Fundamentals of Information Theory

Solution 5

Problem 1 *(10 points)*

Draw the channel diagram of the following discrete channels. Their channel matrix are shown as follows.

(a) A Z channel

$X\backslash Y$	0	1
0	1	0
1	s	$1-s$

(b) A binary erasure channel

$X\backslash Y$	0	E	1
0	$1-s_1-s_2$	s_1	s_2
1	s_2	s_1	$1-s_1-s_2$

(c) A non-symmetric channel

$X\backslash Y$	0	1
0	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{3}{4}$

(d) A semi-symmetric channel

$X\backslash Y$	0	1	2	3
0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$

Solution:

(a) The channel transition diagram is shown in Figure 1.

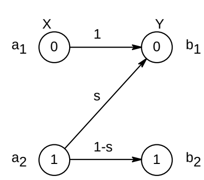

Figure 1: The channel diagram of the Z channel).

(b) The channel transition diagram is shown in Figure 2. This binary erasure channel is a semi-symmetric channel.

(c) The channel transition diagram is shown in Figure 3.

(d) The channel transition diagram is shown in Figure 4.

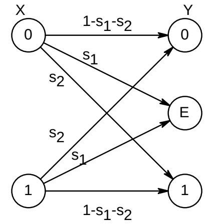

Figure 2: The channel diagram of the binary erasure channel.

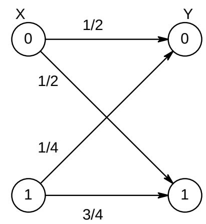

Figure 3: The channel diagram of the non-symmetric channel.

Problem 2 *(10 points) Find the channel capacity of the following discrete memoryless channel, where $Pr\{Z = 0\} = Pr\{Z = a\} = \frac{1}{2}$. The alphabet for x is $X = \{0, 1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.*

Solution: A sum channel.

$$Y = X + Z, X \in \{0, 1\}, Z \in \{0, a\}$$

We have to distinguish various cases depending on the values of a.

$a = 0$ In this case, $Y = X$, and $\max I(X;Y) = \max H(X) = 1$. Hence the capacity is 1 bit per transmission.

$a \neq 0, \pm 1$ In this case, Y has four possible values 0, 1, a and $1+a$. Knowing Y, we know the X which was sent, and hence $H(X|Y) = 0$. Hence $\max I(X;Y) = \max H(X) = 1$, achieved for an uniform distribution on the input X.

$a = 1$ In this case Y has three possible output values, 0, 1 and 2, and the channel is identical to the binary erasure channel discussed in class, with $a = 1/2$. As derived in class, the capacity of this channel is $1 - a = 1/2$ bit per transmission.

$a = -1$ This is similar to the case when $a = 1$ and the capacity here is also $1/2$ bit per transmission.

Problem 3 *(10 points)* ***Erasures and errors in a binary channel.*** *Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ϵ and the probability of erasure be α, so that the channel is illustrated as below:*

(a) Find the capacity of this channel.

(b) Specialize to the case of the binary symmetric channel ($\alpha = 0$).

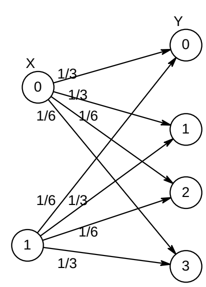

Figure 4: The channel diagram of the semi-symmetric channel.

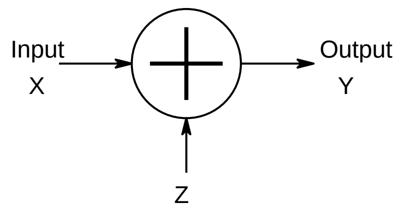

Figure 5: An additive noise channel in Problem 7.2.

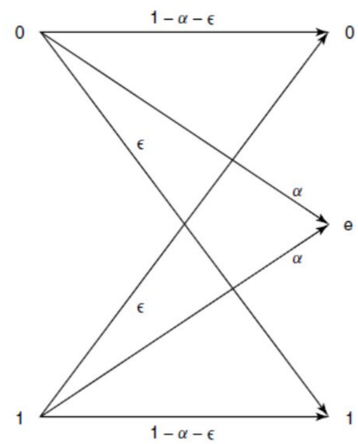

Figure 6: A binary channel with erasures and errors.

(c) Specialize to the case of the binary erasure channel ($\epsilon = 0$).

Solution:

(a) As with the examples in the text, we set the input distribution for the two inputs to be π and $1-\pi$. Then

$$\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)} (H(Y) - H(Y|X)) \\
&= \max_{p(x)} H(Y) - H(1-\epsilon-\alpha, \alpha, \epsilon).
\end{aligned}$$

As in the case of the erasure channel, the maximum value for $H(Y)$ cannot be $\log 3$, since the probability of the erasure symbol is α independent of the input distribution. Thus,

$$\begin{aligned}
H(Y) &= H(\pi(1-\epsilon-\alpha) + (1-\pi)\epsilon, \alpha, (1-\pi)(1-\epsilon-\alpha) + \pi\epsilon) \\
&= H(\alpha) + (1-\alpha) H\left(\frac{\pi+\epsilon-\pi\alpha-2\pi\epsilon}{1-\alpha}, \frac{1-\pi-\epsilon+2\epsilon\pi-\alpha+\alpha\pi}{1-\alpha}\right) \\
&\leq H(\alpha) + (1-\alpha),
\end{aligned}$$

with equality iff $\frac{\pi+\epsilon-\pi\alpha-2\pi\epsilon}{1-\alpha} = \frac{1}{2}$, which can be achieved by setting $\pi = \frac{1}{2}$.
Therefore the capacity of this channel is

$$\begin{aligned}
C &= H(\alpha) + 1 - \alpha - H(1-\alpha-\epsilon, \alpha, \epsilon) \\
&= H(\alpha) + 1 - \alpha - H(\alpha) - (1-\alpha) H\left(\frac{1-\alpha-\epsilon}{1-\alpha}, \frac{\epsilon}{1-\alpha}\right) \\
&= (1-\alpha)\left(1 - H\left(\frac{1-\alpha-\epsilon}{1-\alpha}, \frac{\epsilon}{1-\alpha}\right)\right).
\end{aligned}$$

(b) Setting $\alpha = 0$, we have

$$C = 1 - H(\epsilon),$$

which is the capacity of the binary symmetric channel.

(b) Setting $\epsilon = 0$, we have

$$C = 1 - \alpha,$$

which is the capacity of the binary erasure channel.