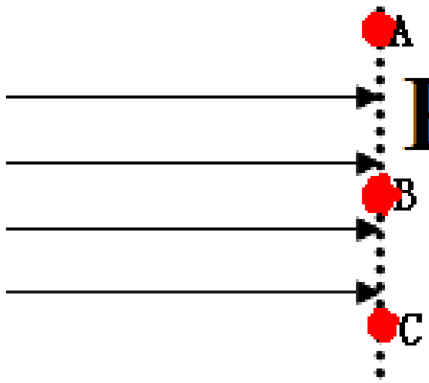
Chapter 8 Plane Wave Propagation

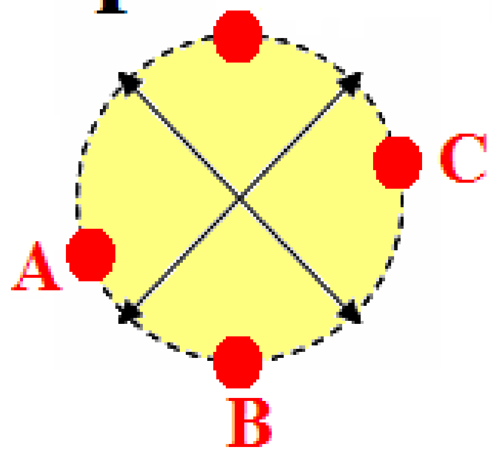
§ 8.1 Introduction

$$\vec{E} = \vec{a}_x E_0 \cos(\underbrace{\omega t + \varphi_0 - kz}_{\textbf{phase}})$$



Plane

spherical



A surface of constant phase:

It is the surface in which a field has the same phase shifts at all points.

A plane wave:

- If the surface of constant phase is a plane, we refer to such a wave as a plane wave.
- In fig.1, the wave is a plane wave.
- In fig.2, the wave is a spherical wave: the surface of constant phase is a spherical surface.

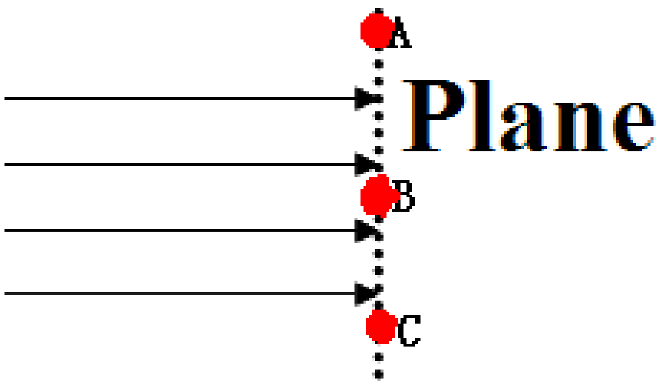


fig.1

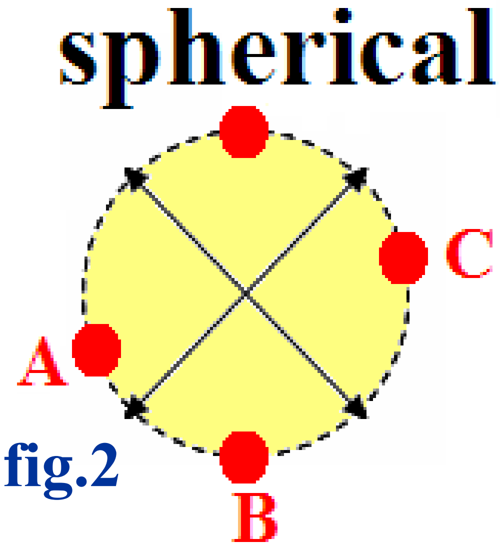
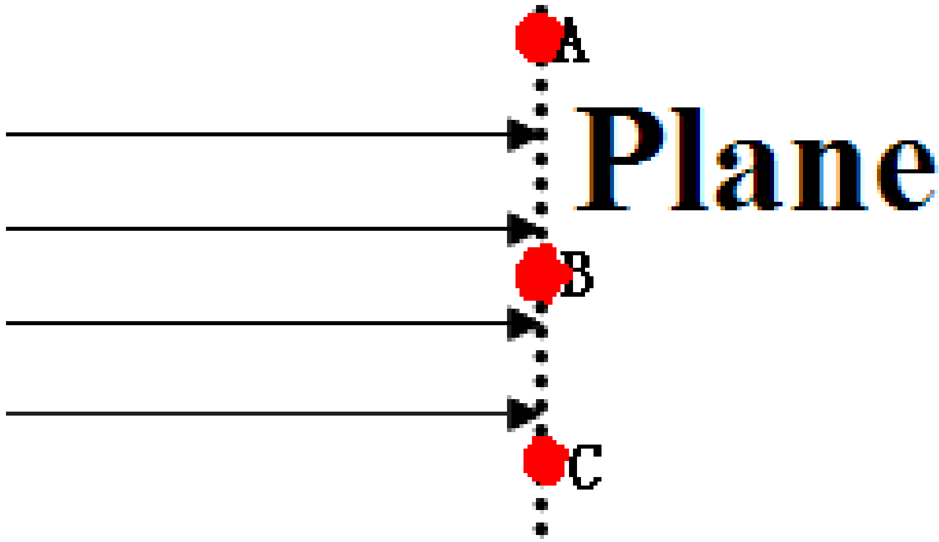


fig.2

A uniform plane wave:

- If a field has the same magnitude and direction in the plane of constant phase, we refer to such a wave as the uniform plane wave.



➢1. plane wave:

in real life, the fields generated by time-varying sources propagate as spherical waves.

However, in a small region far away from the radiating source, the spherical wave may be approximated as a plane wave, that is, one in which **all the field quantities are in a plane normal to the direction of its propagation(the transverse plane).** Consequently, a plane wave does not have any field component in its directi-on of propagation (the longitudinal direction).or the surface in which a field has the same phasor shifts at all points is a plane.

- 2. uniform plane wave

In the family of plane waves, the uniform plane wave
is one of the simplest to investigate and easiest to un-
derstand. The term uniform implies that, at any time,
a field has the same magnitude and direction in a pla-
ne containing it.

In this chapter, we first seek the solution of a uniform plane w-
ave in a medium.

The medium may be an unbounded dielectric medium, a finite-
ly conducting medium, or an ideal conductor. Finally, we intr-
oduce the concept of reflection and transmission of a uniform
plane wave

when it leaves one medium and enters another.

§ 8.2 uniform plane wave in a boundless dielectric medium

(page 307)

- 1. the field quantities

let us consider a dielectric medium in which the conduction current is almost nonexistent in comparison with the displacement current. Such a medium may be treated as a perfect dielectric or lossless medium (σ=0). Thus, by setting σ=0, we obtain Maxwell's equations in phasor form as

$$\nabla\times\dot{\vec{H}} = j\omega\varepsilon\dot{\vec{E}},\quad \nabla\times\dot{\vec{E}} = -j\omega\mu\dot{\vec{H}} \qquad (8.2.1)$$

for a uniform plane wave, a field has the same magnitude and direction in a plane, say plane *xoy*. So a uniform plane wave propagating in the z direction, $\dot{\vec{E}}$ and $\dot{\vec{H}}$ are not functions of *x* and y. that is,

$$\frac{\partial\dot{\vec{E}}}{\partial x} = \frac{\partial\dot{\vec{E}}}{\partial y} = 0 \qquad \frac{\partial\dot{\vec{H}}}{\partial x} = \frac{\partial\dot{\vec{H}}}{\partial y} = 0 \qquad (8.2.2)$$

from equation(8.2.1)and equation(8.2.2), we can get

$$(1) \quad j\omega\varepsilon\dot{E}_x = \frac{\partial\dot{H}_z}{\partial y} - \frac{\partial\dot{H}_y}{\partial z} = -\frac{\partial\dot{H}_y}{\partial z}$$

$$(1') \quad -j\omega\mu\dot{H}_y = \frac{\partial\dot{E}_x}{\partial z} - \frac{\partial\dot{E}_z}{\partial x} = \frac{\partial\dot{E}_x}{\partial z}$$

$$(2)\quad j\omega\varepsilon\dot{E}_y=\frac{\partial\dot{H}_x}{\partial z}-\frac{\partial\dot{H}_z}{\partial x}=\frac{\partial\dot{H}_x}{\partial z}$$

$$(2')\quad -j\omega\mu\dot{H}_x=\frac{\partial\dot{E}_z}{\partial y}-\frac{\partial\dot{E}_y}{\partial z}=-\frac{\partial\dot{E}_y}{\partial z}$$

$$(3)\quad j\omega\varepsilon\dot{E}_z=\frac{\partial\dot{H}_y}{\partial x}-\frac{\partial\dot{H}_x}{\partial y}=0\Rightarrow\dot{E}_z=0$$

$$(3')\quad -j\omega\mu\dot{H}_z=\frac{\partial\dot{E}_y}{\partial x}-\frac{\partial\dot{E}_x}{\partial y}=0\Rightarrow\dot{H}_z=0$$

from(3) and (3’),we can assume that the component of the field quantities $\vec{E}$ and $\vec{H}$ lie in a transverse plane, a plane perpendicular to the direction of propagation of the wave. We refer to such a wave as a plane wave. The

$\vec{E}$ and $\vec{H}$ fields have no components in the longitudinal direction(the direction of wave propagation). **that is, E_z=0 and H_z=0. such a wave is also called a transverse electromagnetic wave (TEM wave)**

- 2. wave equation and its solution

  from (1)and(1’), we can obtain

$$\frac{d^2\dot{E}_x}{dz^2}+\omega^2\mu\varepsilon\dot{E}_x=0 \quad \text{and} \quad \frac{d^2\dot{H}_y}{dz^2}+\omega^2\mu\varepsilon\dot{H}_y=0$$

from (2)and(2'), we can also obtain

$$\frac{d^2 \dot{E}_y}{dz^2} + \omega^2 \mu\varepsilon \dot{E}_y = 0 \quad \text{and} \quad \frac{d^2 \dot{H}_x}{dz^2} + \omega^2 \mu\varepsilon \dot{H}_x = 0$$

Note that we can rewrite these similar wave equations as

$$\frac{d^2 \dot{E}_x}{dz^2} + \beta^2 \dot{E}_x = 0 \qquad (8.2.3)$$

where β is wave number(or the phase constant) and

$$\beta^2 = \omega^2 \mu\varepsilon$$

the general solution of equation(8.2.3) can be given by

$$\frac{d^2\dot{E}_x}{dz^2}+\beta^2\dot{E}_x=0$$

$$\dot{E}_x=\dot{E}_{xf}e^{-j\beta z}+\dot{E}_{xb}e^{j\beta z}$$

$$=E_{xf}e^{j\varphi_{xf}}e^{-j\beta z}+E_{xb}e^{j\varphi_{xb}}e^{j\beta z} \qquad (8.2.4)$$

(8.2.4) can be rewritten in the time domain as

$$E_x(z,t)=\mathrm{Re}[\dot{E}_x e^{j\omega t}]$$

$$=E_{xf}\cos(\omega t-\beta z+\varphi_{xf})+E_{xb}\cos(\omega t+\beta z+\varphi_{xb})$$

if the first term on the right-hand side of (8.2.4) reverts to (hold) its initial magnitude and phase when t and z increase, its phase must be constant, that is,

$$\omega t-\beta z+\varphi_{xf}=\text{constant}$$

obviously, at any given time all points in space having the same phase lie in a plane ($z=z_0$).

Defferentiating the above equation with respect to t, we obtain the speed of a plane of constant p-hase (phase speed) as

$$\upsilon_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

which is greater than zero because ω and β are both posit-ive quantities. The wave propagates in the positive z dire-ction. Hence, the first term on the right-hand side of equa-tion(*8.2.4)* represents a forward-travelling wave. The pha-se velocity of the forward-travelling wave is

$$\vec{\mathbf{v}}_p = \frac{dz}{dt}\vec{a}_z = \frac{\omega}{\beta}\vec{a}_z$$

the wavelength λ:

it is the distance between two planes($z1$ and $z2$) when the phase difference between at any given time is 2π radians. That is,

$$(\omega t - \beta z_1 + \varphi_{xf}) - (\omega t - \beta z_2 + \varphi_{xf}) = 2\pi$$

or

$$\lambda = z_2 - z_1 = 2\pi/\beta$$

similarly, the second term on right-hand side of equation(8.2.4) represents a backward-travelling wave, its velocity is

$$\vec{\mathbf{v}}_p = \frac{dz}{dt}\vec{a}_z = -\frac{\omega}{\beta}\vec{\mathbf{a}}_z$$

time period:

when the phase $\omega\Delta t$ is 2π, the time Δt is called one time period T. that is

$$T=\Delta t=2\pi/\omega$$

➢3. relationships between the electric field and the magnetic field since the backward traveling wave has the same charact eristics as the forward traveling wave, we only study the forward wave propagating along the positive z direction. Then the x and y components of the electric field, in the phasor do main, are

$$\dot{E}_x = \dot{E}_{xf} e^{-j\beta z} = E_{xf} e^{j\varphi_{xf}} e^{-j\beta z}$$
$$= E_{xf} e^{-j(\beta z - \varphi_{xf})}$$

$$\dot{E}_y = \dot{E}_{yf} e^{-j\beta z} = E_{yf} e^{j\varphi_{yf}} e^{-j\beta z}$$
$$= E_{yf} e^{-j(\beta z - \varphi_{yf})}$$

they are functions of z. the z component of the electric field is zero.

Using Maxwell's equation

$$\nabla \times \dot{\vec{\mathbf{E}}} = -j\omega\mu\dot{\vec{\mathbf{H}}}$$

we obtain the *x* and *y* components of the magnetic field as

$$(1')\quad -j\omega\mu\dot{H}_y = \frac{\partial \dot{E}_x}{\partial z} = -j\beta\dot{E}_x \qquad (1')\quad \eta\dot{H}_y = \dot{E}_x$$

$$(2')\quad -j\omega\mu\dot{H}_x = -\frac{\partial \dot{E}_y}{\partial z} = j\beta\dot{E}_y \quad \text{or} \quad (2')\quad -\eta\dot{H}_x = \dot{E}_y$$

$$(3')\quad \dot{H}_z = 0 \qquad (3')\quad \dot{H}_z = 0$$

these equations can also be written in concise form as

$$\dot{\vec{H}} = \vec{\mathbf{a}}_{\mathbf{x}}\dot{H}_x + \vec{\mathbf{a}}_y\dot{H}_y = -\vec{\mathbf{a}}_{\mathbf{x}}\frac{\beta}{\omega\mu}\dot{E}_y + \vec{\mathbf{a}}_y\frac{\beta}{\omega\mu}\dot{E}_x$$

$$= \frac{\beta}{\omega\mu}[\vec{\mathbf{a}}_z \times \vec{\mathbf{a}}_y\dot{E}_y + \vec{\mathbf{a}}_z \times \vec{\mathbf{a}}_x\dot{E}_x] = \frac{\beta}{\omega\mu}\vec{\mathbf{a}}_z \times [\vec{\mathbf{a}}_y\dot{E}_y + \vec{\mathbf{a}}_x\dot{E}_x]$$

$$= \frac{\beta}{\omega\mu}\vec{a}_z \times \dot{\vec{E}} \qquad \because \beta = \omega\sqrt{\mu\varepsilon}$$

$$= \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}\vec{a}_z \times \dot{\vec{E}} = \frac{1}{\eta}\vec{a}_z \times \dot{\vec{E}}$$

or

$$\dot{\vec{E}} = \eta\dot{\vec{H}} \times \vec{a}_z$$

where

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad (8.2.5)$$

has the units of ohms. For this reason, $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is called the intrinsic (or wave) impedance. In general, wave impedance is one complex number. However, for the wave propagating in an ideal dielectric, it is a pure resistance and one real number, the corresponding components of the $\dot{\vec{E}}$ and $\dot{\vec{H}}$ fields are in time phase with each other.

The $\vec{a}_z$ term must be perceived as the unit vector in the direction of wave propagation. For example, if the wave propagates in the y direction, then $\vec{a}_z$ must be replaced by $\vec{a}_y$ to compute one of the unknown fields in terms of the other. The field quantities are normal to the direction of wave propagation, and also the electric field intensity is perpendicular to the magnetic field. That is

$$\dot{\vec{E}} \bullet \dot{\vec{H}} = \dot{\vec{H}} \bullet \vec{a}_z = \dot{\vec{E}} \bullet \vec{a}_z = 0$$

Obviously, the electric field, the magnetic field and $\vec{a}_z$ are mutually orthogonal. That is

$$\vec{a}_E \times \vec{a}_H = \vec{a}_z$$

where $\vec{a}_E$ is the unit vector in the direction of the electric field and $\vec{a}_H$ is the unit vector in the direction of the magnetic field.

The average power density at any point in a perfect dielectric

$$\vec{\mathrm{S}}_{ave} = \frac{1}{T}\int_0^T \vec{\mathrm{S}}\,dt = \frac{1}{T}\int_0^T \vec{E}\times\vec{H}\,dt = \mathrm{Re}[\dot{\vec{\mathrm{S}}}]$$

$$= \frac{1}{2}\mathrm{Re}[\dot{\vec{E}}\times\dot{\vec{H}}^*] = \frac{1}{2}\mathrm{Re}[(\vec{a}_x\dot{E}_x + \vec{a}_y\dot{E}_y)\times(\vec{a}_x\dot{H}^*{}_x + \vec{a}_y\dot{H}^*{}_y)]$$

$$=\frac{1}{2}\mathrm{Re}[\vec{a}_z\dot{E}_x\dot{H}^*{}_y-\vec{a}_z\dot{E}_y\dot{H}^*{}_x]$$

$$=\frac{1}{2}\mathrm{Re}[\vec{a}_z\eta\dot{H}_y\dot{H}^*{}_y+\vec{a}_z\eta\dot{H}_x\dot{H}^*{}_x]$$

$$=\vec{a}_z\frac{\eta}{2}[H_{xf}^2+H_{yf}^2]\quad=\vec{a}_z\frac{\eta}{2}[H_{xf}^2+H_{yf}^2]=\vec{a}_z\frac{\eta}{2}H^2$$

$$=\frac{1}{2}\mathrm{Re}\left[\vec{a}_z\dot{E}_x\left(\frac{\dot{E}_x}{\eta}\right)^*+\vec{a}_z\dot{E}_y\left(\frac{\dot{E}_y}{\eta}\right)^*\right]$$

$$=\vec{a}_z\frac{1}{2\eta}(E_{xf}^2+E_{yf}^2)=\vec{a}_z\frac{1}{2\eta}E^2\qquad(8.2.6)$$

where $E^2=\dot{\vec{E}}\bullet\dot{\vec{E}}^*=(\vec{a}_x\dot{E}_{xf}+\vec{a}_y\dot{E}_{yf})\bullet(\vec{a}_x\dot{E}_{xf}+\vec{a}_y\dot{E}_{yf})^*$

and

$$H^2 = \dot{\vec{H}} \bullet \dot{\vec{H}}^*$$

$$= (\vec{a}_x \dot{H}_{xf} + \vec{a}_y \dot{H}_{yf}) \bullet (\vec{a}_x \dot{H}_{xf} + \vec{a}_y \dot{H}_{yf})^*$$

substituting for η from (3.2.5) in (3.2.6)and expressing the result in terms of the phase velocity as

$$\vec{S}_{ave} = \vec{a}_z \frac{1}{2\eta} E^2 = \vec{a}_z \frac{1}{2\eta} E^2 \frac{\omega}{\beta} \frac{\beta}{\omega}$$

$$= \vec{a}_z \frac{1}{2\sqrt{\frac{\mu}{\varepsilon}}} E^2 \upsilon_p \frac{\omega\sqrt{\mu\varepsilon}}{\omega} = \frac{1}{2} \varepsilon E^2 \vec{\upsilon}_p = \frac{1}{2} \mu H^2 \vec{\boldsymbol{\upsilon}}_p = W_{tave} \vec{\boldsymbol{\upsilon}}_p$$

and the average energy density *Wtave*, the average electric energy density *Weave* and the average magnetic energy density *Wmave* are

$$W_{eave} = \frac{1}{4}\varepsilon E^2 = W_{mave} = \frac{1}{4}\mu H^2$$

$$W_{tave} = W_{eave} + W_{mave}$$

❖summary:

1. a plane wave and a uniform plane wave

2. wave equation and its solution:

a forward traveling wave and a backward traveling wave

$$\dot{E}_x = \dot{E}_{xf} e^{-j\beta z} + \dot{E}_{xb} e^{j\beta z} = E_{xf} e^{j\varphi_{xf}} e^{-j\beta z} + E_{xb} e^{j\varphi_{xb}} e^{j\beta z}$$

3. relationship: $\dot{\vec{\boldsymbol{E}}} = \eta \dot{\vec{\boldsymbol{H}}} \times \vec{\boldsymbol{a}}_z$ and $\dot{\vec{\boldsymbol{H}}} = \frac{1}{\eta} \vec{\boldsymbol{a}}_z \times \dot{\vec{\boldsymbol{E}}}$

where $\vec{\mathbf{a}}_z$ is the direction of the wave propagation, η is wave impedance.

4. Characteristics:

Phase: $\omega t-\beta z+\varphi_0$ and $\omega t-\beta z+\varphi_0$ =constant

Phase constant: $\beta^2=\omega^2\mu\varepsilon$ or $\beta=\omega\sqrt{\mu\varepsilon}$

phase speed and phase velocity: $\upsilon_p=\dfrac{dz}{dt}=\dfrac{\omega}{\beta}$ and

$$\vec{\boldsymbol{v}}_p=\frac{dz}{dt}=\frac{\omega}{\beta}\vec{\boldsymbol{a}}_z$$

time period: $T=2\pi/\omega$

wavelength: $\lambda=2\pi/\beta$

wave impedance: $\eta=\sqrt{\dfrac{\mu}{\varepsilon}}$

Poynting's vector, average power density:

$$\vec{\mathbf{S}}_{ave} = \vec{a}_z \frac{1}{2}\eta H^2 = \vec{a}_z \frac{1}{2\eta} E^2$$

$$\vec{S}_{ave} = W_{tave} \vec{\boldsymbol{\upsilon}}_p$$

Average energy density:

$$W_{eave} = \frac{1}{4}\varepsilon E^2 = W_{mave} = \frac{1}{4}\mu H^2$$

$$W_{tave} = W_{eave} + W_{mave}$$

If a uniform plane wave propagates in free space, we can simply replace μ with μ_0 *and* ε with ε_0 in all the preceding equations.

free space (or vacuum) is a special case of a dielectric medium in which $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ that is

Phase constant: $\beta_0{}^2 = \omega^2 \mu_0 \varepsilon_0$

$$or \quad \beta_0 = \omega\sqrt{\mu_0 \varepsilon_0} = \omega / c$$

$c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s is the speed of light.

$$\vec{\upsilon}_p = \frac{dz}{dt}\vec{a}_z = \frac{\omega}{\beta_0}\vec{a}_z = c\vec{a}_z$$

$$\upsilon_p = \frac{dz}{dt} = \frac{\omega}{\beta_0} = c$$ and

phase speed and phase velocity:

time period: $T = 2\pi/\omega$

wavelength: $\lambda_0 = 2\pi/\beta_0 = c/f$

wave impedance: $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\Omega$

The end