§ 8.5 Plane wave in a conducting medium

❑ 1. Maxwell's equations in a conducting medium

let us consider a general case of wave propagation in a medium having finite conductivity σ, permeability μ, and permittivity ε. Thus, we obtain Maxwell's equations in phasor form as

$$\begin{aligned}\nabla\times\dot{\vec{H}} &= \dot{\vec{J}}_v + j\omega\varepsilon\dot{\vec{E}} \\ &= \sigma\dot{\vec{E}} + j\omega\varepsilon\dot{\vec{E}} = j\omega(\varepsilon - j\frac{\sigma}{\omega})\dot{\vec{E}} = j\omega\varepsilon_c\dot{\vec{E}} \\ \nabla\times\dot{\vec{E}} &= -j\omega\mu\dot{\vec{H}} \qquad (8.5.1)\end{aligned}$$

where

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) \qquad (8.5.2)$$

is called the complex permittivity of the medium. The complex permittivity is a function of frequency and often given in the literature as

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

where ε' is the permittivity ($\varepsilon_r \varepsilon_0$), *and* $\omega\varepsilon''$ is the conductivity of the medium. The term $\frac{\sigma}{\omega\varepsilon}$ is referred to as the loss tangent. That is

$$\tan\delta = \frac{\sigma}{\omega\varepsilon}$$

equation (8.5.1) can rewritten as

$$\nabla \times \dot{\vec{H}} = \dot{\vec{J}}_v + j\omega\varepsilon\dot{\vec{E}} = \dot{\vec{J}}_c + \dot{\vec{J}}_d$$

where

$$\dot{\vec{\mathbf{J}}}_c = \sigma\dot{\vec{\mathbf{E}}}$$

$$\nabla\times\dot{\vec{H}} = \dot{\vec{J}}_v + j\omega\varepsilon\dot{\vec{E}} = \dot{\vec{J}}_c + \dot{\vec{J}}_d$$

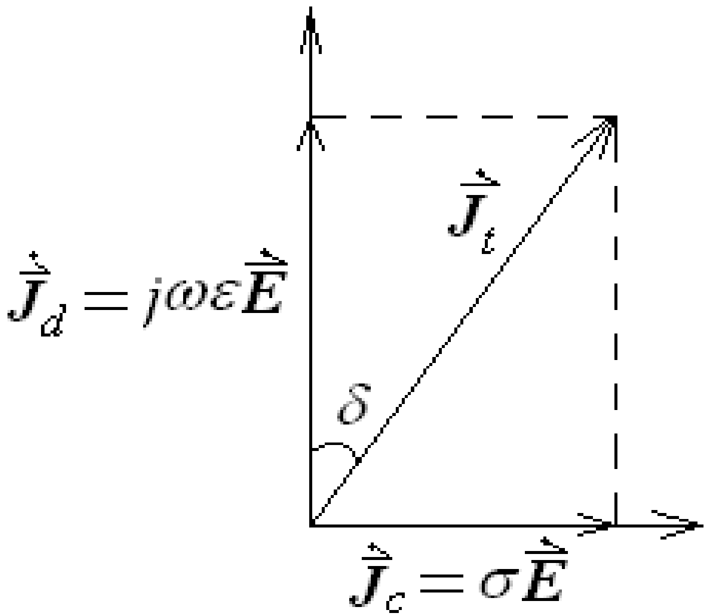
is the conduction current density, and

$$\dot{\vec{J}}_d = j\omega\varepsilon\dot{\vec{E}}$$

is the displacement current density.

Thus we can obtain the total current density $\dot{\vec{J}}_t$, that is $\dot{\vec{J}}_t = \dot{\vec{J}}_c + \dot{\vec{J}}_d$

using $\dot{\vec{E}}$ as a reference phasor, we can sketch the phasor diagram in terms of the three current densities.



2. the general wave equation and its solution

similarly, for a uniform plane wave, a field has the same magnitude and direction in a plane, say plane *xoy*. So a uniform plane wave propagating in the z direction, $\dot{\vec{\mathbf{E}}}$ and $\dot{\vec{\boldsymbol{H}}}$ are not functions of *x* and y.

that is, $$\frac{\partial \dot{\vec{E}}}{\partial x} = \frac{\partial \dot{\vec{E}}}{\partial y} = 0, \quad \frac{\partial \dot{\vec{H}}}{\partial x} = \frac{\partial \dot{\vec{H}}}{\partial y} = 0 \qquad (8.5.3)$$

from equation(8.5.1)

$$\nabla \times \dot{\vec{H}} = j\omega\varepsilon_c \dot{\vec{E}}, \quad \nabla \times \dot{\vec{E}} = -j\omega\mu \dot{\vec{H}}$$

we can obtain

$$\nabla \times \dot{\vec{H}} = j\omega\varepsilon_c \dot{\vec{E}}, \qquad \nabla \times \dot{\vec{E}} = -j\omega\mu\dot{\vec{H}}$$

$$(1)\quad j\omega\varepsilon_c \dot{E}_x = \frac{\partial \dot{H}_z}{\partial y} - \frac{\partial \dot{H}_y}{\partial z} = -\frac{\partial \dot{H}_y}{\partial z} \qquad (1')\quad -j\omega\mu\dot{H}_y = \frac{\partial \dot{E}_x}{\partial z} - \frac{\partial \dot{E}_z}{\partial x} = \frac{\partial \dot{E}_x}{\partial z}$$

$$(2)\quad j\omega\varepsilon_c \dot{E}_y = \frac{\partial \dot{H}_x}{\partial z} - \frac{\partial \dot{H}_z}{\partial x} = \frac{\partial \dot{H}_x}{\partial z} \qquad (2')\quad -j\omega\mu\dot{H}_x = \frac{\partial \dot{E}_z}{\partial y} - \frac{\partial \dot{E}_y}{\partial z} = -\frac{\partial \dot{E}_y}{\partial z}$$

$$(3)\quad j\omega\varepsilon_c \dot{E}_z = \frac{\partial \dot{H}_y}{\partial x} - \frac{\partial \dot{H}_x}{\partial y} = 0 \qquad (3')\quad -j\omega\mu\dot{H}_z = \frac{\partial \dot{E}_y}{\partial x} - \frac{\partial \dot{E}_x}{\partial y} = 0$$

$$\Rightarrow \dot{E}_z = 0 \qquad\qquad \Rightarrow \dot{H}_z = 0$$

from(3) and (3’),we can assume that the component of the field quantities $\vec{\mathbf{E}}$ and $\vec{H}$ lie in a transverse plane, a plane perpendicular to the direction of propagation of the wave. We refer to such a wave as a plane wave. The $\vec{\mathbf{E}}$ and $\vec{H}$ fields have no components in the longitudinal direction(the direction of wave propa gation).that is, Ez=0 and Hz=0. such a wave is also ca lled a transverse electromagnetic wave (TEM wave)

from (1)and(1'), we can obtain

$$\frac{d^2\dot{E}_x}{dz^2}+\omega^2\mu\varepsilon_c\dot{E}_x=0 \quad \text{and} \quad \frac{d^2\dot{H}_y}{dz^2}+\omega^2\mu\varepsilon_c\dot{H}_y=0$$

from (2)and(2'), we can also obtain

$$\frac{d^2\dot{E}_y}{dz^2}+\omega^2\mu\varepsilon_c\dot{E}_y=0 \quad \text{and} \quad \frac{d^2\dot{H}_x}{dz^2}+\omega^2\mu\varepsilon_c\dot{H}_x=0$$

Note that we can rewrite these similar wave equations as

$$\frac{d^2\dot{E}_x}{dz^2} - \gamma^2\dot{E}_x = 0 \qquad (8.5.4)$$

where γ is the propagation constant and

$$\gamma^2 = -\omega^2\mu\varepsilon_c \qquad \text{or}$$

$$\gamma = j\omega\sqrt{\mu\varepsilon_c} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

the general solution of equation(8.5.4) can be given by

$$\dot{E}_x = \dot{E}_{xf}e^{-\gamma z} + \dot{E}_{xb}e^{\gamma z}$$

$$= \dot{E}_{xf}e^{-\alpha z}e^{-j\beta z} + \dot{E}_{xb}e^{\alpha z}e^{j\beta z}$$

$$= E_{xf} e^{-\alpha z} e^{j\varphi_{xf}} e^{-j\beta z} + E_{xb} e^{\alpha z} e^{j\varphi_{xb}} e^{j\beta z} \quad (8.5.5)$$

(8.5.5) can be rewritten in the time domain as

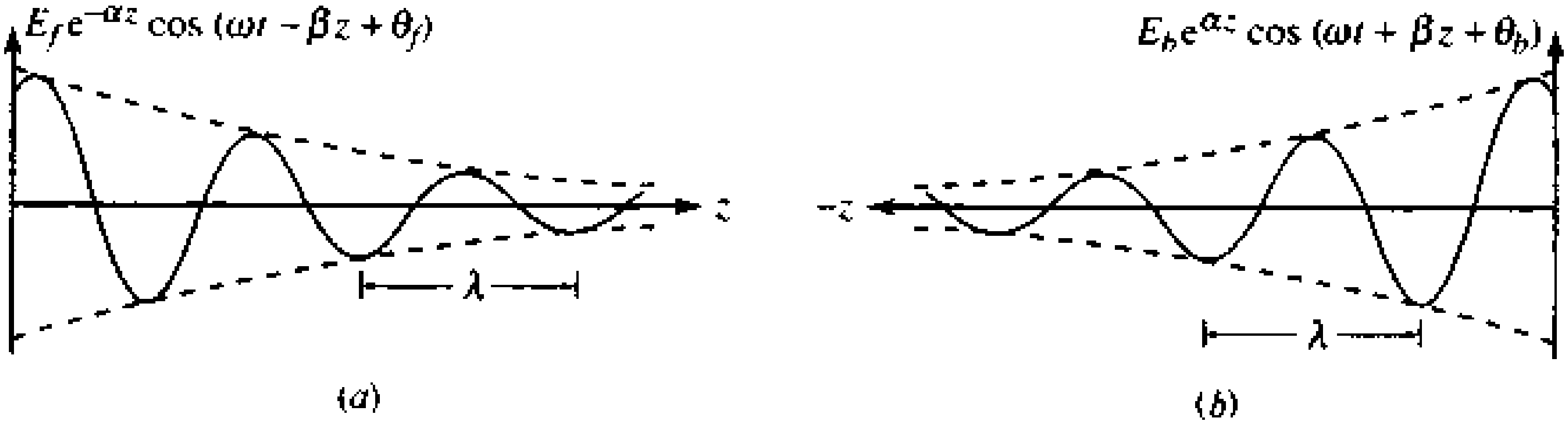
$$E_x(z,t) = \mathrm{Re}[\dot{E}_x e^{j\omega t}]$$
$$= E_{xf} e^{-\alpha z} \cos(\omega t - \beta z + \varphi_{xf}) + E_{xb} e^{\alpha z} \cos(\omega t + \beta z + \varphi_{xb})$$
$$(8.5.6)$$

α is called the attenuation constant, ***β*** is called the phase constant.

- 3. propagation characteristics

(1) attenuated wave

The first term on the right-hand side of (8.5.6) represents a time-harmonic uniform plane wave propagating in thepositive z direction(forward- travelling wave). The factor e signifies that the wave attenuates as it proceeds in the *z* direction. The second term represents a backward-traveling wave, which also attenuates as it makes its headway in the negative z direction, as shown in the following figure.



(a) forward- and (b) backward-travelling waves in a conducting medium at time t= constant

the solution of a wave equation in a conducting medium is an attenuated (damped)wave.

If letting l is the distance traveled by the forward-traveling wave in a conducting medium, the amplitude falls to $|\vec{E}_2|$ from $|\vec{E}_1|$, we obtain

$$|\vec{E}_2| = |\vec{E}_1| e^{-\alpha l}$$

therefore, α is called the attenuation constant, β is called the phase constant.

❑(2) Skin depth

how far can the wave propagate in a conducting medium before its amplitude becomes insignificant? The question is usually answered in terms of skin depth.

The skin depth is the distance traveled by the Wave in a conducting medium at which its amplitude falls to 1/*e* of its value on the surface of that conducting medium. If we denote the skin depth by δc, the amplitude of the wave falls to 1/*e* when $\alpha\delta_c$=1 thus,

$$\delta_c = 1/\alpha$$

the amplitude of the wave reduces to less than 1% after the wave has penetrated a distance equal to 5δc. thereafter, the wave is assumed to be completely attenuated.

- **(3) propagation constant γ**

from $\gamma = j\omega\sqrt{\mu\varepsilon_c} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$

we can obtain

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \qquad (8.5.7)$$

$$2\alpha\beta = \omega\mu\sigma \qquad (8.5.8)$$

and
$$\alpha^2 - \beta^2 = -\omega^2 \mu\varepsilon \qquad (8.5.7)$$
$$2\alpha\beta = \omega\mu\sigma \qquad (8.5.8)$$

from (8.5.7)and (8.5.8), we have

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$
$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}$$

obviously, α and β are functions of angle frequency ω, and also they increase when σ increases. When σ=0, α=0 and $\beta = \omega\sqrt{\mu\varepsilon}$ are the same as the perfect dielectric's.

- (4)wavelength

the wavelength, in a conducting medium, is

$$\lambda = 2\pi/\beta$$

where β is the imaginary part of propagation constant γ. Note that , in a conducting medium, $\beta \neq \omega\sqrt{\mu\varepsilon}$

- (5) the phase velocity

$$E_x(z,t) = \mathrm{Re}[\dot{E}_x e^{j\omega t}]$$

$$= E_{xf} e^{-\alpha z} \cos(\omega t - \beta z + \varphi_{xf}) + E_{xb} e^{\alpha z} \cos(\omega t + \beta z + \varphi_{xb})$$

by setting the phase of the wave equal to a constant and differentiating with respect to t, we obtain the phase speed of the forward- traveling wave as

$$u_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

(6)the magnetic field intensity

using Maxwell's equation $\nabla \times \dot{\vec{E}} = -j\omega\omega\dot{\vec{H}}$

$$(1') \quad -j\omega\mu\dot{H}_y = \frac{\partial \dot{E}_x}{\partial z} - \frac{\partial \dot{E}_z}{\partial x} = \frac{\partial \dot{E}_x}{\partial z}$$

$$(2') \quad -j\omega\mu\dot{H}_x = \frac{\partial \dot{E}_z}{\partial y} - \frac{\partial \dot{E}_y}{\partial z} = -\frac{\partial \dot{E}_y}{\partial z}$$

$$(3') \quad -j\omega\mu\dot{H}_z = \frac{\partial \dot{E}_y}{\partial x} - \frac{\partial \dot{E}_x}{\partial y} = 0 \Rightarrow \dot{H}_z = 0$$

if $\dot{\vec{E}} = \vec{a}_x \dot{E}_x$,**the magnetic field intensity can be given by**

$$\dot{\vec{H}} = \frac{\gamma}{j\omega\mu}[\dot{E}_{xf} e^{-\gamma z}]\vec{\boldsymbol{a}}_y \qquad (8.5.9)$$

the intrinsic impedance η_c **of the conducting medium is**

$$\eta_c = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{j\omega\sqrt{\mu\varepsilon_c}} = \frac{j\omega\mu}{j\omega\sqrt{\mu(\varepsilon - j\sigma/\omega)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta_c| e^{j\theta_\eta}$$

where $|\eta_c|$ is the magnitude of the intrinsic impedance η_c and θ_η is its phase angle. Now we can express (*8.5.9)* in terms of η_c as

$$\dot{\vec{H}} = \frac{1}{\eta_c}[\dot{E}_{xf} e^{-\gamma z}]\vec{a}_y$$

$$= \frac{1}{|\eta_c|}[\dot{E}_{xf} e^{-\gamma z} e^{-j\theta_\eta}]\vec{a}_y$$

$$= \frac{1}{|\eta_c|}[E_{xf} e^{-\alpha z} e^{-j\beta z} e^{j\varphi_{xf}} e^{-j\theta_\eta}]\vec{a}_y$$

in the phasor form, and as

$$H_y(z,t) = \mathrm{Re}[\dot{H}_y e^{j\omega t}]$$
$$= \frac{1}{|\eta_c|} E_{xf} e^{-\alpha z} \cos(\omega t - \beta z + \varphi_{xf} - \theta_\eta) - \frac{1}{|\eta_c|} E_{xb} e^{\alpha z} \cos(\omega t + \beta z + \varphi_{xb} - \theta_\eta)$$

in the time domain.

- **(7) wave impedance**

$$\eta_c = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{j\omega\sqrt{\mu\varepsilon_c}} = \frac{j\omega\mu}{j\omega\sqrt{\mu(\varepsilon - j\sigma/\omega)}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{1}{\sqrt{1 - j\sigma/(\omega\varepsilon)}}$$
$$= \eta \frac{\sqrt{1 + j\sigma/(\omega\varepsilon)}}{\sqrt{1+\sigma^2/(\omega\varepsilon)^2}} = \eta \frac{\sqrt{\sqrt{1+\sigma^2/(\omega\varepsilon)^2}\, e^{j\varphi}}}{\sqrt{1+\sigma^2/(\omega\varepsilon)^2}} = \eta \frac{e^{j\varphi/2}}{\sqrt[4]{1+\sigma^2/(\omega\varepsilon)^2}} = |\eta_c| e^{j\theta_\eta}$$

where

$$\frac{\eta}{\sqrt[4]{1+\sigma^2/(\omega\varepsilon)^2}} = |\eta_c| \le \eta$$

is the magnitude of the intrinsic impedance, which the wave impedance of a conducting medium is smaller than that of its perfect dielectric, and its phase angle

$$\theta_\eta = \frac{1}{2}\varphi = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\varepsilon}$$

- **(8) the average power density** $\vec{S}_{ave}$

$$\vec{\mathrm{S}}_{ave} = \frac{1}{2}\mathrm{Re}\,[\dot{\vec{E}} \times \dot{\vec{H}}^*]$$

$$= \frac{1}{2}\mathrm{Re}\,[\vec{a}_x \dot{E}_x \times \vec{a}_y \dot{H}_y^*]$$

$$= \vec{\mathbf{a}}_z \frac{1}{2} \mathrm{Re}$$

$$\left[[E_{xf} e^{-\alpha z} e^{j(\varphi_{xf} - \beta z)} + E_{xb} e^{\alpha z} e^{j(\varphi_{xb} + \beta z)}] \bullet [\frac{1}{|\eta_c|} E_{xf} e^{-\alpha z} e^{j(\varphi_{xf} - \beta z - \theta_\eta)} - \frac{1}{|\eta_c|} E_{xb} e^{\alpha z} e^{j(\varphi_{xb} + \beta z - \theta_\eta)}]^* \right]$$

$$= \vec{\mathbf{a}}_z \frac{1}{2} \mathrm{Re}$$

$$\left[[E_{xf} e^{-\alpha z} e^{j(\varphi_{xf} - \beta z)} + E_{xb} e^{\alpha z} e^{j(\varphi_{xb} + \beta z)}] \bullet [\frac{1}{|\eta_c|} E_{xf} e^{-\alpha z} e^{-j(\varphi_{xf} - \beta z - \theta_\eta)} - \frac{1}{|\eta_c|} E_{xb} e^{\alpha z} e^{-j(\varphi_{xb} + \beta z - \theta_\eta)}] \right]$$

$$
\begin{aligned}
&= \vec{a}_z \frac{1}{2|\eta_c|} \mathrm{Re}\,[E_{xf}^2 e^{-2\alpha z} e^{j\theta_\eta} - E_{xf} e^{j(\varphi_{xf}-\beta z)} E_{xb} e^{-j(\varphi_{xb}+\beta z-\theta_\eta)} \\
&\qquad + E_{xb} e^{j(\varphi_{xb}+\beta z)} E_{xf} e^{-j(\varphi_{xf}-\beta z-\theta_\eta)} - E_{xb}^2 e^{2\alpha z} e^{j\theta_\eta}\,] \\
&= \vec{a}_z \frac{1}{2|\eta_c|} \mathrm{Re}\,[E_{xf}^2 e^{-2\alpha z} e^{j\theta_\eta} - E_{xf} E_{xb} e^{j(\varphi_{xf}-\varphi_{xb}-2\beta z)} e^{j\theta_\eta} \\
&\qquad + E_{xb} E_{xf} e^{j(\varphi_{xb}-\varphi_{xf}+2\beta z)} e^{j\theta_\eta} - E_{xb}^2 e^{2\alpha z} e^{j\theta_\eta}\,] \\
&= \vec{a}_z \frac{1}{2|\eta_c|} \mathrm{Re}\,[E_{xf}^2 e^{-2\alpha z} e^{j\theta_\eta} - E_{xf} E_{xb} e^{j\theta_\eta} (e^{j(\varphi_{xf}-\varphi_{xb}-2\beta z)} \\
&\qquad - e^{-j(\varphi_{xf}-\varphi_{xb}-2\beta z)}) - E_{xb}^2 e^{2\alpha z} e^{j\theta_\eta}\,]
\end{aligned}
$$

$$= \vec{a}_z \frac{1}{2|\eta_c|} \mathrm{Re}\,[E_{xf}^2 e^{-2\alpha z} e^{j\theta_\eta} - E_{xf} E_{xb} e^{j\theta_\eta} \left(2j \sin\,(\varphi_{xf} - \varphi_{xb} - 2\beta z)\right)$$

$$- E_{xb}^2 e^{2\alpha z} e^{j\theta_\eta}\,]$$

$$= \vec{a}_z \frac{1}{2|\eta_c|} E_{xf}^2 e^{-2\alpha z} \cos\theta_\eta - \vec{a}_z \frac{1}{|\eta_c|} E_{xf} E_{xb} \sin\,(2\beta z + \varphi_{xb} - \varphi_{xf})\sin\theta_\eta$$

$$- \vec{\mathbf{a}}_z \frac{1}{2|\eta_c|} E_{xb}^2 e^{2\alpha z} \cos\theta_\eta$$

$$= \vec{\mathbf{S}}_{fave} + \vec{\mathbf{S}}_{fbave} + \vec{\mathbf{S}}_{bave}$$

where

$$\vec{\mathbf{S}}_{fave} = \vec{a}_z \frac{1}{2|\eta_c|} E_{xf}^2 e^{-2\alpha z} \cos\theta_\eta$$

represents the average power density in the forward-travelling wave,

$$\vec{\mathbf{S}}_{bave} = -\vec{a}_z \frac{1}{2|\eta_c|} E_{xb}^2 e^{2\alpha z} \cos\theta_\eta$$

yields the average power density in the back-travelling wave, and

$$\vec{\mathbf{S}}_{fbave} = -\vec{\mathbf{a}}_z \frac{1}{|\eta_c|} E_{xf} E_{xb} \sin(2\beta z + \varphi_{xb} - \varphi_{xf}) \sin\theta_\eta$$

is the average power density due to the cross-couple between the forward and the backward waves. Note the cross-coupling between the two waves varies as sin θ_η**.**

Thus the cross-coupling term disappears when θ_η=0 (a condition that is true only when the medium is a perfect dielectric. That is

$$\vec{S}_{fbave} = -\vec{a}_z \frac{1}{|\eta_c|} E_{xf} E_{xb} \sin(2\beta z + \varphi_{xb} - \varphi_{xf}) \sin\theta_\eta = 0$$

when θ_η=0

Example 8.3 you read by yourselves.

(9)electromagnetic energy density

for a forward-travelling wave,

$$\dot{\vec{E}} = \vec{a}_x \dot{E}_x = \dot{E}_{xf} e^{-\gamma z} = E_{xf} e^{j\varphi_{xf}} e^{-\alpha z} e^{-j\beta z}$$

the average electric energy density is

$$W_{eave} = \frac{1}{T}\int_0^T \frac{1}{2}\varepsilon E^2(z,t)dt$$
$$= \frac{1}{4}\varepsilon \vec{E} \bullet \vec{E}* = \frac{1}{4}\varepsilon \dot{E}_x \bullet \dot{E}_x^* = \frac{1}{4}\varepsilon E_{xf}^2 e^{-2\alpha z}$$

in terms of Maxwell' equation $\nabla \times \dot{\vec{E}} = -j\omega\mu\dot{\vec{H}}$

, the magnetic Field intensity is

$$\dot{\vec{H}} = \frac{1}{\eta_c}\vec{a}_z \times \dot{\vec{E}} = \frac{1}{\eta_c}\dot{E}_{xf}e^{-\gamma z}\vec{\mathbf{a}}_y$$

$$= \frac{1}{|\eta_c|}\dot{E}_{xf}e^{-\alpha z}e^{-j\beta z}e^{-j\theta_\eta}\vec{\mathbf{a}}_y$$

thus, the average magnetic energy density is

$$W_{mave} = \frac{1}{T}\int_0^T \frac{1}{2}\mu H_y(z,t)dt = \frac{1}{4}\mu H_{yf}^2 e^{-2\alpha z}$$

$$= \frac{1}{4}\mu\frac{1}{|\eta_c|^2}E_{xf}^2 e^{-2\alpha z} = \frac{1}{4}\mu\frac{1}{\eta^2}\sqrt{1+\sigma^2/(\omega\varepsilon)^2}E_{xf}^2 e^{-2\alpha z}$$

$$= \frac{1}{4}\varepsilon\sqrt{1+\sigma^2/(\omega\varepsilon)^2}E_{xf}^2 e^{-2\alpha z} > \frac{1}{4}\varepsilon E_{xf}^2 e^{-2\alpha z} = W_{eave}$$

because $$\frac{\eta}{\sqrt[4]{1+\sigma^2/(\omega\varepsilon)^2}} = |\eta_c| \le \eta$$

that is, *Weave<Wmave*, the reason is that the conduction current $\dot{\vec{J}} = \sigma\dot{\vec{E}}$ **produce the another magnetic field.**