§ 2.8 The gradient and the directional derivative of a scalar field (function)

➢ 1.Directional derivative

(1) concept

Before we begin our discussion of our object, it is important to define the derivative of a function of one or more variables.

The derivative of a scalar function *f* (*x*) with respect to *x* is defined as

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \qquad \textbf{(2.8-1)}$$

$\frac{df}{dx}$ **is the rate of change of the scalar function *f* in the direction of the unit vector** $\vec{a}_x$

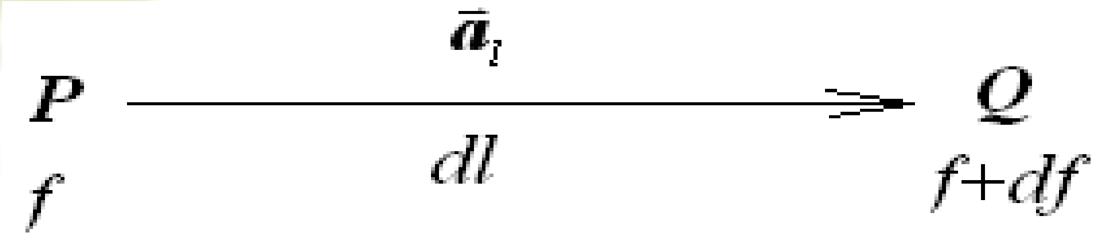
let *f* (*x*, *y*, *z*) be a real-valued differentiable function of *x*, *y*, and *z*.The partial derivative of *f* (*x*, *y*, *z*) from point P to M along the *x* axes (or along $\vec{a}_x$), can be written as

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x} \qquad \textbf{(2.8-2)}$$

The derivative of the function *f*(*l*) from point P to Q along the direction of $d\vec{l}$(or $\vec{a}_l$), can be written as

$$\frac{df}{dl} = \lim_{\Delta l \to 0} \frac{f(l+\Delta l) - f(l)}{\Delta l} = \lim_{\Delta l \to 0} \frac{\Delta f}{\Delta l} \qquad \textbf{(2.8-3)}$$

$\dfrac{df}{dl}$ **is called the directional derivative of** f **along** $\vec{a}_l$



In fact, $\dfrac{df}{dl}$ **is the rate of change of the scalar function**

f **in the direction of the unit vector** $\vec{a}_l$

Obviously,

$$\begin{array}{ccc} P & \xrightarrow[\quad dl \quad]{\quad \vec{a}_l \quad} & Q \\ f & & f+df \end{array} \qquad \frac{df}{dl} = \lim_{\Delta l \to 0} \frac{f(l+\Delta l) - f(l)}{\Delta l} = \lim_{\Delta l \to 0} \frac{\Delta f}{\Delta l}$$

$$\Big\downarrow \quad \vec{a}_l = \vec{a}_x \qquad\qquad \Big\downarrow$$

$$\begin{array}{ccc} P & \xrightarrow[\quad dx \quad]{\quad \vec{a}_x \quad} & Q \\ f & & f+df \end{array} \qquad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

$\frac{df}{dx}$ **is the rate of change of the scalar function *f* in the direction of the unit vector** $\vec{a}_x$

$\frac{df}{dl}$ **is the rate of change of the scalar function *f* in the direction of the unit vector** $\vec{a}_l$

Both of them are the rate of change of the scalar function *f* from a point to another point in space.

- **(2)Some subjects about relative calculus**

the differential change in *f* from point P to Q, can be written as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left(\frac{\partial f}{\partial x}\vec{\boldsymbol{a}}_x + \frac{\partial f}{\partial y}\vec{\boldsymbol{a}}_y + \frac{\partial f}{\partial z}\vec{\boldsymbol{a}}_z\right) \bullet \left(dx\vec{\boldsymbol{a}}_x + dy\vec{\boldsymbol{a}}_y + dz\vec{\boldsymbol{a}}_z\right) \qquad \textbf{(2.8-4)}$$

in terms of the differential length element

$$d\vec{\boldsymbol{l}} = dx\vec{\boldsymbol{a}}_x + dy\vec{\boldsymbol{a}}_y + dz\vec{\boldsymbol{a}}_z$$

from point P to Q, we can rewrite (2.8-4) as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(\frac{\partial f}{\partial x}\vec{\boldsymbol{a}}_x + \frac{\partial f}{\partial y}\vec{\boldsymbol{a}}_y + \frac{\partial f}{\partial z}\vec{\boldsymbol{a}}_z\right) \bullet d\vec{\boldsymbol{l}}$$

$$df = \left(\frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z \right) \bullet d\vec{l}$$

or

$$\frac{df}{dl} = \left(\frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z \right) \bullet \frac{d\vec{l}}{dl} = \vec{G} \bullet \vec{a}_l$$

where

$$\vec{a}_l = \frac{d\vec{l}}{dl}$$

is a unit vector from P to Q in the direction of

$d\vec{l}$ (or $\vec{a}_l$)

and

$$\vec{G} = \frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z$$

➢2. The gradient of a scalar function

➢（1）Concept

Since $\frac{df}{dl} = \left(\frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z\right) \bullet \frac{d\vec{l}}{dl} = \vec{G} \bullet \vec{a}_l$

$$= \left|\vec{G}\right| \vec{a}_n \bullet \vec{a}_l \qquad (\because \quad \vec{a}_n \bullet \vec{a}_l = \left|\vec{a}_n\right|\left|\vec{a}_l\right| \cos\theta)$$

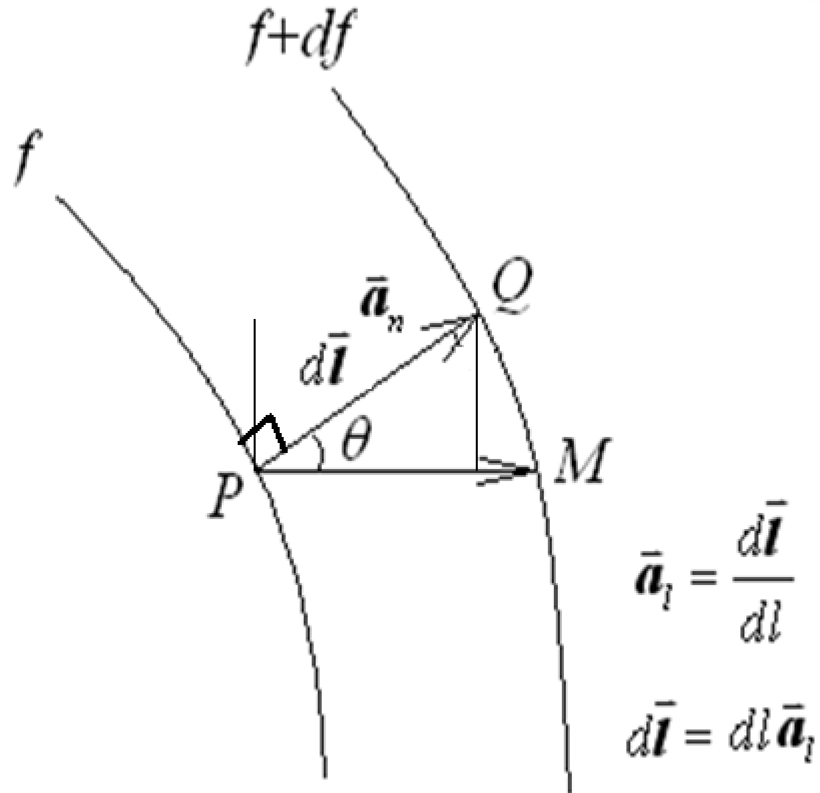
$$= \left|\vec{G}\right| \cos\theta$$

$$\leq \left|\vec{G}\right| \qquad \textbf{(2.8-5)}$$

from (2.8-5) it is evident that the rate of change in the function f is maximum when $d\vec{l}$ (or $\vec{a}_l$) and $\vec{G}=\left|\vec{G}\right|\vec{a}_n$ are collinear. That is

$$\left.\frac{df}{dl}\right|_{\max}=\left|\vec{G}\right|$$

there exists a surface passing through P on which f is constant.
Similarly, there exists a surface passing through Q on which $f+df$ is constant.



➢2. The gradient of a scalar function

For the ratio *df/dl* to be maximum, the distance *dl* from P to Q must be minimum. In other words *df/dl* is maximum when $d\vec{l}$ (or $\vec{a}_l$) is normal to the surface $f(x, y, z)$ =constant. This, in turn, implies that

$\vec{G} = |\vec{G}|\vec{a}_n$ is normal to the surface=constant

$\vec{G} = |\vec{G}|\vec{a}_n$, by definition, is the gradient of $f(x, y, z)$

(a)Direction: it is normal to the surface on which the given function is constant.

(b)It points in the direction in which the given function changes most rapidly with position.

(c)Its magnitude gives the maximum rate of change of the given function per unit distance.

(d)The directional derivative of a function at a point in any direction is equal to the dot product of the gradient of the function and the unit vector in that direction.

We write the gradient of *f* (*x*, *y*, *z*) as ∇ *f* (*x*, *y*, *z*)

$$\nabla = \vec{\boldsymbol{a}}_x \frac{\partial}{\partial x} + \vec{\boldsymbol{a}}_y \frac{\partial}{\partial y} + \vec{\boldsymbol{a}}_z \frac{\partial}{\partial z}$$

$$\nabla f = \left(\vec{\boldsymbol{a}}_x \frac{\partial}{\partial x} + \vec{\boldsymbol{a}}_y \frac{\partial}{\partial y} + \vec{\boldsymbol{a}}_z \frac{\partial}{\partial z} \right) f = \frac{\partial f}{\partial x} \vec{\boldsymbol{a}}_x + \frac{\partial f}{\partial y} \vec{\boldsymbol{a}}_y + \frac{\partial f}{\partial z} \vec{\boldsymbol{a}}_z$$

In the cylindrical coordinate system:

$$\nabla = \frac{\partial}{\partial \rho} \vec{\boldsymbol{a}}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} \vec{\boldsymbol{a}}_\phi + \frac{\partial}{\partial z} \vec{\boldsymbol{a}}_z$$

$$\nabla f = \frac{\partial f}{\partial \rho} \vec{\boldsymbol{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \vec{\boldsymbol{a}}_\phi + \frac{\partial f}{\partial z} \vec{\boldsymbol{a}}_z$$

(e)The directional derivative of a function at a point in any direction is equal to the dot product of the gradient of the function and the unit vector in that direction.

$$df/dl = \nabla f(x, y, z) \bullet \vec{a}_l$$

or

$$df = \nabla f(x, y, z) \bullet \vec{a}_l dl$$

$$= \nabla f(x, y, z) \bullet d\vec{l}$$

In the spherical coordinate system:

$$\nabla f = \frac{\partial f}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\vec{a}_\phi$$

❖ **Example 1**

Considering $f(x, y)$ **, the directional derivative can be given by**

$$\frac{df}{dl} = \lim_{\Delta l \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{\Delta l}$$

since

$$f(x+\Delta x, y+\Delta y) - f(x, y) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + o(\Delta l)$$

Thus,

$$\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{\Delta l} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta l} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta l} + \frac{o(\Delta l)}{\Delta l}$$

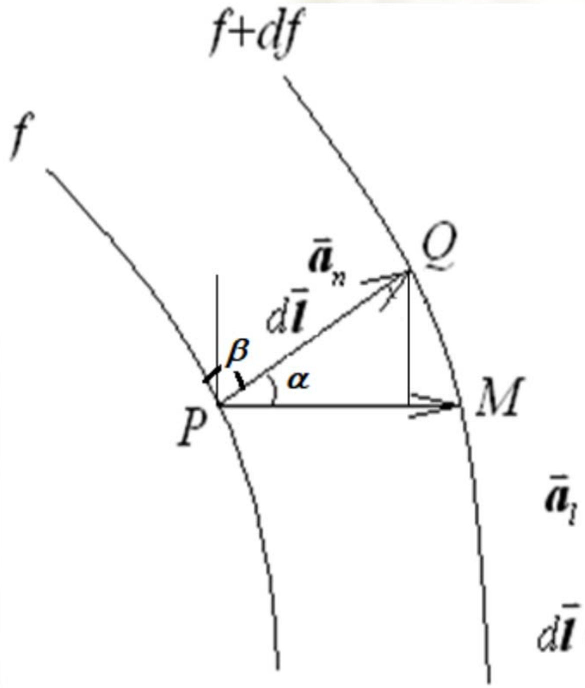
And

$$\frac{df}{dl} = \lim_{\Delta l\to 0}\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{\Delta l} = \lim_{\Delta l\to 0}\left(\frac{\partial f}{\partial x}\frac{\Delta x}{\Delta l} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta l} + \frac{o(\Delta l)}{\Delta l}\right)$$

$$= \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

$$= \left(\vec{a}_x\frac{\partial f}{\partial x} + \vec{a}_y\frac{\partial f}{\partial y}\right)\bullet\left(\vec{a}_x\cos\alpha + \vec{a}_y\cos\beta\right)$$

$$= \left(\vec{a}_x\frac{\partial f}{\partial x} + \vec{a}_y\frac{\partial f}{\partial y}\right)\bullet\vec{a}_l$$



$$\bar{a}_l = \frac{d\bar{l}}{dl}$$

$$d\bar{l} = dl\,\bar{a}_l$$

Where α is the angle between the vector $d\vec{l}$ (or $\vec{a}_l$) and the x axis.

And β is the angle between the vector $d\vec{l}$ (or $\vec{a}_l$) and the y axis.

Review:
The directional derivative of *f(x,y,z)* varies with the direction of *dl*: namely, many values.

There exists a maximum among these values. The maximum and its direction are defined as the gradient of *f* (*x,y,z*) .

➢（2）Examples and Calculus

Examples 2：Find the gradient of $f(x, y, z)=6x^2y^3+e^z$ at the Point $P(2,-1,0)$ and directional derivative of it along $\overrightarrow{OP}$. (Page 43)—exercise /homework

Examples 3:

A vector $\vec{R}=\vec{r}-\vec{r}\,'$, $\vec{r}$ **is the position vector of a field point** $P(x,y,z)$**(the vector from the origin to the point *P*),** $\vec{r}\,'$ **is the position vector of a source point** $P'(x',y',z')$

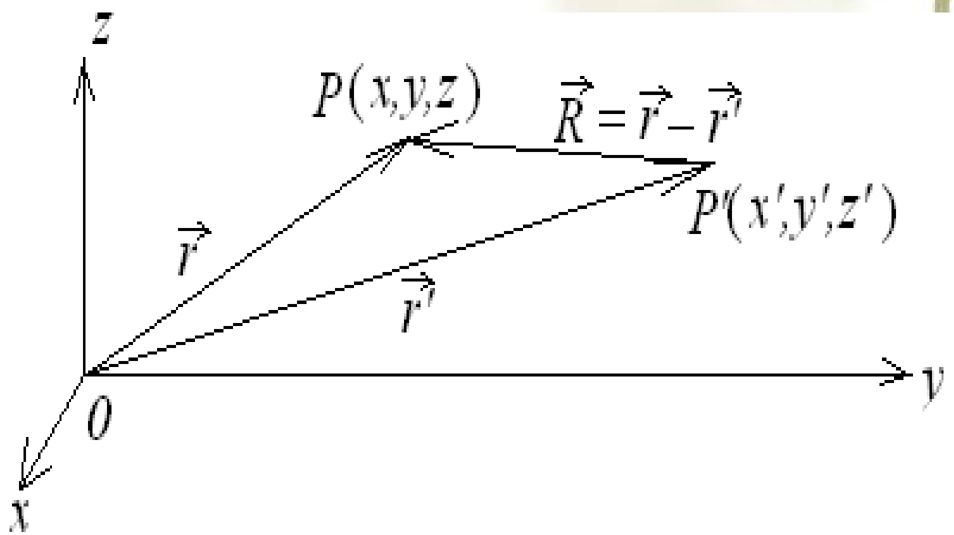
The gradient of the scalar function 1/R with respect to *x*, *y* and *z* variables can be written as $\nabla(1/R)$, The gradient of the scalar function 1/R with respect to x', y' and z'.

x', y' *and* z' **variables can be written as** $\nabla'(1/R)$**, We will generally use primed letters for the coordinates of the source point and unprimed letters for points at which the desired quantity is to be determined.**

Prove:

（1） $\nabla\left(\dfrac{1}{R}\right) = -\dfrac{\vec{R}}{R^3}$

（2） $\nabla'\left(\dfrac{1}{R}\right) = \dfrac{\vec{R}}{R^3}$



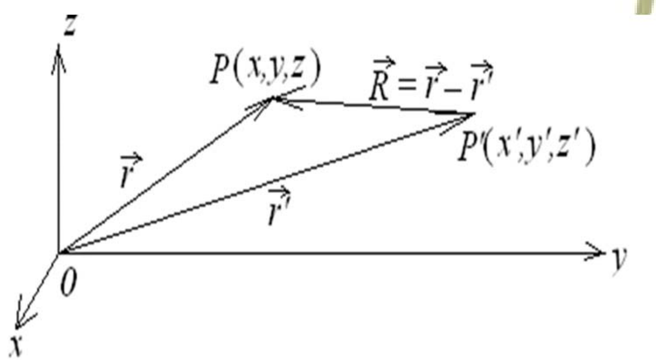
Solution:

Since $P(x,y,z)$ **is given in rectangular coordinates, we can obtain**

$$\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z \;,\; \vec{r}\,' = x'\vec{a}_x + y'\vec{a}_y + z'\vec{a}_z$$

$$\vec{R} = \vec{r} - \vec{r}\,' = \vec{a}_x (x-x') + \vec{a}_y (y-y') + \vec{a}_z (z-z')$$

$$R = \left|\vec{R}\right| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$



$$\nabla\left(\frac{1}{R}\right) = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right)\left(\frac{1}{R}\right)$$

$$= \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right)\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= -\frac{(x-x')\vec{a}_x + (y-y')\vec{a}_y + (z-z')\vec{a}_z}{\left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)^3} = -\frac{\vec{R}}{R^3}$$

while

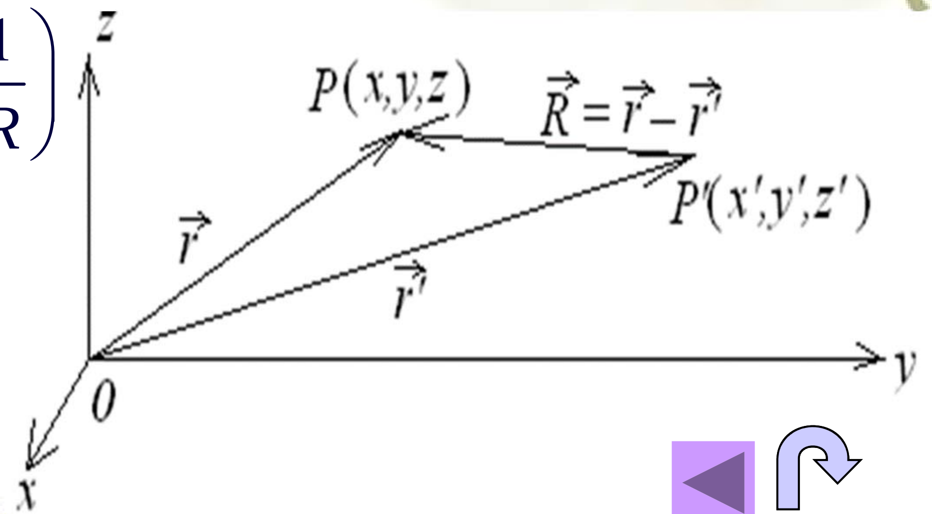
$$\nabla'\left(\frac{1}{R}\right)=\left(\frac{\partial}{\partial x'}\vec{a}_x+\frac{\partial}{\partial y'}\vec{a}_y+\frac{\partial}{\partial z'}\vec{a}_z\right)\left(\frac{1}{R}\right)$$

$$=\left(\frac{\partial}{\partial x'}\vec{a}_x+\frac{\partial}{\partial y'}\vec{a}_y+\frac{\partial}{\partial z'}\vec{a}_z\right)\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$=-\frac{-(x-x')\vec{a}_x-(y-y')\vec{a}_y-(z-z')\vec{a}_z}{\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)^3}=\frac{\vec{R}}{R^3}$$

That is

$$\nabla\left(\frac{1}{R}\right)=-\frac{\vec{R}}{R^3}=-\nabla'\left(\frac{1}{R}\right)$$



- **Exercises**

P 66, T2.33.

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