{C} = \vec{A}\times\vec{B} = 4\vec{a}_x - 10\vec{a}_y - 8\vec{a}_z = C\,\vec{a}_n$ $\quad C = \sqrt{4^2+10^2+8^2} = 13.416$

$\vec{a}_n = 0.298\,\vec{a}_x - 0.745\,\vec{a}_y - 0.596\,\vec{a}_z$, $\quad A = 3.742, \quad B = 3.742$

$$\left.\begin{array}{l} \cos\alpha = \dfrac{\vec{A}\cdot\vec{B}}{AB} = -0.286 \Rightarrow \alpha = 106.6^\circ \text{ or } \alpha = -106.6^\circ \\ \sin\alpha = \dfrac{|A\times B|}{AB} = \dfrac{C}{AB} = 0.958 \Rightarrow \alpha = 73.4^\circ \text{ or } \alpha = 106.6^\circ \end{array}\right\} \Rightarrow \alpha = 106.6^\circ$$

Scalar projection of $\vec{A}$ onto B: $\vec{A}\cdot\vec{a}_B = \dfrac{\vec{A}\cdot\vec{B}}{B} = -1.069$

Vector projection of A onto B: $-1.069\,\vec{a}_B = -1.069\,\dfrac{\vec{B}}{B}$

$= -0.286[\vec{a}_x - 2\vec{a}_y + 3\vec{a}_z]$

Problem 2.7 $\vec{r}_1 = \vec{OP} = 5\vec{a}_x + 12\vec{a}_y + \vec{a}_z$, $\vec{r}_2 = \vec{OQ} = 2\vec{a}_x - 3\vec{a}_y + \vec{a}_z$

$\vec{PQ} = \vec{r} = \vec{r}_2 - \vec{r}_1 = -3\vec{a}_x - 15\vec{a}_y \Rightarrow r = PQ = 15.3$

since $\vec{PQ}$ has no z-component, it is in the xy plane.

P(5, 12, 1) and Q(2, -3, 1).

Problem 2.8 since $\vec{A}+\vec{C} = \vec{B}$ they form a triangle.

since $\vec{A}\cdot\vec{C} = 0$, $\vec{A}\perp C$.

Area $= \frac{1}{2}|\vec{A}\times\vec{C}| = \frac{1}{2}|-5\vec{a}_x - 5\vec{a}_y - 20\vec{a}_z| = 10.6$

Problem 2.9 $\vec{A}\cdot\vec{B} = 30 + 10 - 40 = 0 \Rightarrow \vec{A}\perp\vec{B}$

Problem 2.10

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} -2 & -3 & 1 \\ 2 & -5 & 3 \\ 4 & 2 & 6 \end{vmatrix}$$

$= 48$

Problem 2.11

$\vec{C} = C\,\vec{a}_n = \vec{A}\times\vec{B}$

$= 2\vec{a}_x + 6\vec{a}_y + 10\vec{a}_z$

$C = \sqrt{2^2+6^2+10^2} = 11.832$

$\vec{a}_n = \dfrac{\vec{C}}{C} = 0.17\vec{a}_x + 0.51\vec{a}_y + 0.85\vec{a}_z$

Problem 2.12 $\quad \vec{A} = \vec{PQ} = 2\,\vec{a}_x + \vec{a}_y + 4\vec{a}_z\ , \quad \vec{B} = \vec{PS} = 4\,\vec{a}_x + 6\vec{a}_y + 8\,\vec{a}_z$

$$\vec{A}\times\vec{B} = -16\,\vec{a}_x + 8\,\vec{a}_z \Rightarrow \text{Area} = \tfrac{1}{2}|\vec{A}\times\vec{B}| = 8.94$$

Problem 2.13 $\quad \vec{A} = 4\,\vec{a}_x - 3\,\vec{a}_y + \vec{a}_z\ , \quad \vec{B} = 2\,\vec{a}_x + \vec{a}_y - \vec{a}_z$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{8-3-1}{AB} = 0.32 \qquad A = 5.1\ ,\quad B = 2.45$$

$$\theta = 71.33^\circ$$

Problem 2.14 $\quad \vec{A} = 3\,\vec{a}_\rho + 5\,\vec{a}_\phi - 4\,\vec{a}_z\ , \quad \vec{B} = 2\,\vec{a}_\rho + 4\,\vec{a}_\phi + 3\,\vec{a}_z$

$$\vec{A}+\vec{B} = 5\,\vec{a}_\rho + 9\,\vec{a}_\phi - \vec{a}_z \qquad A = 7.071 \qquad B = 5.385$$

$$\vec{A}\cdot\vec{B} = 6 + 20 - 12 = 14 \qquad \cos\alpha = \frac{\vec{A}\cdot\vec{B}}{AB} = 0.368 \Rightarrow \alpha = 68.43^\circ$$

$$\vec{A}\times\vec{B} = 31\,\vec{a}_\rho - 17\,\vec{a}_\phi + 2\,\vec{a}_z$$

$$|\vec{A}\times\vec{B}| = 35.412 \qquad \vec{a}_n = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = 0.875\,\vec{a}_\rho - 0.48\,\vec{a}_\phi + 0.056\,\vec{a}_z$$

Scalar projection of $\vec{A}$ onto $\vec{B}$: $\vec{A}\cdot\vec{a}_B = \dfrac{\vec{A}\cdot\vec{B}}{B} = 2.6$

Vector projection of $\vec{A}$ onto $\vec{B}$: $2.6\,\vec{a}_B = 0.483[2\,\vec{a}_\rho + 4\vec{a}_\phi + 3\,\vec{a}_z]$

Problem 2.15 $\quad P(5, 30^\circ, 5) = P(5\cos 30^\circ, 5\sin 30^\circ, 5) = P(4.33, 2.5, 5)$

$$Q(2, 60^\circ, 4) = Q(2\cos 60^\circ, 2\sin 60^\circ, 4) = Q(1, 1.732, 4)$$

$$\vec{r} = \vec{PQ} = -3.33\,\vec{a}_x - 0.768\,\vec{a}_y - \vec{a}_z \Rightarrow r = PQ = 3.56$$

Problem 2.16 $\quad \vec{A} = 2\,\vec{a}_\rho + 3\,\vec{a}_\phi$ @ $P(1, 90^\circ, 2) \Rightarrow \vec{A} = 2\,\vec{a}_y - 3\,\vec{a}_x$

$$\vec{B} = -3\,\vec{a}_\rho + 10\,\vec{a}_z \ @\ (2, \pi, 3) \Rightarrow \vec{B} = 3\,\vec{a}_x + 10\,\vec{a}_z$$

$$\vec{A}+\vec{B} = 2\,\vec{a}_y + 10\,\vec{a}_z \qquad \vec{A}\cdot\vec{B} = -9 \qquad A = \sqrt{13} \qquad B = \sqrt{109}$$

$$\cos\theta = \vec{A}\cdot\vec{B}/AB \Rightarrow \theta = 103.83^\circ \qquad \vec{A}\times\vec{B} = 20\,\vec{a}_x + 30\,\vec{a}_y - 6\,\vec{a}_z$$

Problem 2.17 $\quad \vec{A} = -7\,\vec{a}_r + 2\,\vec{a}_\theta + \vec{a}_\phi\ , \quad \vec{B} = \vec{a}_r - 2\vec{a}_\theta + 4\,\vec{a}_\phi$

$$2\vec{A} - 3\vec{B} = -17\,\vec{a}_r + 10\,\vec{a}_\theta - 10\,\vec{a}_\phi \qquad A = 7.3485 \qquad B = 4.5826$$

$$\vec{A}\cdot\vec{B} = -7 - 4 + 4 = -7 \qquad \vec{A}\times\vec{B} = 10\,\vec{a}_r + 29\,\vec{a}_\theta + 12\,\vec{a}_\phi$$

$$\cos\alpha = \vec{A}\cdot\vec{B}/AB = -0.208 \Rightarrow \alpha = 108^\circ$$

$$\vec{a}_n = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = 0.304\,\vec{a}_r + 0.88\,\vec{a}_\theta + 0.364\,\vec{a}_\phi$$

Problem 2.18 Transform $\vec{A}$ and $\vec{B}$. $P(2,45°,45°)$, $Q(10,90°,90°)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin 45° \cos 45° & \cos 45° \cos 45° & -\sin 45° \\ \sin 45° \sin 45° & \cos 45° \sin 45° & \cos 45° \\ \cos 45° & -\sin 45° & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$$

$A_x = -3.207 \quad A_y = -1.793 \quad A_z = -6.363$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \Rightarrow$$

$B_x = -4,\ B_y = 1,\ B_z = 2$

$\vec{A} = -3.207\,\vec{a}_x - 1.793\,\vec{a}_y - 6.363\,\vec{a}_z$

$\vec{B} = -4\,\vec{a}_x + \vec{a}_y + 2\,\vec{a}_z$

$2\vec{A} - 3\vec{B} = 5.586\,\vec{a}_x - 6.586\,\vec{a}_y - 18.726\,\vec{a}_z$, $\quad \vec{A}\cdot\vec{B} = -1.691$

$\vec{A}\times\vec{B} = 2.777\,\vec{a}_x + 31.866\,\vec{a}_y - 10.379\,\vec{a}_z$, $\quad A = 7.348,\ B = 4.583$

$|\vec{A}\times\vec{B}| = 33.629 \Rightarrow \vec{a}_n = \dfrac{\vec{A}\times\vec{B}}{33.629} = 0.083\,\vec{a}_x + 0.948\,\vec{a}_y\ 0.309\,\vec{a}_z$

$\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB} = -0.05 \Rightarrow \theta = 92.88°$

Problem 2.19 $P(10, \pi/4, \pi/3) \Rightarrow P(3.54, 6.12, 7.07)$

$Q(2, \pi/2, \pi) \Rightarrow Q(-2, 0, 0)$

$\vec{PQ} = -5.54\,\vec{a}_x - 6.12\,\vec{a}_y - 7.07\,\vec{a}_z \Rightarrow |\vec{PQ}| = 10.87$

Problem 2.20 $f = 12xy + z \qquad P(0,0,0) \longrightarrow Q(1,1,0) \quad z=0 \quad dz=0$

Straight line: $y = x$

a) $\displaystyle\int_c f\,d\vec{l} = \int_c 12xy[dx\,\vec{a}_x + dy\,\vec{a}_y]$

$\displaystyle = \vec{a}_x\int 12xy\,dx + \vec{a}_y\int 12xy\,dy$

$\displaystyle = \vec{a}_x\int_0^1 12x^2dx + a_y\int_0^1 12y^2dy$

$= 4\vec{a}_x + 4\vec{a}_y$

b) $\displaystyle\int_c f\,dl = \int_c f\sqrt{dx^2+dy^2}$

$\displaystyle = \int_c f\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$\displaystyle = 16.97\int_c xy\,dx = 16.97\int_0^1 x^2dx$

$= 5.66$

Problem 2.21 $\vec{E} = \dfrac{10}{\rho}\,\vec{a}_\rho \qquad V_{ab} = -\displaystyle\int_b^a \vec{E}\cdot d\vec{l} = -\int_{0.8}^{0.1}\frac{10}{\rho}\,d\rho$

$a = 10\text{ cm} = 0.1\text{ m}$

$b = 80\text{ cm} = 0.8\text{ m}$

$= -10\ln\left[\dfrac{0.1}{0.8}\right] = 10\ln(8)$

$= 20.79\ \text{V}$

Problem 2.22 $\quad n_e = 300\rho\cos^2\phi$ electrons/m² $\quad$ disc radius = 20 m

$$N = \int_s n_e\, ds = 300\int_0^{20}\rho^2 d\rho\int_0^{2\pi}\cos^2\phi\, d\phi$$

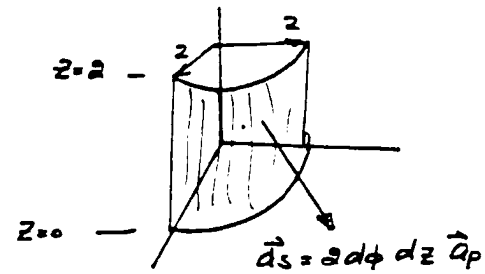
$$= 100\,\rho^3\Big|_0^{20}\left[\tfrac{1}{2}\phi + \tfrac{1}{4}\sin 2\phi\right]_0^{2\pi} = 2{,}513{,}274 \text{ electrons}$$

$$Q = -1.6\times10^{-19}\times 2{,}513{,}274 = -4.02\times10^{-13}\ C \quad \text{or} \quad 0.402\ \text{PC}$$

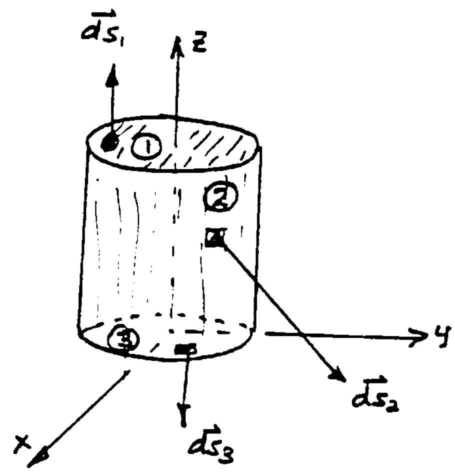
Problem 2.23 $\quad f = xyz \qquad x = 2\cos\phi \quad y = 2\sin\phi$

$$\int_s f\, ds = \int_0^{\pi/2} 8\sin\phi\cos\phi\, d\phi\int_0^1 z\, dz$$

$$= 8\ \frac{\sin^2\phi}{2}\Big|_0^{\pi/2}\ \frac{z^2}{2}\Big|_0^1 = 2$$



Problem 2.24 $\quad \vec{F} = x^3\vec{a}_x + x^2y\,\vec{a}_y + x^2z\,\vec{a}_z$



Surface 1: $\vec{ds}_1 = \rho\, d\phi\, d\rho\ \vec{a}_z$

$$\int_{s_1}\vec{F}\cdot\vec{ds}_1 = \iint x^2 z\ \rho\, d\rho\, d\phi$$

$z = 2 \qquad x = \rho\cos\phi$

Thus

$$\int_{s_1}\vec{F}\cdot\vec{ds}_1 = 2\int_0^4\rho^3 d\rho\int_0^{2\pi}\cos^2\phi\, d\phi$$

$$= 128\pi$$

Surface 3: $\vec{ds}_3 = -\rho\, d\rho\, d\phi\ \vec{a}_z$

$z = 0 \quad \Rightarrow$

$$\int_{s_3}\vec{F}\cdot\vec{ds}_3 = 0$$

Surface 2: $\vec{ds}_2 = 4\, d\phi\, dz\ \vec{a}_\rho$

$$\vec{F}\cdot\vec{ds}_2 = (x^3\,\vec{a}_x\cdot\vec{a}_\rho + x^2y\,\vec{a}_y\cdot\vec{a}_\rho)\,4\,d\phi\, dz$$

$x = 4\cos\phi \qquad y = 4\sin\phi$

$$\vec{F}\cdot\vec{ds}_2 = [64\cos^4\phi + 64\cos^2\phi\sin^2\phi]\cdot 4\,d\phi\, dz$$

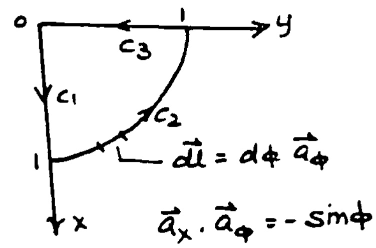
$$= 256\cos^2\phi\, d\phi\, dz$$

$$\int_{s_3}\vec{F}\cdot\vec{ds}_3 = 256\int_0^{2\pi}\cos^2\phi\, d\phi\int_0^2 dz$$

$$= 512\pi$$

$$\oint_s \vec{F}\cdot\vec{ds} = 128\pi + 0 + 512\pi = 640\pi$$

Problem 2.25 $\vec{F} = x\,\vec{a}_x$



Along C_1: $\int_{C_1} \vec{F}\cdot\vec{dl} = \int_0^1 x\,dx = 0.5$

Along C_2: $\int_{C_2} \vec{F}\cdot\vec{dl} = -\int_0^{\pi/2} \sin\phi\cos\phi\,d\phi = -0.5$

$\vec{dl} = d\phi\,\vec{a}_\phi$ $\quad \vec{a}_x\cdot\vec{a}_\phi = -\sin\phi$ $\quad x = \cos\phi$

Along C_3: $\int_{C_3} \vec{F}\cdot\vec{dl} = +\int_1^0 x\,\vec{a}_x\cdot\vec{a}_y\,dy = 0 \Rightarrow \oint \vec{F}\cdot\vec{dl} = 0$

Problem 2.26 $\vec{F} = xy\,\vec{a}_x$ $\quad \vec{dl} = 2\,d\theta\,\vec{a}_\theta$ $\quad \vec{a}_x\cdot\vec{a}_\theta = \cos\theta\cos\phi = 0.5\cos\theta$

$\vec{F}\cdot\vec{dl} = 2xy\,\vec{a}_x\cdot\vec{a}_\theta$ $\quad \phi = 60^\circ$

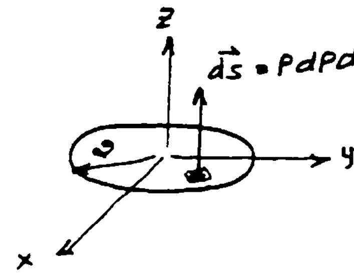
$x = 2\sin\theta\cos\phi = \sin\theta$

$y = 2\sin\theta\sin\phi = 1.732\sin\theta$

$$\int_C \vec{F}\cdot\vec{dl} = 1.732\int_0^\pi \sin^2\theta\cos\theta\,d\theta = 0.577\,\sin^3\theta\Big|_0^\pi = 0$$

Problem 2.27 $\vec{D} = (2+16\rho^2)\,\vec{a}_z$

$\vec{ds} = \rho\,d\rho\,d\phi\,\vec{a}_z$



$$\int_S \vec{D}\cdot\vec{ds} = \int_0^2 (2+16\rho^2)\,\rho\,d\rho\int_0^{2\pi} d\phi = 136\pi$$

Problem 2.28 $\vec{D} = (2+16r^2)\,\vec{a}_z$ $\quad \vec{ds} = 4\sin\theta\,d\theta\,d\phi\,\vec{a}_r$

$\vec{a}_z\cdot\vec{a}_r = \cos\theta$

$$\int_S \vec{D}\cdot\vec{ds} = \int_0^{\pi/2} (2+16\times 4)\,4\sin\theta\cos\theta\,d\theta\int_0^{2\pi} d\phi = 264\pi$$

Problem 2.29 $\vec{D} = 10\cos\phi\,\vec{a}_\rho$ $\quad \vec{ds} = \rho\,d\rho\,dz\,\vec{a}_z$ $\quad \vec{ds}\cdot\vec{D} = 0$

$$\int_S \vec{D}\cdot\vec{ds} = 0$$

Problem 2.30

$B = 10\cos\theta\,\vec{a}_r$ $\quad \vec{ds} = 4\sin\theta\,d\theta\,d\phi\,\vec{a}_r$

$$\int \vec{D}\cdot\vec{ds} = 40\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta\int_0^{2\pi} d\phi = 40\pi$$

Problem 2.31

$$Q = \int_v \rho\,dv = k\int_0^a r^4\,dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi = 0.8\pi k a^5 \quad C$$

Problem 2.32 $\vec{F} = xy^2\vec{a}_x + (x^2+1)y\,\vec{a}_y$

a) $\oint \vec{F}.\vec{dl} = ?$ $\vec{dl} = 3d\phi\,\vec{a}_\phi$ $\vec{a}_x.\vec{a}_\phi = -\sin\phi$

$\vec{F}.\vec{dl} = [3xy^2 d\phi\;\vec{a}_x.\vec{a}_\phi + (x^2+1)y\;\vec{a}_y.\vec{a}_\phi](3d\phi)$ $\vec{a}_y.\vec{a}_\phi = \cos\phi$

$\oint \vec{F}.\vec{dl} = \int_0^{2\pi}(-81\cos\phi\sin^3\phi + 81\sin\phi\cos^3\phi + 9\sin\phi\cos\phi)\,d\phi$ $x = 3\cos\phi,\; y = 3\sin\phi$

$= 0$

b) $\vec{F}.\vec{ds} = 0 \quad \because \vec{ds} = \rho\,d\phi\,d\rho\,\vec{a}_z$ and $F_z = 0$.

Problem 2.33 $f = x^3y^2z \Rightarrow \nabla f = 3x^2y^2z\,\vec{a}_x + 2x^3yz\,\vec{a}_y + x^3y^2\,\vec{a}_z$

$= 540\,\vec{a}_x + 240\,\vec{a}_y + 72\,\vec{a}_z \quad @ (2,3,5)$

$\nabla^2 f = \nabla\cdot(\nabla f) = 6xy^2z + 2x^3z = 620 \quad @ (2,3,5)$

Problem 2.34 $\nabla\phi = \frac{1}{\rho}\vec{a}_\phi \;,\quad \nabla\times[\vec{a}_z \ln\rho] = -\frac{1}{\rho}\vec{a}_\phi$

a) Hence, $\nabla\phi + \nabla\times(\vec{a}_z \ln\rho) = 0$

$\nabla(\ln\rho) = \frac{1}{\rho}\vec{a}_\rho \qquad \nabla\times(\vec{a}_z\phi) = \frac{1}{\rho}\vec{a}_\rho$

b) Hence, $\nabla(\ln\rho) - \nabla\times(\vec{a}_z\phi) = 0$

Problem 2.35 $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\vec{a}_r, \qquad \nabla\phi = \frac{1}{r\sin\theta}\vec{a}_\phi$

$\nabla\times(\cos\theta\,\nabla\phi) = \nabla\times\left(\frac{\cos\theta}{r\sin\theta}\vec{a}_\phi\right) = -\frac{1}{r^2}\vec{a}_r$

a) Hence, $\nabla(1/r) - \nabla\times(\cos\theta\,\nabla\phi) = 0$

$\nabla\theta = \frac{1}{r}\vec{a}_\theta \Rightarrow \frac{r\nabla\theta}{\sin\theta} = \frac{1}{\sin\theta}\vec{a}_\theta, \quad \nabla\times\left(\frac{r\nabla\theta}{\sin\theta}\right) = \frac{1}{r\sin\theta}\vec{a}_\phi$

b) Hence $\nabla\phi - \nabla\times\left(\frac{r\nabla\theta}{\sin\theta}\right) = 0$

Problem 2.36 $\vec{E} = yz\,\vec{a}_x + xz\,\vec{a}_y + xy\,\vec{a}_z$

$\nabla.\vec{E} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0 \Rightarrow$ Solenoidal or Continuous field

$$\nabla\times\vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \vec{a}_x[x-x] + \vec{a}_y[y-y] + \vec{a}_z[z-z] = 0$$

$\because \nabla\times\vec{E} = 0$, $\vec{E}$ is conservative field

Problem 2.37 Let $\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$

$$\nabla \cdot \nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0$$

$$\nabla \times \nabla f = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{a}_x \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right] + \vec{a}_y \left[\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right] + \vec{a}_z \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right] = 0$$

Problem 2.38

a) $\Phi = V_o \phi \ln(\frac{\rho}{a})$ $\quad \vec{E} = -\nabla \Phi = -\left[\frac{\partial \Phi}{\partial \rho} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi} \vec{a}_\phi \right] = -\frac{V_o \phi}{\rho} \vec{a}_\rho - \frac{V_o}{\rho} \ln(\frac{\rho}{a}) \vec{a}_\phi$

$$\nabla \cdot \vec{E} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (-V_o \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(-\frac{V_o}{\rho} \ln(\frac{\rho}{a}) \right) = 0 \Rightarrow$$

$$\rho_v = \epsilon_o \nabla \cdot \vec{E} = 0$$

b) $\Phi = V_o r \cos\theta$ $\quad \nabla \Phi = \frac{\partial}{\partial r} (V_o r \cos\theta) \vec{a}_r + \frac{1}{r} \frac{\partial}{\partial \theta} (V_o r \cos\theta) \vec{a}_\theta$

$$= V_o \cos\theta \, \vec{a}_r - V_o \sin\theta \, \vec{a}_\theta$$

$$\vec{E} = -\nabla \Phi = -V_o \cos\theta \, \vec{a}_r + V_o \sin\theta \, \vec{a}_\theta$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (-V_o r^2 \cos\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (V_o \sin^2\theta)$$

$$= -\frac{2}{r} V_o \cos\theta + \frac{2}{r} V_o \cos\theta = 0$$

$$\rho_v = \epsilon_o \nabla \cdot \vec{E} = 0$$

c) $\Phi = V_o r \sin\theta$ $\quad \nabla \Phi = V_o \sin\theta \, \vec{a}_r + V_o \cos\theta \, \vec{a}_\theta$

$$\vec{E} = -\nabla \Phi = -V_o \sin\theta \, \vec{a}_r - V_o \cos\theta \, \vec{a}_\theta$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (-V_o r^2 \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (-V_o \sin\theta \cos\theta)$$

$$= -\frac{2}{r} V_o \sin\theta - \frac{V_o}{r \sin\theta} \left[\cos^2\theta - \sin^2\theta \right] = -\frac{V_o}{r \sin\theta}$$

$$\rho_v = \epsilon_o \nabla \cdot \vec{E} = -\frac{V_o \epsilon_o}{r \sin\theta}$$

Problem 2.39

a) $$\nabla(fg) = \vec{a}_x\left[f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right] + \vec{a}_y\left[f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right] + \vec{a}_z\left[f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right]$$

$$= f\left[\frac{\partial g}{\partial x}\vec{a}_x + \frac{\partial g}{\partial y}\vec{a}_y + \frac{\partial g}{\partial z}\vec{a}_z\right] + g\left[\frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z\right]$$

$$= f\nabla g + g\nabla f$$

b) $$f\vec{A} = fA_x\,\vec{a}_x + fA_y\,\vec{a}_y + f\,\vec{a}_z A_z$$

$$\nabla\cdot(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= A_x\frac{\partial f}{\partial x} + f\frac{\partial A_x}{\partial x} + A_y\frac{\partial f}{\partial y} + f\frac{\partial A_y}{\partial y} + A_z\frac{\partial f}{\partial z} + f\frac{\partial A_z}{\partial z}$$

$$= (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z)\cdot\left(\frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z\right) + f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \vec{A}\cdot\nabla f + f\,\nabla\cdot\vec{A}$$

c)

$$\nabla\times f\vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix}$$

$$= \vec{a}_x\left[\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right] + \vec{a}_y\left[\frac{\partial}{\partial z}(fA_x) - \frac{\partial}{\partial x}(fA_z)\right] + \vec{a}_z\left[\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right]$$

$$= \vec{a}_x\left[f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right]$$
$$+ \vec{a}_y\left[f\frac{\partial A_x}{\partial z} + A_x\frac{\partial f}{\partial z} - f\frac{\partial A_z}{\partial x} - A_z\frac{\partial f}{\partial x}\right]$$
$$+ \vec{a}_z\left[f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right]$$

$$= f\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a}_z\right]$$
$$+ \left(\frac{\partial f}{\partial y}A_z - \frac{\partial f}{\partial z}A_y\right)\vec{a}_x + \left(\frac{\partial f}{\partial z}A_x - \frac{\partial f}{\partial x}A_z\right)\vec{a}_y$$
$$+ \left(\frac{\partial f}{\partial x}A_y - \frac{\partial f}{\partial y}A_x\right)\vec{a}_z$$

$$= f\,\nabla\times\vec{A} + \nabla f\times\vec{A}$$

Problem 2.40 $\quad x = \rho\cos\phi \Rightarrow dx = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$ ①

$y = \rho\sin\phi \Rightarrow dy = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$ ②

From ① and ②: $\quad d\rho = \cos\phi\, dx + \sin\phi\, dy$, and $d\phi = -\frac{1}{\rho}\sin\phi\, dx + \frac{1}{\rho}\cos\phi\, dy$

However, $d\rho = \frac{\partial\rho}{\partial x}dx + \frac{\partial\rho}{\partial y}dy \Rightarrow \frac{\partial\rho}{\partial x} = \cos\phi$ and $\frac{\partial\rho}{\partial y} = \sin\phi$

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \Rightarrow \frac{\partial\phi}{\partial x} = -\frac{1}{\rho}\sin\phi$ and $\frac{\partial\phi}{\partial y} = \frac{1}{\rho}\cos\phi$

Thus, $\frac{\partial}{\partial x} = \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial x} = \cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}$

$\frac{\partial}{\partial y} = \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial y} = \sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}$

Problem 2.41

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$= \cos^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\rho} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} + \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi}$$

$$- \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} + \frac{\sin\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(\sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$= \sin^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} - \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi}$$

$$+ \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\phi\partial\rho} + \frac{\cos^2\phi}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\cos^2\phi\frac{\partial^2}{\partial\phi^2} - \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} = \frac{1}{\rho^2}\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial\phi^2}\right]$$

Problem 2.42 $\quad \vec{E} = E_0\cos\theta\,\vec{a}_r - E_0\sin\theta\,\vec{a}_\theta$

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_0\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-E_0\sin^2\theta\right) = 0$$

$$\nabla\times\vec{E} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\,\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ E_0\cos\theta & -E_0 r\sin\theta & 0 \end{vmatrix}$$

$$= \frac{\vec{a}_\phi}{r^2\sin\theta}\cdot r\sin\theta\left[-E_0\sin\theta + E_0\sin\theta\right] = 0$$

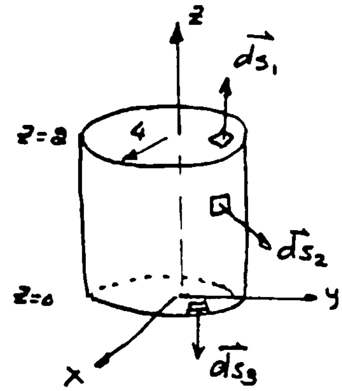
<u>Problem 2.43</u> $\quad \nabla\cdot\vec{E} = 0$ from Problem 2.42. Thus, $\int_v \nabla\cdot\vec{E}\,dv = 0$

$$\oint_s \vec{E}\cdot\vec{ds} = \int_0^{\pi} (E_0\cos\theta)\,r^2\sin\theta\,d\theta \int_0^{2\pi} d\phi \quad \text{when } r = b$$

$$= b^2 E_0 \pi \; \sin^2\theta \Big|_0^{\pi} = 0$$

<u>Problem 2.44</u> $\quad \vec{F} = x^3\vec{a}_x + x^2y\,\vec{a}_y + x^2z\,\vec{a}_z$

$$\nabla\cdot\vec{F} = 3x^2 + x^2 + x^2 = 5x^2$$



$$\int_v \nabla\cdot\vec{F}\,dv = \int_v 5x^2\,dv = 5\int_0^4 \rho^3 d\rho \int_0^{2\pi}\cos^2\phi\,d\phi \int_0^2 dz$$

$$= \frac{5}{4}\rho^4\Big|_0^4 (\pi)(2) = 640\pi$$

Surface s_1: $\int_{s_1}\vec{F}\cdot\vec{ds}_1 = \int_{s_1} x^2 z\,\rho\,d\rho\,d\phi$ where $z = 2$

$$= 2\int_0^4 \rho^3 d\rho \int_0^{2\pi}\cos^2\phi\,d\phi = 128\pi$$

on $\vec{s}_3$:
$\vec{a}_x\cdot\vec{a}_\rho = \cos\phi$
$a_y\cdot\vec{a}_\rho = \sin\phi$
$x = 4\cos\phi$
$y = 4\sin\phi$

Surface s_3: $\int_{s_3}\vec{F}\cdot\vec{ds}_3 = \int_{s_3} x^2 z\,\rho\,d\rho\,d\phi = 0 \quad \because z = 0$

Surface s_2: $\int_{s_2}\vec{F}\cdot\vec{ds}_2 = \int_{s_2}\left[x^3\,\vec{a}_x\cdot\vec{a}_\rho + x^2y\,\vec{a}_y\cdot\vec{a}_\rho\right]\rho\,d\phi\,dz \qquad \rho = 4$

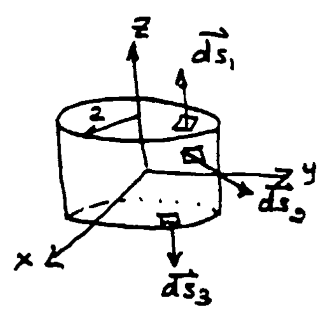
$$= 256\int_0^{2\pi}\cos^4\phi\,d\phi\int_0^2 dz + 256\int_0^{2\pi}\cos^2\phi\,\sin^2\phi\,d\phi\int_0^2 dz$$

$$= 256\int_0^{2\pi}\cos^2\phi\,d\phi\int_0^2 dz = 256(\pi)(2) = 512\pi$$

$$\oint \vec{F}\cdot\vec{ds} = 128\pi + 512\pi + 0 = 640\pi$$

<u>Problem 2.45</u> $\quad \vec{A} = (12 + 6\rho^2)\,z\,\vec{a}_z$

$$\nabla\cdot\vec{A} = 12 + 6\rho^2$$



$$\int_v \nabla\cdot\vec{A}\,dv = \int_0^2 (12+6\rho^2)\rho\,d\rho\int_0^{2\pi}d\phi\int_{-1}^{1}dz = 192\pi$$

$$\int_{s_1}\vec{A}\cdot\vec{ds}_1 = \int_0^2 (12+6\rho^2)\rho\,d\rho\int_0^{2\pi}d\phi = 96\pi$$

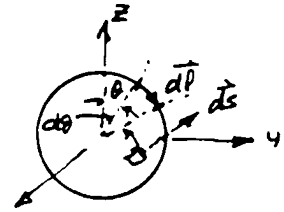
$\vec{ds}_1 = \rho\,d\rho\,d\phi\,\vec{a}_z$ at $z=1$
$\vec{ds}_2 = 2\,d\phi\,dz\,\vec{a}_\rho$ at $\rho = 2$
$\vec{ds}_3 = -\rho\,d\rho\,d\phi\,\vec{a}_z$ at $z = -1$

$$\int_{s_2}\vec{A}\cdot\vec{ds}_2 = 0$$

$$\int_{s_3}\vec{A}\cdot\vec{ds}_3 = \int_0^2 (12+6\rho^2)\rho\,d\rho\int_0^{2\pi}d\phi = 96\pi$$

$$\oint_s \vec{A}\cdot\vec{ds} = 192\pi = \int_v \nabla\cdot\vec{A}\,dv$$

Problem 2.46 $\vec{F} = 3y^2\vec{a}_x + 4z\,\vec{a}_y + 6y\,\vec{a}_z$



$\vec{a}_z\cdot\vec{a}_\theta = \cos(90+\theta) = -\sin\theta$ $\quad\left.\begin{array}{l} z = 2\cos\theta \\ y = 2\sin\theta \end{array}\right\}$ at $r=2$

$\vec{a}_y\cdot\vec{a}_\theta = \cos\theta \qquad \vec{a}_x\cdot\vec{a}_\theta = 0$

$$\oint_c \vec{F}\cdot d\vec{l} = \oint_c \left(6y^2 d\theta\, \vec{a}_x\cdot\vec{a}_\theta + 8z\, d\theta\, \vec{a}_y\cdot\vec{a}_\theta + 12y\, d\theta\, \vec{a}_z\cdot\vec{a}_\theta\right)$$

$$= \int_0^{2\pi} 16\cos^2\theta\, d\theta - \int_0^{2\pi} 24\sin^2\theta\, d\theta = -8\pi$$

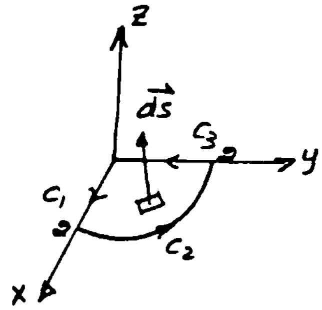
$d\vec{l} = 2\,d\theta\,\vec{a}_\theta$

$d\vec{s} = -r\,dr\,d\theta\,\vec{a}_x$

$$(\nabla\times\vec{F})\cdot\vec{a}_x = +2 \Rightarrow \int_s \nabla\times\vec{F}\cdot d\vec{s} = -2\int_0^2 r\,dr\int_0^{2\pi} d\theta = -8\pi$$

Thus, $\oint_c \vec{F}\cdot d\vec{l} = -8\pi = \int_s (\nabla\times\vec{F})\cdot d\vec{s}$

Problem 2.47 $\vec{F} = \frac{x}{\rho}\,\vec{a}_x$



$d\vec{s} = \rho\, d\rho\, d\phi\, \vec{a}_z$

$$\nabla\times\vec{F}\cdot d\vec{s} = \left[\vec{a}_y \frac{\partial}{\partial z}\left(\frac{x}{\rho}\right) - \vec{a}_z \frac{\partial}{\partial y}\left(\frac{x}{\rho}\right)\right]\cdot \rho\, d\rho\, d\phi\, \vec{a}_z$$

$$= -\frac{\partial}{\partial y}\left(\frac{x}{\rho}\right)\rho\, d\rho\, d\phi$$

$x = \rho\cos\phi\,,\ y = \rho\sin\phi \qquad \rho = \sqrt{x^2+y^2}$

$$\frac{\partial}{\partial y}\left(\frac{x}{\rho}\right) = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{\rho^3} = -\frac{\sin\phi\cos\phi}{\rho}$$

$$\int_s \nabla\times\vec{F}\cdot d\vec{s} = \int_0^2 d\rho \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = 1$$

Path c_1: $d\vec{l}_1 = dx\,\vec{a}_x \Rightarrow \int_{c_1} \vec{F}\cdot d\vec{l}_1 = \int_0^2 \frac{x}{\rho}\,dx = \int_0^2 dx = 2 \qquad (\rho = x \ \because \phi = 0)$

Path c_2: $d\vec{l}_2 = 2\,d\phi\,\vec{a}_\phi \qquad \rho = 2 \qquad \vec{a}_x\cdot a_\phi = -\sin\phi$

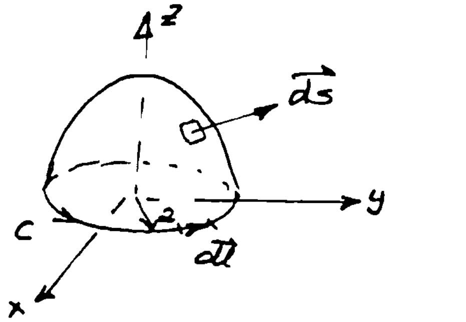
$\vec{F}\cdot d\vec{l}_2 = \frac{x}{2}\cdot 2\,d\phi\, \vec{a}_x\cdot\vec{a}_\phi \qquad x = 2\cos\phi$

$$\int_{c_2} \vec{F}\cdot d\vec{l}_2 = -2\int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = -1$$

Path c_3: $d\vec{l}_3 = dy\,\vec{a}_y \qquad \vec{F}\cdot d\vec{l}_3 = 0 \qquad \int_{c_3} \vec{F}\cdot d\vec{l}_3 = 0$

Thus, $\oint \vec{F}\cdot d\vec{l} = 2 - 1 + 0 = 1 = \int_s \nabla\times\vec{F}\cdot d\vec{s}$

Problem 2.48 $\quad \vec{F} = 100\cos\theta\ \vec{a}_r$



on path c, $\vec{dl} = 2\,d\phi\ a_\phi$

$$\oint_c \vec{F}\cdot\vec{dl} = 0$$

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 100\cos\theta & 0 & 0 \end{vmatrix}$$

$\vec{ds} = r^2\sin\theta\,d\theta\,d\phi\ \vec{a}_r$ at $r=2$

$$= \frac{100}{r}\sin\theta\ \vec{a}_\phi \quad \Rightarrow \int_s \nabla\times\vec{F}\cdot\vec{ds} = 0$$

Problem 2.49 $\quad -\frac{1}{2}\le x\le\frac{1}{2},\quad -\frac{1}{2}\le y\le\frac{1}{2},\quad -\frac{1}{2}\le z\le\frac{1}{2}$

$f = x^2 \Rightarrow \nabla f = 2x\,\vec{a}_x$ and $\nabla^2 f = 2$

$g = y^2 \Rightarrow \nabla g = 2y\,\vec{a}_y$ and $\nabla^2 g = 2 \qquad \nabla f\cdot\nabla g = 0 \Rightarrow$

$$\int_v f\nabla^2 g\,dv = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx\int_{-1/2}^{1/2} dy\int_{-1/2}^{1/2} dz = \frac{1}{6} \qquad \int_v \nabla f\cdot\nabla g\,dv = 0$$

$$\oint_s f\nabla g\cdot\vec{ds} = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x^2 y\,dx\int_{-1/2}^{1/2} dz\Big|_{y=\frac{1}{2}} - \int_{-\frac{1}{2}}^{1/2} 2x^2 y\,dx\int_{-1/2}^{1/2} dz\Big|_{y=-\frac{1}{2}} = \frac{1}{6}$$

Hence, $\int_v f\nabla^2 g\,dv + \int_v \nabla f\cdot\nabla g\,dv = \oint_s f\nabla g\cdot\vec{ds}$. Green's 1st Identity =

$$\int_v g\nabla^2 f\,dv = \int_{-1/2}^{1/2} 2y^2 dy\int_{-1/2}^{1/2} dx\int_{-1/2}^{1/2} dz = \frac{1}{6} \Rightarrow \int_v (f\nabla^2 g - g\nabla^2 f)\,dv = 0 \quad (1)$$

$$\oint_s g\nabla f\cdot\vec{ds} = \int_{-1/2}^{1/2} 2xy^2 dy\int_{-1/2}^{1/2} dz\Big|_{x=\frac{1}{2}} - \int_{-1/2}^{1/2} 2xy^2 dy\int_{-1/2}^{1/2} dz\Big|_{x=-\frac{1}{2}} = \frac{1}{6}$$

$$\oint [f\nabla g - g\nabla f]\cdot\vec{ds} = 0 \quad (2)$$

Thus, from (1) and (2),

$$\int_v (f\nabla^2 g - g\nabla^2 f)\,dv = \oint_s (f\nabla g - g\nabla f)\cdot\vec{ds},$$ Green's 2nd Identity