§ 2.10 The circulation and the curl of a Vector field

- **1. The circulation of a vector field**
  - **(1) Concept**

The line integral of a vector field $\vec{F}$ around a closed path *c* is called the circulation of $\vec{F}$, as can be

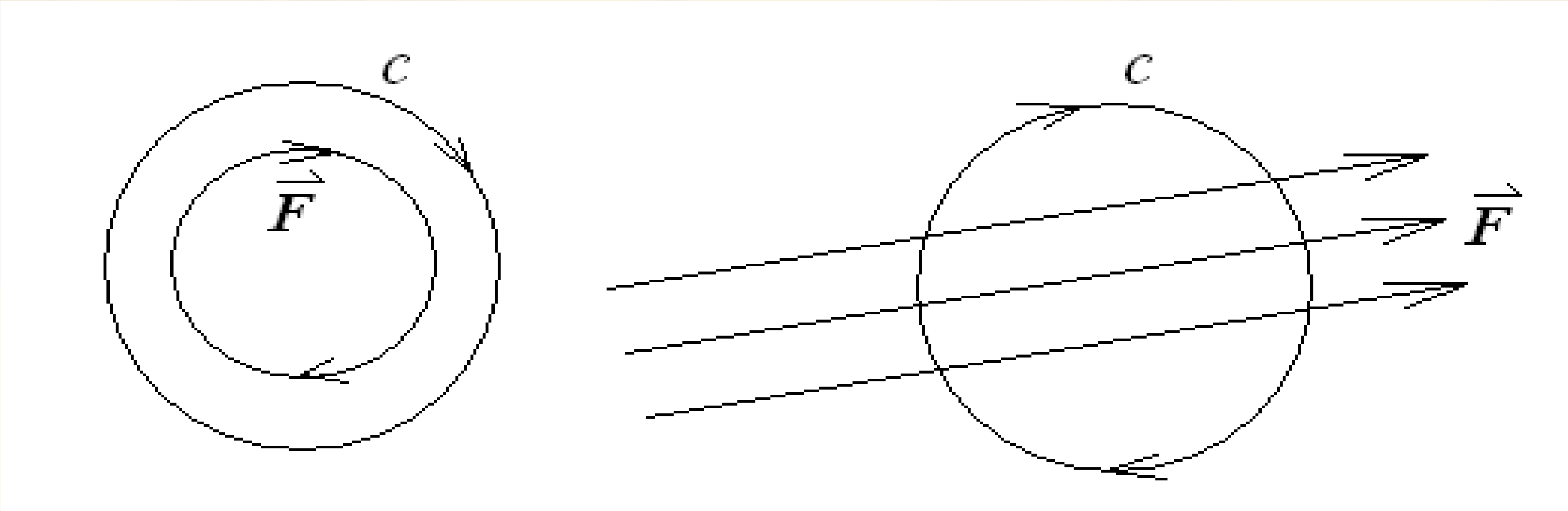
expressed by $\oint_c \vec{F} \bullet d\vec{l} = \oint_c F \cos\theta dl$

for a given curve *c*, the path from a point to another is a vector $d\vec{l}$. It is called differential

length vector $d\vec{l}$

(2) The expressed characteristics

$\oint_c \vec{F} \bullet d\vec{l} = \oint_c F\cos\theta dl \neq 0$, that is that the projection of $\vec{F}$ on the curve c is nonzero, rotational sources exist.



$\oint_c \vec{F} \bullet d\vec{l} = \oint_c F\cos\theta dl = 0$ the sum of the projection components of $\vec{F}$ on the curve c vanishes, rotational sources do not exist.

Example: fluid velocity $\vec{v}$, in a given region rotational source exists ,we can obtain

$$\oint_c \vec{v} \bullet d\vec{l} = \oint_c v\cos\theta dl \neq 0$$

•in a given region bounding by the curve *c* a rotational source exists.

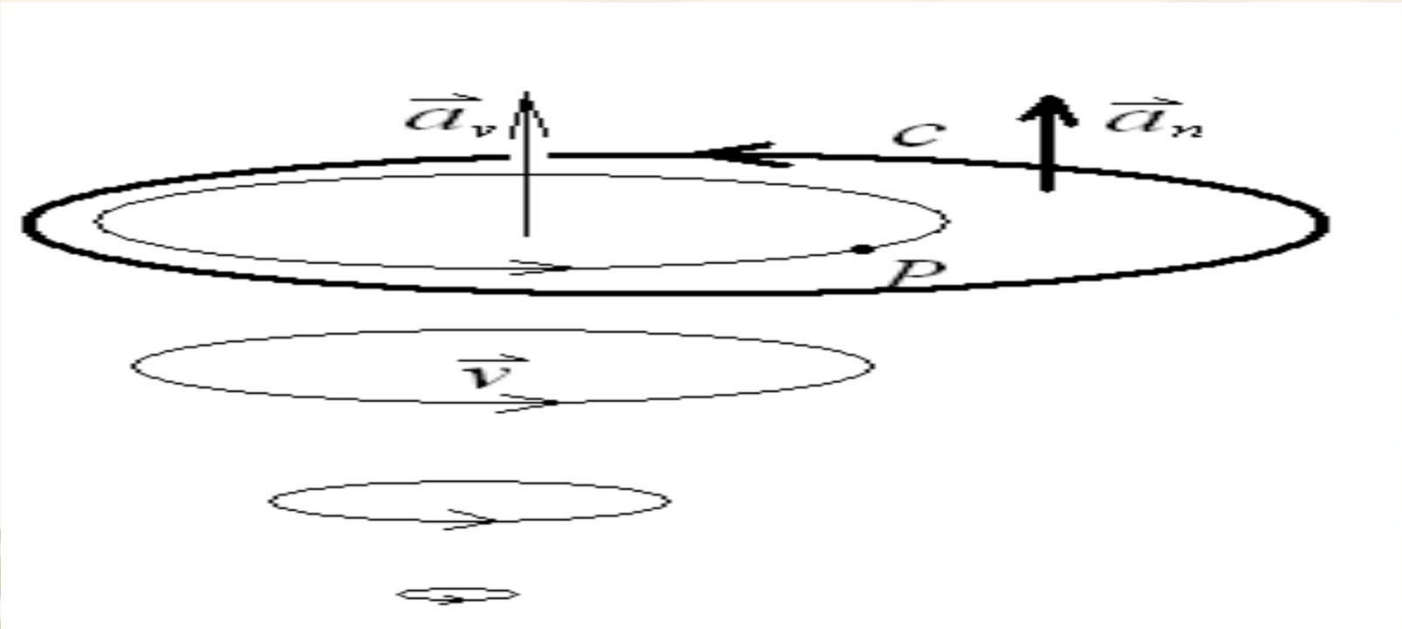
While

$$\oint_c \vec{v} \bullet d\vec{l} = \oint_c v\cos\theta dl = 0$$

•in a given region bounding by the curve *c* a rotational source does not exist .

The circulation can specifies the fact that in a given region bounding by the curve *c* a rotational source exists or does not, but it can not specify the fact that at a given point a rotational source exists or does not.

- (3)the circulation density of a vector field



in order to define the distribution of rotational sources in a given region, we can let the closed curve *c* reduced, the surface bounded by *c* approaches zero. Taking the limit of the circulation:

$$\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s}$$

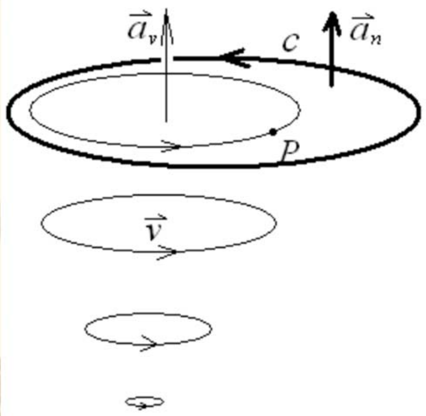
①can specify rotational sources at points on the surface bounded by the curve *c*.

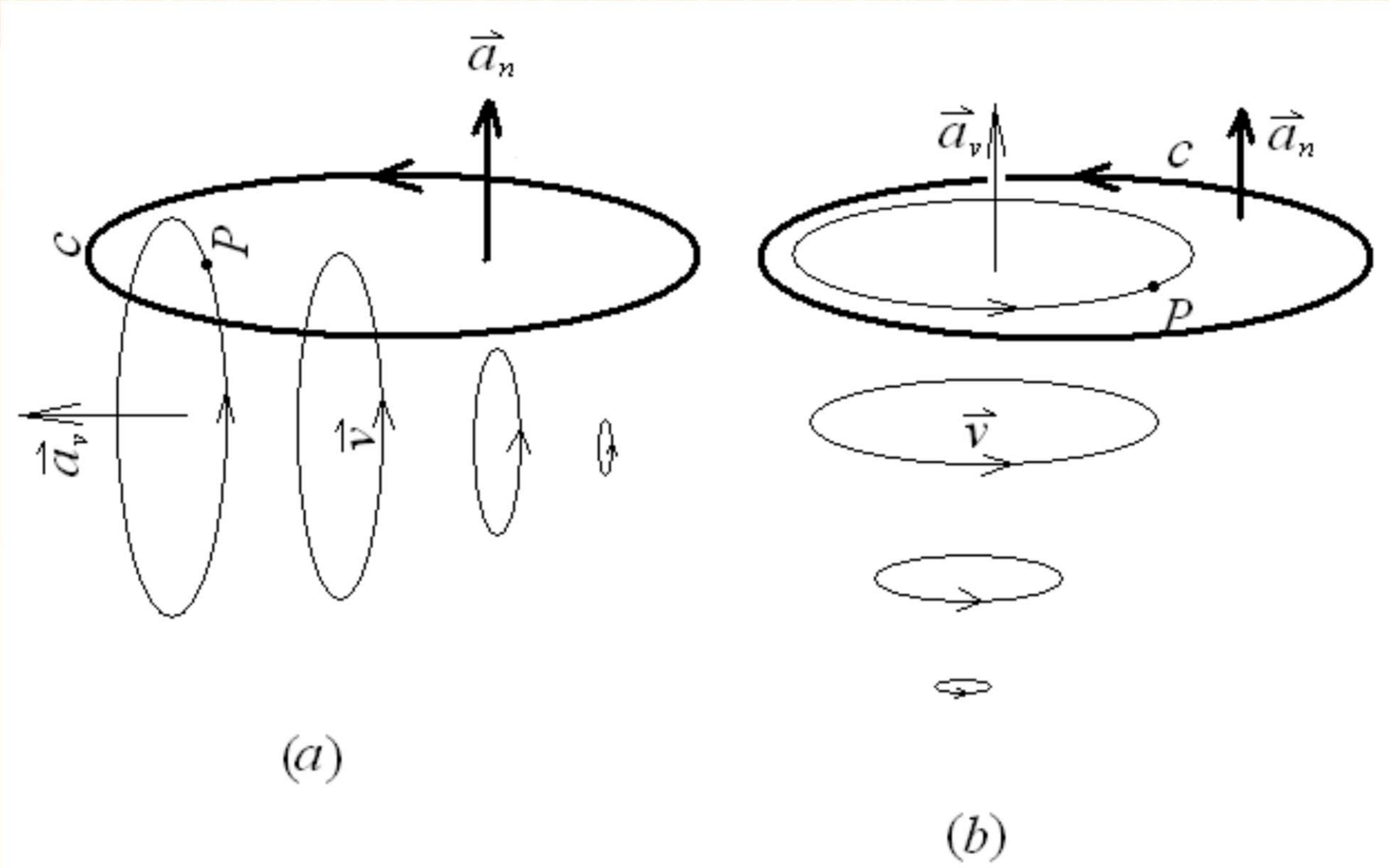
$$\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s}$$

=0， irrotational field, or conservative field

≠0， rotational field， or solenoidal field

② can not specify the rotational sources at a certain point in a three-dimensionally space.





The flow of water can provide an excellent example of a rotational velocity field of the flow.

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(a) The plane (surface) on which a rotational velocity field of the flow exist is normal to the plane(surface) bounded by the curve *c*:

$$\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} = 0, \text{ irrotational}$$

(b) The plane (surface) on which a rotational velocity field of the flow exist is parallel to the plane(surface) bounded by the curve *c*:

$$\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} \neq 0, \text{ rotational}$$

and the value is maximum. Obviously, the limit changes with the direction of the plane bounded by the curve *c*

$$\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s}$$

Can not completely specify a rotational source at a certain point in a 3-dimensional space.

2. the curl of a vector field

(1) concept

magnitude: the maximum of the circulation density

direction: the direction of the surface bounded by c

when $\lim\limits_{\Delta s \to 0} \dfrac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s}$ **is maximum.**

It can be expressed by

$$rot\vec{F} = \vec{a}_n \left[\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} \right]_{\max} = \text{curl } \vec{\mathbf{F}}$$

$$rot\vec{F} = \vec{a}_n \left[\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} \right]_{max} = \text{curl } \vec{\mathbf{F}}$$

$rot\vec{F} = 0$, $\vec{\mathbf{F}}$ **is irrotational vector field or conservative field**

$rot\vec{F} \neq 0$, $\vec{\mathbf{F}}$ **is rotational vector field or solenoidal field**

It can specify rotational velocity intensity and its direction at a point in space. It is a vector.

•The curl of a vector field:

It is a vector. Its magnitude is the maximum of the circulation density, its direction $\vec{a}_n$ is the same with the direction of the surface bounded by a closed path *c* when the maximum of the circulation density is obtained.

However, when the direction of a certain surface bounded by a closed path c is not the same with $\vec{a}_n$

Supposing θ is the angle between them, we will obtain

$$\mathrm{rot}_N \vec{\mathrm{F}} = \lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} = \mathrm{rot}\ \vec{\mathrm{F}} \bullet \vec{a}_n = \left| \mathrm{rot}\ \vec{F} \right| \cos\theta$$

Rewiew ：

circulation density(many values)

curl (only one ,maximum)

directional derivative

df/dl(many values)

gradient(only one , maximum)

scalar quantity

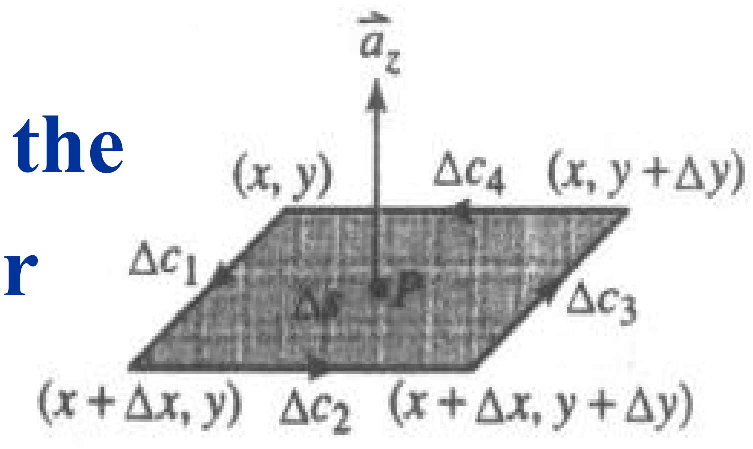
vector quantity

- (2) calculation and representation

$$\mathrm{rot}\vec{F} = \nabla\times\vec{F}$$

in rectangular coordinate system

- Supposing a point P is within the small surface Δs bounded by path Δc, as illustrated in the following figure.



- The line integral of $\vec{F}$ along the closed path Δc, consists of four separate paths:

$$\oint_{\Delta c} \vec{F} \bullet d\vec{l} = \oint_{\Delta c1} \vec{F} \bullet d\vec{l} + \oint_{\Delta c2} \vec{F} \bullet d\vec{l} + \oint_{\Delta c3} \vec{F} \bullet d\vec{l} + \oint_{\Delta c4} \vec{F} \bullet d\vec{l}$$

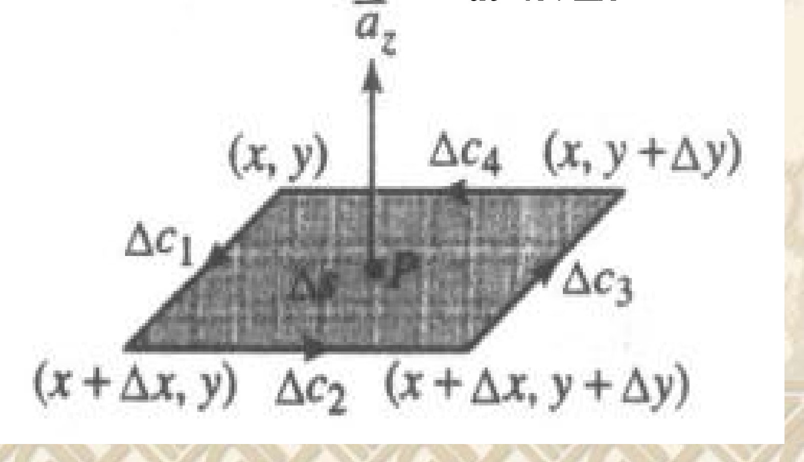
Now we evaluate each of the four integrals of the above equation separately.

Along the path Δc_1 :

$$\oint_{\Delta c1} \vec{F} \bullet d\vec{l} = \int_{x}^{x+\Delta x} \left[F_x \vec{a}_x + F_y \vec{a}_y + F_z \vec{a}_z \right] \bullet dx \vec{a}_x = \left[F_x \Delta x \right]_{at\ \ y}$$

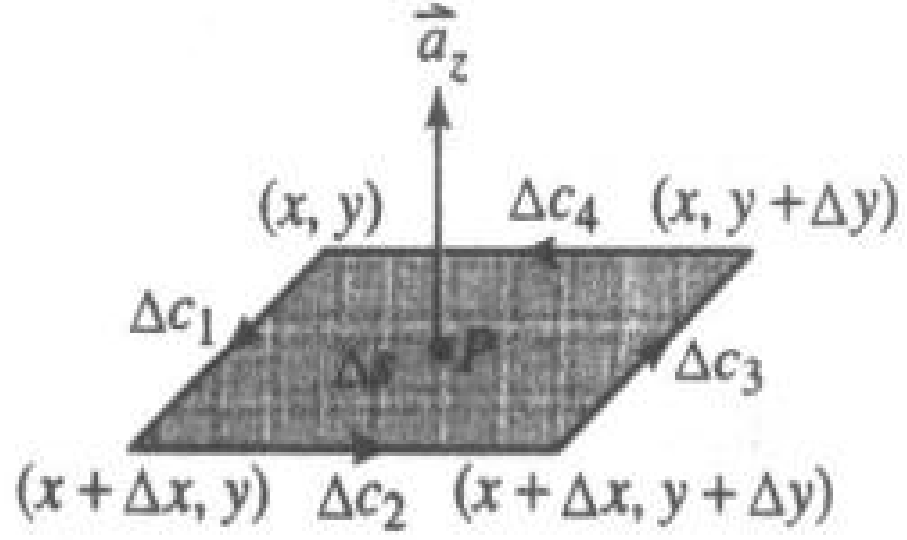
- Where $F_x \Delta x$ is to be evaluated at y, and we have made the assumption that the component is approximately constant from x to $x+ \Delta x$. This assumption is in accordance with the mean value theorem. We will make similar assumptions for the other components of $\vec{F}$.
- The line integration along the path Δc_2 :

$$\oint_{\Delta c2} \vec{F} \bullet d\vec{l} = \int_{y}^{y+\Delta y} \left[F_x \vec{a}_x + F_y \vec{a}_y + F_z \vec{a}_z \right] \bullet dy \vec{a}_y = \left[F_y \Delta y \right]_{at \;\; x+\Delta x}$$



❖ **The line integration along the path Δc_3 :**

$$\oint_{\Delta c3} \vec{\boldsymbol{F}} \bullet d\vec{\boldsymbol{l}} = \int_{x+\Delta x}^{x} \left[F_x \vec{\boldsymbol{a}}_x + F_y \vec{\boldsymbol{a}}_y + F_z \vec{\boldsymbol{a}}_z\right] \bullet [-dx\vec{\boldsymbol{a}}_x] = -\left[F_x \Delta x\right]_{at \quad y+\Delta y}$$



❖ **Finally, for the path Δc_4,**

$$\oint_{\Delta c4} \vec{\boldsymbol{F}} \bullet d\vec{\boldsymbol{l}} = \int_{y+\Delta y}^{y} \left[F_x \vec{\boldsymbol{a}}_x + F_y \vec{\boldsymbol{a}}_y + F_z \vec{\boldsymbol{a}}_z\right] \bullet [-dy\vec{\boldsymbol{a}}_y] = -\left[F_y \Delta y\right]_{at \quad x}$$

❖ **Thus, the line integration along a closed path *c* can be obtained:**

$$\oint_{\Delta c} \vec{\boldsymbol{F}} \bullet d\vec{\boldsymbol{l}} = [F_x \Delta x]_{at\ \ y} - [F_x \Delta x]_{at\ \ y+\Delta y} + [F_y \Delta y]_{at\ \ x+\Delta x} - [F_y \Delta y]_{at\ \ x}$$

- However, in the limit $\Delta x \to 0$, $\Delta y \to 0$,we can write

$$[F_x \Delta x]_{at\ \ y} - [F_x \Delta x]_{at\ \ y+\Delta y} = -\frac{\partial F_x}{\partial y} \Delta x \Delta y$$

$$[F_y \Delta y]_{at\ \ x+\Delta x} - [F_y \Delta y]_{at\ \ x} = \frac{\partial F_y}{\partial x} \Delta x \Delta y$$

by using the Taylor series expansion and neglecting the higher-order terms.

Therefore,

$$\oint_{\Delta c} \vec{\boldsymbol{F}} \bullet d\vec{\boldsymbol{l}} = \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] \Delta x \Delta y$$

- We shall note that the direction of the surface bounded by the closed path c is the same with $\vec{a}_z$. The differential surface element is $d\vec{s}_z = dxdy\vec{a}_z$

- The circulation density

$$\lim_{\Delta s\to 0}\frac{\oint_c \vec{F}\bullet d\vec{l}}{\Delta s}=\lim_{\Delta s\to 0}\frac{[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}]\Delta x\Delta y}{\Delta x\Delta y}=\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}$$

- If the direction of the surface bounded by the closed path c is at the same with $\vec{a}_x$, namely, the differential surface element is

$$d\vec{s}_x = dydz\vec{a}_x$$

Applying the same method, we can obtain the circulation density

$$\lim_{\Delta s\to 0}\frac{\oint_c \vec{F}\bullet d\vec{l}}{\Delta s}=\lim_{\Delta s\to 0}\frac{[\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}]\Delta y\Delta z}{\Delta y\Delta z}=\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}$$

❖ **In fact, Changing *z* into *x*, *x* into *y*, *y* into *z*, *the above equation can be obtained. Because***

$$d\vec{s}_z = dxdy\vec{a}_z$$

$$d\vec{s}_x = dydz\vec{a}_x$$

$$d\vec{s}_y = dxdz\vec{a}_y$$

- **If the direction of the surface bounded by the closed path c is at the same with $\vec{a}_y$, namely, the differential surface element is**

$$d\vec{s}_y = dxdz\vec{a}_y$$

Applying the same method, we can obtain the circulation density

$$\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} = \lim_{\Delta s \to 0} \frac{[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}]\Delta x \Delta z}{\Delta x \Delta z} = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$$

❖ **If the direction of the surface bounded by the closed path c is arbitrary $\vec{a}_n$, we can have**

$$rot\vec{F} = \vec{a}_n \left[\lim_{\Delta s \to 0} \frac{\oint_c \vec{F} \bullet d\vec{l}}{\Delta s} \right]_{\max} = \textbf{curl } \vec{\mathbf{F}}$$

$$= \vec{a}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \vec{a}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \vec{a}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$= \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \right) \times \left(\vec{a}_x F_x + \vec{a}_y F_y + \vec{a}_z F_z \right)$$

$$= \nabla \times \vec{F}$$

❖ Review:

$$\nabla \times \vec{F} = \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \right) \times \vec{F}$$

$$= \vec{a}_x \times \frac{\partial}{\partial x} \vec{F} + \vec{a}_y \times \frac{\partial}{\partial y} \vec{F} + \vec{a}_z \times \frac{\partial}{\partial z} \vec{F}$$

$$= \vec{a}_x \times \frac{\partial}{\partial x} \left(\vec{a}_x F_x + \vec{a}_y F_y + \vec{a}_z F_z \right) + \vec{a}_y \times \frac{\partial}{\partial y} \left(\vec{a}_x F_x + \vec{a}_y F_y + \vec{a}_z F_z \right)$$

$$+ \vec{a}_z \times \frac{\partial}{\partial z} \left(\vec{a}_x F_x + \vec{a}_y F_y + \vec{a}_z F_z \right)$$

$$= \vec{a}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) + \vec{a}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \vec{a}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right)$$

$$=\left(\vec{a}_x\frac{\partial}{\partial x}+\vec{a}_y\frac{\partial}{\partial y}+\vec{a}_z\frac{\partial}{\partial z}\right)\times\left(\vec{a}_xF_x+\vec{a}_yF_y+\vec{a}_zF_z\right)$$

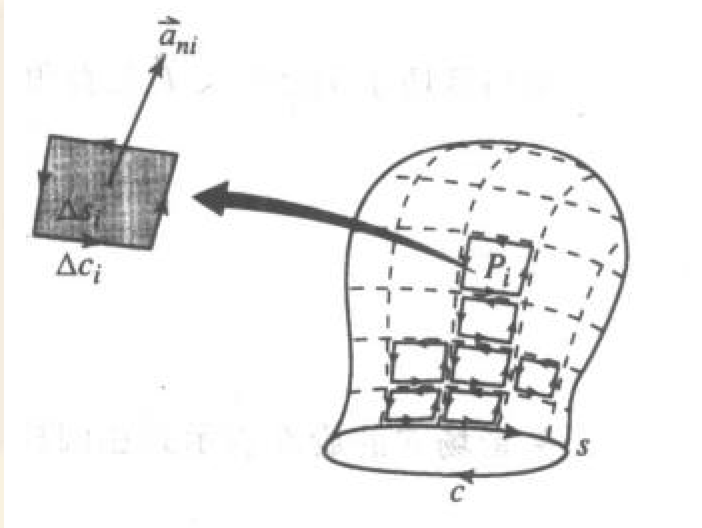
$$=\begin{vmatrix}\vec{a}_x & \vec{a}_y & \vec{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$$

- **3. stokes` theorem**

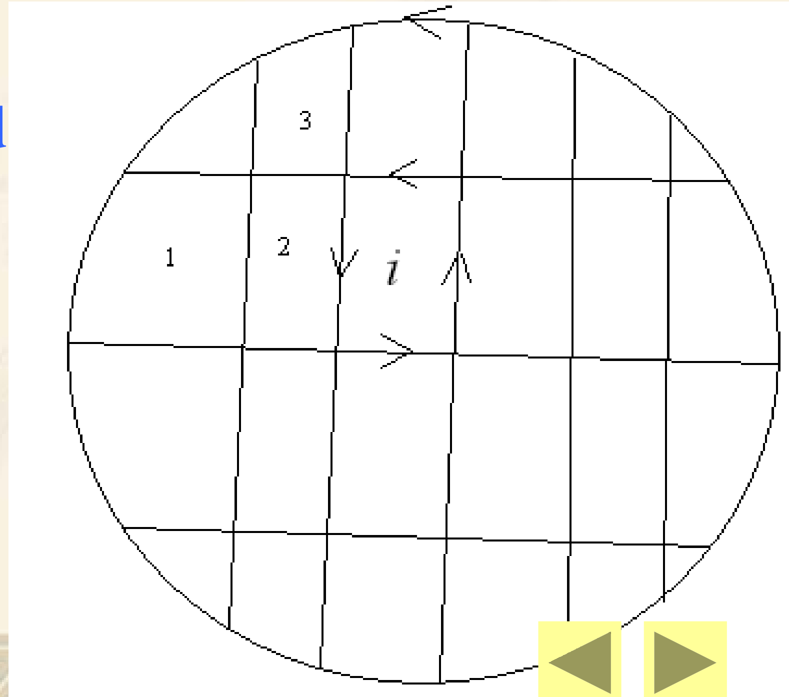
It states that the integral of the normal component of the curl of a vector field $\vec{F}$ over an area is equal to the line integral of the vector field $\vec{F}$ along the curve bounding the area.

Its mathematic representation is

$$\int_s \left(\nabla \times \vec{F}\right) \bullet d\vec{s} = \oint_c \vec{F} \bullet d\vec{l}$$



let us divide the surface area s into n elementary surface areas (cells) such that an ith cell has an area Δs_i with unit normal $\vec{a}_{ni}$ and bounded by a closed path Δc_i enclosing a point P.



n **elementary surface area:**

$$\oint_{c1}\vec{F}\bullet d\vec{l}+\oint_{c2}\vec{F}\bullet d\vec{l}+\oint_{c3}\vec{F}\bullet d\vec{l}+\cdots+\oint_{ci}\vec{F}\bullet d\vec{l}+\cdots=\oint_{c}\vec{F}\bullet d\vec{l}$$

the line integrals along adjacent elementary areas cancel because the length vectors are directed in opposite direct-ions. The only contribution is from the integrating over t-he path *c*. While

$$\oint_{c1}\vec{F}\bullet d\vec{l}+\oint_{c2}\vec{F}\bullet d\vec{l}+\oint_{c3}\vec{F}\bullet d\vec{l}+\cdots+\oint_{ci}\vec{F}\bullet d\vec{l}+\cdots$$

$$=\mathrm{rot}_{\mathrm{N}}\vec{F}ds_1+\mathrm{rot}_{\mathrm{N}}\vec{F}ds_2+\mathrm{rot}_{\mathrm{N}}\vec{F}ds_3+\cdots+\mathrm{rot}_{\mathrm{N}}\vec{F}ds_i+\cdots$$

$$=\mathrm{rot}\vec{F}\bullet\vec{a}_{n1}ds_1+\mathrm{rot}\vec{F}\bullet\vec{a}_{n2}ds_2+\mathrm{rot}\vec{F}\bullet\vec{a}_{n3}ds_3+\cdots+\mathrm{rot}\vec{F}\bullet\vec{a}_{ni}ds_i+\cdots$$

$$= \mathrm{rot}\vec{F} \bullet d\vec{s}_1 + \mathrm{rot}\vec{F} \bullet d\vec{s}_2 + \mathrm{rot}\vec{F} \bullet d\vec{s}_3 + \cdots + \mathrm{rot}\vec{F} \bullet d\vec{s}_i + \cdots$$

$$= \int_s \mathrm{rot}\vec{F} \bullet d\vec{s}$$

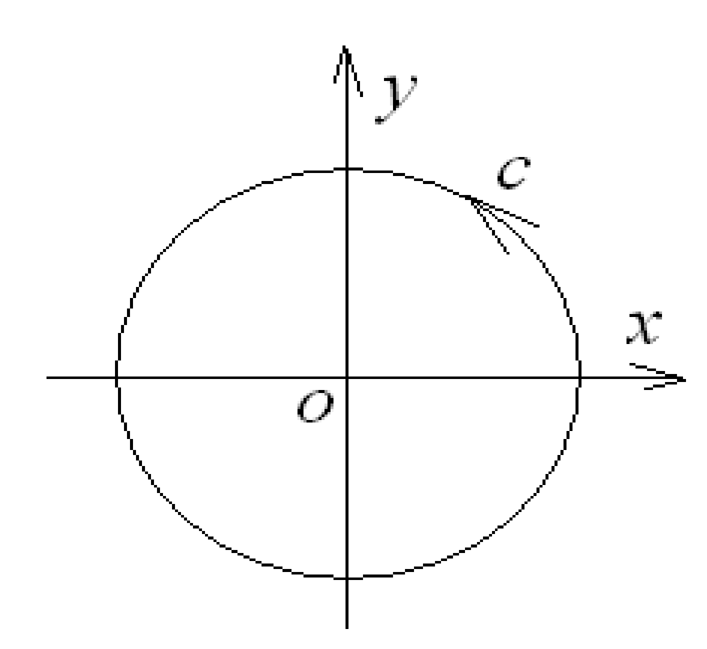
$$= \int_s (\nabla \times \vec{F}) \bullet d\vec{s}$$

Example:

A vector $\vec{\mathbf{F}} = \vec{a}_x x^2 + \vec{a}_y xy^2$**, the curve** c **is**

$x^2+y^2=a.^2$ **prove the stokes' theorem:**

$$\int_s \left(\nabla \times \vec{F}\right) \bullet d\vec{s} = \oint_c \vec{F} \bullet d\vec{l}$$



❖**solution:**

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy^2 & 0 \end{vmatrix} = \vec{a}_z y^2$$

$$\int_s \left(\nabla \times \vec{\boldsymbol{F}}\right) \bullet d\vec{\boldsymbol{s}} = \int_s \left(\nabla \times \vec{\boldsymbol{F}}\right) \bullet \vec{\boldsymbol{a}}_n ds = \int_s \vec{\boldsymbol{a}}_z y^2 \bullet \vec{\boldsymbol{a}}_z ds$$

$$= \int_s y^2 ds = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2$$

$$= \int_{-a}^{a} \frac{2}{3}\left(\sqrt{a^2 - x^2}\right)^3 dx = \int_{-\pi/2}^{\pi/2} \frac{2}{3} a^4 \cos^4 \theta d\theta$$

$$= \frac{\pi}{4} a^4$$

$$\oint_c \vec{F} \bullet d\vec{l} = \oint_c (\vec{a}_x x^2 + \vec{a}_y xy^2) \bullet (\vec{a}_x dx + \vec{a}_y dy) = \oint_c x^2 dx + xy^2 dy$$

$$= \int_0^{2\pi} a^2 \cos^2 \theta da \cos\theta + a^3 \cos\theta \sin^2 \theta da \sin\theta = \int_0^{2\pi} a^3 \cos\theta \sin^2 \theta da \sin\theta$$

$$= \int_0^{2\pi} a^4 \cos^2 \theta \sin^2 \theta d\theta = \int_0^{2\pi} a^4 \frac{1+\cos 2\theta}{2} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4} (1-\cos^2 2\theta) d\theta = \int_0^{2\pi} \frac{a^4}{4} \sin^2 2\theta d\theta$$

$$= \frac{\pi}{4} a^4$$

- **Exercises:**

page 66, T2.41, T2.42