§ 7.13 Time-harmonic fields

1.concept

One of the most important cases of time-varying electromagnetic fields is the time-harmonic (sinusoidal) field. In this type of field, the excitation source varies sinusoidally in time with a single frequency. In a linear system, a sinusoidally varying source generates fields that also vary

sinusoidally in time at all points in the system.
If the electric field $\vec{E}$ is given as

$$\vec{\mathbf{E}}(x,y,z,t)=\vec{\mathbf{a}}_{\mathbf{x}}E_x(x,y,z,t)+\vec{\mathbf{a}}_{\mathbf{y}}E_y(x,y,z,t)+\vec{\mathbf{a}}_{\mathbf{z}}E_z(x,y,z,t)$$

the time-harmonic variations of these components may be written as

$$E_x(x,y,z,t)=E_x(r,t)=E_{x0}(r)\cos[\omega t+\varphi_x(r)]$$

$$E_y(x,y,z,t)=E_y(r,t)=E_{y0}(r)\cos[\omega t+\varphi_y(r)]$$

$$E_z(x,y,z,t)=E_z(r,t)=E_{z0}(r)\cos[\omega t+\varphi_z(r)]$$

where $E_{x0}(r), E_{y0}(r)$ *and* $E_{z0}(r)$ **are the amplitudes of the components of the** $\vec{E}$ **field along the** $\vec{a}_x, \vec{a}_y$ *and* $\vec{a}_z$ **directions, respectively.**

We have also used a shorthand notation(*r*) to imply that the fields are functions of space coordinates *x*, *y* and *z*. In addition, $\varphi_x(r), \varphi_y(r)$ *and* $\varphi_z(r)$ **are the phase shifts of the *x*, *y* and *z* components of the field** $\vec{E}$ **at a given point (*x*, *y* , *z*) in space.**

The amplitude of each component is now a function of space only. We can also write each component as

$$E_x(x,y,z,t) = E_x(r,t) = E_{x0}(r)\cos[\omega t + \varphi_x(r)]$$

$$= \mathrm{Re}[E_{x0}(r)e^{j\varphi_x(r)}e^{j\omega t}] = \mathrm{Re}[\dot{E}_x(r)e^{j\omega t}]$$

$$\dot{E}_x(r) = E_{x0}(r)e^{j\varphi_x(r)}$$

$$E_y(x,y,z,t) = E_y(r,t) = E_{y0}(r)\cos[\omega t + \varphi_y(r)]$$

$$= \mathrm{Re}[E_{y0}(r)e^{j\varphi_y(r)}e^{j\omega t}] = \mathrm{Re}[\dot{E}_y(r)e^{j\omega t}]$$

$$\dot{E}_y(r) = E_{y0}(r)e^{j\varphi_x(r)}$$

$$E_x(x,y,z,t) = E_x(r,t) = E_{x0}(r)\cos[\omega t + \varphi_x(r)]$$
$$= \mathrm{Re}[E_{x0}(r)e^{j\varphi_x(r)}e^{j\omega t}]$$

$$E_y(x,y,z,t) = E_y(r,t) = E_{y0}(r)\cos[\omega t + \varphi_y(r)] = \mathrm{Re}[E_{y0}(r)e^{j\varphi_y(r)}e^{j\omega t}]$$

$$E_z(x,y,z,t) = E_z(r,t) = E_{z0}(r)\cos[\omega t + \varphi_z(r)]$$
$$= \mathrm{Re}[E_{z0}(r)e^{j\varphi_z(r)}e^{j\omega t}]$$

where Re stands for the real part of the complex function enclosed in the brackets. If we define

$$\dot{E}_x(r) = E_{x0}(r)e^{j\varphi_x(r)}$$

$$\dot{E}_y(r) = E_{y0}(r)e^{j\varphi_y(r)}$$

$$\dot{E}_z(r) = E_{z0}(r)e^{j\varphi_z(r)}$$

(they are the complex functions of space only)

then the scalar components of the $\vec{E}$ field can be written as

$$E_x(x,y,z,t) = E_x(r,t) = \mathrm{Re}[\dot{E}_x(r)e^{j\omega t}]$$

$$E_y(x,y,z,t) = E_y(r,t) = \mathrm{Re}[\dot{E}_y(r)e^{j\omega t}]$$

$$E_z(x,y,z,t) = E_z(r,t) = \mathrm{Re}[\dot{E}_z(r)e^{j\omega t}]$$

the $\vec{E}$ field can now be written as

$$\begin{aligned}
&\vec{\mathbf{E}}(x,y,z,t) \\
&= \vec{\mathbf{a}}_{\mathbf{x}} E_x(x,y,z,t) + \vec{\mathbf{a}}_{\mathbf{y}} E_y(x,y,z,t) + \vec{\mathbf{a}}_{\mathbf{z}} E_z(x,y,z,t) \\
&= \vec{\mathbf{a}}_{\mathbf{x}} \mathrm{Re}[\dot{E}_x(r)e^{j\omega t}] + \vec{\mathbf{a}}_{\mathbf{y}} \mathrm{Re}[\dot{E}_y(r)e^{j\omega t}] + \vec{\mathbf{a}}_{\mathbf{z}} \mathrm{Re}[\dot{E}_z(r)e^{j\omega t}] \\
&= \mathrm{Re}\{[\vec{\mathbf{a}}_{\mathbf{x}} \dot{E}_x(r) + \vec{\mathbf{a}}_{\mathbf{y}} \dot{E}_y(r) + \vec{\mathbf{a}}_{\mathbf{z}} \dot{E}_z(r)]e^{j\omega t}\} \\
&= \mathrm{Re}[\dot{\vec{E}}(r)e^{j\omega t}]
\end{aligned}$$

$\dot{E}_x(r), \dot{E}_y(r)$ *and* $\dot{E}_z(r)$ **are said to be the phasor equivalents of** $E_x(r), E_y(r)$ *and* $E_z(r)$**, while** $\dot{\vec{E}}(r)$ **is the phasor equivalent of** $\vec{E}$ **the space dependency is included in** $\dot{\vec{E}}(r)$ **and the time dependency is retained in the implicit form.**

Since the time rate of change of the $\vec{E}$ field is

$$\frac{\partial \vec{E}(r,t)}{\partial t} = \frac{\partial}{\partial t}\mathrm{Re}[\dot{\vec{E}}(r)e^{j\omega t}]$$

$$= \mathrm{Re}[\ j\omega \dot{\vec{E}}(r)e^{j\omega t}]$$

which states that the differentiation with respect to time in the time domain yields a factor $j\omega$ in the phasor domain. Similarly, we can show that the integration with respect to time becomes division by $j\omega$.

❖Summary

$$E_x = \mathrm{Re}[\dot{E}_x e^{j\omega t}]$$
$$E_y = \mathrm{Re}[\dot{E}_y e^{j\omega t}]$$
$$E_z = \mathrm{Re}[\dot{E}_z e^{j\omega t}]$$

$$\vec{\boldsymbol{E}} = \mathrm{Re}[\,\dot{\vec{\boldsymbol{E}}}\, e^{j\omega t}\,]$$

$$\dot{\vec{\boldsymbol{E}}} = \vec{\boldsymbol{a}}_{\boldsymbol{x}} \dot{E}_x + \vec{\boldsymbol{a}}_{\boldsymbol{y}} \dot{E}_y + \vec{\boldsymbol{a}}_{\boldsymbol{z}} \dot{E}_z$$

We can always express a field in the time domain by multiplying its counterpart in the phasor or frequency domain by $e^{j\omega t}$ and taking its real part only

❑2. Maxwell's equations in phasor form

$$(1)\quad \nabla\times\vec{\mathbf{H}}=\vec{\mathbf{J}}+\frac{\partial\vec{\mathbf{D}}}{\partial t}$$

$$\nabla\times\mathrm{Re}[\dot{\vec{\mathbf{H}}}e^{j\omega t}]=\mathrm{Re}[\dot{\vec{\mathbf{J}}}e^{j\omega t}]+\frac{\partial}{\partial t}\mathrm{Re}[\dot{\vec{\mathbf{D}}}e^{j\omega t}]$$

$$\nabla\times\mathrm{Re}[\dot{\vec{\mathbf{H}}}e^{j\omega t}]=\mathrm{Re}[\dot{\vec{\mathbf{J}}}e^{j\omega t}]+\mathrm{Re}[j\omega\dot{\vec{\mathbf{D}}}e^{j\omega t}]$$

$$\mathrm{Re}\{\nabla\times[\dot{\vec{\mathbf{H}}}e^{j\omega t}]\}=\mathrm{Re}\{[\dot{\vec{\mathbf{J}}}e^{j\omega t}]+[j\omega\dot{\vec{\mathbf{D}}}e^{j\omega t}]\}$$

$$\mathrm{Re}\{[\nabla\times\dot{\vec{\boldsymbol{H}}}]e^{j\omega t}\}=\mathrm{Re}\{[\dot{\vec{\boldsymbol{J}}}+j\omega\dot{\vec{\boldsymbol{D}}}]e^{j\omega t}\}$$

$$\nabla\times\dot{\vec{\mathbf{H}}}=\dot{\vec{\mathbf{J}}}+j\omega\dot{\vec{\mathbf{D}}}$$

$$\mathrm{Re}\{[\nabla \times \dot{\vec{\boldsymbol{H}}}]e^{j\omega t}\} = \mathrm{Re}\{[\dot{\vec{\boldsymbol{J}}} + j\omega \dot{\vec{\boldsymbol{D}}}]e^{j\omega t}\}$$

$$\mathrm{Re}\{[x + jy]e^{j\omega t}\} = \mathrm{Re}\{[x' + jy']e^{j\omega t}\}$$

$$\mathrm{Re}\{[x + jy](\cos(\omega t) + j\sin(\omega t))\} = \mathrm{Re}\{[x' + jy'](\cos(\omega t) + j\sin(\omega t))\}$$

$$x\cos(\omega t) - y\sin(\omega t) = x'\cos(\omega t) - y'\sin(\omega t)$$

Since time t always goes on, $0 \le \cos(\omega t) \le 1$, $0 \le \sin(\omega t) \le 1$, a large number of values within [0,1],can be arbitrary.

Thus, *x=x'; y=y'*

the time rate of change $\dfrac{\partial}{\partial t} \Longrightarrow j\omega$

we can obtain the phasor forms of the four Maxwell's equations in the point (differential) and integral forms are

$$(1)\quad \nabla \times \dot{\vec{\mathbf{H}}} = \dot{\vec{\mathbf{J}}} + j\omega\dot{\vec{\mathbf{D}}} \Longrightarrow \oint_l \dot{\vec{\mathbf{H}}} \bullet d\vec{\mathbf{l}} = \int_s (\dot{\vec{\mathbf{J}}} + j\omega\dot{\vec{\mathbf{D}}}) \bullet d\vec{\mathbf{s}}$$

$$(2)\quad \nabla \times \dot{\vec{\mathbf{E}}} = -j\omega\dot{\vec{\mathbf{B}}} \Longrightarrow \oint_l \dot{\vec{\mathbf{E}}} \bullet d\vec{\mathbf{l}} = -j\omega\int_s \dot{\vec{\mathbf{B}}} \bullet d\vec{\mathbf{s}}$$

$$(3)\quad \nabla \bullet \dot{\vec{\mathbf{D}}} = \dot{\rho}_v \Longrightarrow \oint_s \dot{\vec{\mathbf{D}}} \bullet d\vec{\mathbf{s}} = \int_v \dot{\rho}_v dv$$

$$(4)\quad \nabla \bullet \dot{\vec{\mathbf{B}}} = 0 \Longrightarrow \oint_s \dot{\vec{\mathbf{B}}} \bullet d\vec{\mathbf{s}} = 0$$

$$(5)\quad \nabla \bullet \dot{\vec{\mathbf{J}}} = -j\omega\dot{\rho}_v \Longrightarrow \oint_s \dot{\vec{\mathbf{J}}} \bullet d\vec{\mathbf{s}} = -j\omega\int_v \dot{\rho}_v dv$$

the constitutive relationships in the phasor form are

$$(6)\quad \dot{\vec{\mathbf{D}}} = \varepsilon \dot{\vec{\mathbf{E}}}$$

$$(7)\quad \dot{\vec{\mathbf{J}}} = \sigma \dot{\vec{\mathbf{E}}}$$

$$(8)\quad \dot{\vec{\mathbf{B}}} = \mu \dot{\vec{\mathbf{H}}}$$

3. Poynting Theorem in phasor form

In terms of mathematic knowledge , we have equation

$$\dot{\vec{\mathbf{E}}} \bullet (\nabla \times \dot{\vec{\mathbf{H}}}^{*}) - \dot{\vec{\mathbf{H}}}^{*} \bullet (\nabla \times \dot{\vec{\mathbf{E}}}) = -\nabla \bullet (\dot{\vec{\mathbf{E}}} \times \dot{\vec{\mathbf{H}}}^{*})$$

we will understand these signs in the followings:

$$\dot{\vec{\mathbf{E}}} \bullet (\nabla \times \dot{\vec{\mathbf{H}}}^{*}) - \dot{\vec{\mathbf{H}}}^{*} \bullet (\nabla \times \dot{\vec{\mathbf{E}}}) = -\nabla \bullet (\dot{\vec{\mathbf{E}}} \times \dot{\vec{\mathbf{H}}}^{*})$$

1) The scalar product of the conjugate of Maxwell's equation (1) with $\dot{\vec{E}}$ yields

$$\dot{\vec{\mathbf{E}}} \bullet (\nabla \times \dot{\vec{\mathbf{H}}}^{*}) = \dot{\vec{\mathbf{E}}} \bullet (\dot{\vec{\mathbf{J}}}^{*} - j\omega \dot{\vec{\mathbf{D}}}^{*}) \qquad (7.122)$$

where * represents the conjugate of a field quantity.

2)Similarly, the scalar product of Maxwell's equation (2) with $\dot{\vec{H}}^{*}$ yields

$$\dot{\vec{\mathbf{H}}}^{*} \bullet (\nabla \times \dot{\vec{\mathbf{E}}}) = -j\omega \dot{\vec{\mathbf{H}}}^{*} \bullet \dot{\vec{\mathbf{B}}} \qquad (7.121)$$

Subtracting equation(7.121) from equation(7.122), we obtain

$$\dot{\mathbf{E}} \bullet (\nabla \times \dot{\mathbf{H}}^{*}) - \dot{\mathbf{H}}^{*} \bullet (\nabla \times \dot{\mathbf{E}}) = -\nabla \bullet (\dot{\mathbf{E}} \times \dot{\mathbf{H}}^{*})$$

$$= \dot{\mathbf{E}} \bullet (\dot{\mathbf{J}}^{*} - j\omega \dot{\mathbf{D}}^{*}) + j\omega \dot{\mathbf{H}}^{*} \bullet \dot{\mathbf{B}}$$

$$= \dot{\mathbf{E}} \bullet \dot{\mathbf{J}}^{*} + j\omega(\dot{\mathbf{B}} \bullet \dot{\mathbf{H}}^{*} - \dot{\mathbf{E}} \bullet \dot{\mathbf{D}}^{*})$$

Is divided by 2 both sides, we can obtain

$$-\nabla \bullet \left(\frac{1}{2} \dot{\boldsymbol{E}} \times \dot{\boldsymbol{H}}^{*} \right) = \frac{1}{2} \dot{\mathbf{E}} \bullet \dot{\mathbf{J}}^{*} + j\omega\left(\frac{1}{2} \dot{\mathbf{B}} \bullet \dot{\mathbf{H}}^{*} - \frac{1}{2} \dot{\mathbf{E}} \bullet \dot{\mathbf{D}}^{*}\right)$$

Using the definition of the complex Poynting vector or complex power density

$$\dot{\vec{\mathbf{S}}} = \frac{1}{2}\dot{\vec{\mathbf{E}}} \times \dot{\vec{\mathbf{H}}}^*$$

we obtain

$$-\nabla \bullet \dot{\vec{\mathbf{S}}} = \frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{J}}}^* + j\omega\left(\frac{1}{2}\dot{\vec{\mathbf{B}}} \bullet \dot{\vec{\mathbf{H}}}^* - \frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{D}}}^*\right)$$

$$-\nabla \bullet \dot{\vec{\mathbf{S}}} = \frac{1}{2}\dot{\vec{\boldsymbol{E}}} \bullet \dot{\vec{\boldsymbol{J}}}^* + j2\omega\left(\frac{1}{4}\dot{\vec{\boldsymbol{B}}} \bullet \dot{\vec{\boldsymbol{H}}}^* - \frac{1}{4}\dot{\vec{\boldsymbol{E}}} \bullet \dot{\vec{\boldsymbol{D}}}^*\right) \qquad (7.125)$$

equation(7.125) is known as the complex Poynting theorem in the differential form. Integrating over volume *v* bounded by surfaces and applying the divergence theorem, we obtain the complex Poynting theorem in integral form as

$$-\oint_s \dot{\vec{\mathbf{S}}} \bullet d\vec{\mathbf{s}} = \int_v \left(\frac{1}{2} \dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{J}}}^* \right) dv + j2\omega \int_v \left(\frac{1}{4} \dot{\vec{\mathbf{B}}} \bullet \dot{\vec{\mathbf{H}}}^* - \frac{1}{4} \dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{D}}}^* \right) dv \tag{7.126}$$

for a time-vary field($\vec{E}$, $\vec{H}$) the field changes with time.

For example,

$$\vec{\mathbf{E}}(x,y,z,t) = \vec{\mathbf{a}}_{\mathbf{x}} E_x(x,y,z,t) + \vec{\mathbf{a}}_{\mathbf{y}} E_y(x,y,z,t) + \vec{\mathbf{a}}_{\mathbf{z}} E(x,y,z,t)$$

$$E_x(x,y,z,t) = E_x(r,t) = E_{x0}(r)\cos[\omega t + \varphi_x(r)]$$

$$= \mathrm{Re}[E_{x0}(r) e^{j\varphi_x(r)} e^{j\omega t}]$$

$$E_y(x,y,z,t) = E_y(r,t) = E_{y0}(r)\cos[\omega t + \varphi_y(r)]$$

$$= \mathrm{Re}[E_{y0}(r) e^{j\varphi_y(r)} e^{j\omega t}]$$

$$E_z(x,y,z,t) = E_z(r,t) = E_{z0}(r)\cos[\omega t + \varphi_z(r)]$$

$$= \mathrm{Re}[E_{z0}(r)e^{j\varphi_z(r)}e^{j\omega t}]$$

That means that the magnitude and direction of the time-varying field at time t_1 may be different from the magnitude and direction of the time-varying field at another time t_2. thus, the energy density(W_e, W_m) and the power density ($\vec{S}$)also changes with time.

In general, we should have the time-average energy density and the power density. That is,

$$W_{eave} = \frac{1}{T}\int_0^T W_e(t)dt = \frac{1}{T}\int_0^T \left(\frac{1}{2}\varepsilon E^2(t)\right)dt$$

$$= \frac{1}{T}\int_0^T \left(\frac{1}{2}\varepsilon E^2 \cos^2(\omega t + \varphi_E(r))\right)dt$$

$$= \frac{1}{4}\varepsilon E^2$$

$$= \frac{1}{4}\varepsilon \dot{\mathbf{E}} \bullet \dot{\mathbf{E}}^* = \frac{1}{4}\dot{\mathbf{D}} \bullet \dot{\mathbf{E}}^* = \frac{1}{4}\dot{\mathbf{E}} \bullet \dot{\mathbf{D}}^*$$

(since the permittivity ε is a real number)

$$W_{mave} = \frac{1}{T}\int_0^T W_m(t)dt = \frac{1}{T}\int_0^T \left(\frac{1}{2}\mu H^2(t)\right)dt$$

$$= \frac{1}{T}\int_0^T \left(\frac{1}{2}\mu H^2 \cos^2(\omega t + \varphi_H(r))\right)dt$$

$$= \frac{1}{4}\mu H^2$$

$$= \frac{1}{4}\mu \dot{\vec{\mathbf{H}}} \bullet \dot{\vec{\mathbf{H}}}^* = \frac{1}{4}\dot{\vec{\mathbf{B}}} \bullet \dot{\vec{\mathbf{H}}}^*$$

similarly, the time-average power dissipated within the conductive medium can also be given by

$$P_{lave} = \frac{1}{T}\int_0^T P_l(t)dt = \frac{1}{T}\int_0^T \sigma E^2(t)dt$$

$$= \frac{1}{T}\int_0^T \left(\sigma E^2 \cos^2(\omega t + \varphi_E(r))\right)dt$$

$$= \frac{1}{2}\sigma E^2$$

$$= \frac{1}{2}\sigma \dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{E}}}^* = \frac{1}{2}\dot{\vec{\mathbf{J}}} \bullet \dot{\vec{\mathbf{E}}}^* = \frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{J}}}^*$$

(the conductivity σ is a real number,

Plave* is a volume power density. W/m^3*)

$$\vec{\mathbf{S}}_{ave} = \frac{1}{T}\int_0^T \vec{\mathrm{S}}(t)dt = \frac{1}{T}\int_0^T [\vec{\mathbf{E}}(t)\times\vec{\mathbf{H}}(t)]dt$$

$$= \frac{1}{T}\int_0^T [\mathrm{Re}(\dot{\vec{\mathbf{E}}}e^{j\omega t})\times\mathrm{Re}(\dot{\vec{\mathbf{H}}}e^{j\omega t})]dt$$

$$= \frac{1}{T}\int_0^T \left(\frac{1}{2}\mathrm{Re}[(\dot{\vec{\mathbf{E}}}e^{j\omega t})\times(\dot{\vec{\mathbf{H}}}e^{j\omega t})^*]\right)dt$$

$$= \frac{1}{T}\int_0^T \left(\frac{1}{2}\mathrm{Re}[(\dot{\vec{\mathbf{E}}}e^{j\omega t})\times(\dot{\vec{\mathbf{H}}}^*e^{-j\omega t})]\right)dt$$

$$= \frac{1}{2}\mathrm{Re}(\dot{\vec{\mathbf{E}}}\times\dot{\vec{\mathbf{H}}}^*)$$

$$= \mathrm{Re}(\dot{\vec{\mathbf{S}}})$$

$\dot{\vec{\mathbf{S}}} = \frac{1}{2}\dot{\vec{\mathbf{E}}} \times \dot{\vec{\mathbf{H}}}^*$ **therefore, equation (7.125) and (7.126) can be rewritten as**

$$-\nabla \bullet \dot{\vec{\mathbf{S}}} = \frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{J}}}^* + j\omega\left(\frac{1}{2}\dot{\vec{\mathbf{B}}} \bullet \dot{\vec{\mathbf{H}}}^* - \frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{D}}}^*\right) \quad (7.125\text{a})$$

$$-\nabla \bullet \dot{\vec{\mathbf{S}}} = \frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{J}}}^* + j2\omega\left(\frac{1}{4}\dot{\vec{\mathbf{B}}} \bullet \dot{\vec{\mathbf{H}}}^* - \frac{1}{4}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{D}}}^*\right) \quad (7.125\text{a})$$

$$-\nabla \bullet \dot{\vec{\mathbf{S}}} = P_{lave} + j2\omega(W_{mave} - W_{eave}) \quad (7.125\text{b})$$

$$-\oint_s \dot{\vec{\mathbf{S}}} \bullet d\vec{\mathbf{s}} = \int_v \left(\frac{1}{2}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{J}}}^*\right) dv + j2\omega \int_v \left(\frac{1}{4}\dot{\vec{\mathbf{B}}} \bullet \dot{\vec{\mathbf{H}}}^* - \frac{1}{4}\dot{\vec{\mathbf{E}}} \bullet \dot{\vec{\mathbf{D}}}^*\right) dv$$

(7.126a)

$$-\oint_s \dot{\vec{\mathbf{S}}} \bullet d\vec{\mathbf{s}} = \int_v P_{lave}\, dv + j2\omega \int_v (W_{mave} - W_{eave})\, dv \quad (7.126\text{b})$$

4.General wave equations

Let us consider a uniform but source-free medium having dielectric constant ε, magnetic permeability μ, and conductivity σ. The medium is considered to be source free as long as it does not contain the charges and currents necessary to generate the fields. However , the conduction current density as determined by Ohm's law($\vec{J}=\sigma\vec{E}$) can exist in a finitely conducting medium. Under these conditions, Maxwell's equations are

$$(1)\quad \nabla\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial}{\partial t}\vec{\mathbf{D}} = \sigma\vec{\mathbf{E}} + \frac{\partial}{\partial t}\vec{\mathbf{D}}$$

$$(2)\quad \nabla \times \vec{\mathbf{E}} = -\frac{\partial}{\partial t}\vec{\mathbf{B}}$$

$$(3)\quad \nabla \bullet \vec{\mathbf{D}} = 0$$

$$(4)\quad \nabla \bullet \vec{\mathbf{B}} = 0$$

Instead of four variables, the preceding coupled equations are in terms of two variables($\vec{E}, \vec{H}$) by applying the constitutive equations

$\vec{D} = \varepsilon\vec{E}$ **and** $\vec{B} = \mu\vec{H}$

$$(1)\quad \nabla \times \vec{\mathbf{H}} = \sigma\vec{\mathbf{E}} + \varepsilon\frac{\partial}{\partial t}\vec{\mathbf{E}}$$

$$(2)\quad \nabla \times \vec{\mathbf{E}} = -\mu\frac{\partial}{\partial t}\vec{\mathbf{H}}$$

$$(3)\quad \nabla \bullet \vec{\mathbf{E}} = 0$$

$$(4)\quad \nabla \bullet \vec{\mathbf{H}} = 0$$

Let us now obtain an equation in terms of one variable, say the $\vec{E}$ field only. To do this, we take the curl of equation(2) and obtain

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{\mathbf{H}}$$

$$= -\mu \frac{\partial}{\partial t}\left(\sigma \vec{\mathbf{E}} + \varepsilon \frac{\partial}{\partial t} \vec{\mathbf{E}} \right)$$

$$= -\mu\sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

Using the vector identity

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla(\nabla \bullet \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}}$$

and substituting $\nabla \bullet \vec{E} = 0$ **, we have**

$$\nabla^2 \vec{\mathbf{E}} - \mu\sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0 \qquad (7.j1)$$

which is a set of three scalar equations, one for each component of the $\vec{E}$ **field, in a conducting medium. We can also obtain a similar set of three scalar equations in terms of the** $\vec{H}$ **field as**

$$\nabla^2 \vec{\mathbf{H}} - \mu\sigma \frac{\partial \vec{\mathbf{H}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = 0 \qquad (7.j2)$$

the set of six independent equations given by (7.j1) and (7.j2) are known as the general wave equations. They govern the behavior of all electromagnetic fields in a uniform but source-free conducting medium.

❖Disscussion:

❑**(1)the second-order term in the second-order differential equation indicates that the fields decay (lose energy) as they propagate through the medium. For this reason, a conducting medium is called a lossy medium.**

- (2)The conduction current is almost nonexistent in c-omparison with the displacement current. Such a m-edium may be treated as a perfect dielectric or lossl-ess medium ($\sigma = 0$). Thus, by setting $\sigma = 0$ in *(7.j1)* and (7.j2), we obtain the wave equations for a loss-less medium as

$$\nabla^2 \vec{\mathbf{E}} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0$$

$$\nabla^2 \vec{\mathbf{H}} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = 0$$

相量形式，复数形式

- (3)The general wave equations in phasor form

Equation(2.6.5) can be rewritten in phasor form as

$$\nabla^2 \dot{\vec{\mathbf{E}}} - j\omega\mu\sigma \dot{\vec{\mathbf{E}}} - \mu\varepsilon (j\omega)(j\omega)\dot{\vec{\mathbf{E}}} = 0$$

$$\nabla^2 \dot{\vec{\mathbf{E}}} - j\omega\mu\sigma\dot{\vec{\mathbf{E}}} + \omega^2 \mu\varepsilon\dot{\vec{\mathbf{E}}} = 0$$

$$\nabla^2 \dot{\vec{\mathbf{E}}} + \omega^2 \mu\varepsilon_c \dot{\vec{\mathbf{E}}} = 0$$

$$\nabla^2 \dot{\vec{\mathbf{E}}} + k^2 \dot{\vec{\mathbf{E}}} = 0 \qquad (7.j3)$$

where ε_c is complex permittivity for the conducting medium, it can be written as

$$\varepsilon_c = \varepsilon - j\sigma/\omega$$

and where k is the propagation constant. It is a complex quantity.

$$K^2 = \omega^2 \mu \varepsilon_c$$

Similarly, and equation(2.6.6) can be rewritten as

$$\nabla^2 \dot{\vec{\mathbf{H}}} + k^2 \dot{\vec{\mathbf{H}}} = 0 \qquad (7.j4)$$

Equation(7.j3) and equation(7.j4) can also be derived from Maxwell’s equations in phasor form.(Exercise 2.)