Chapter 7 Time-varying electromagnetic fields

1.Faraday’s law of electromagnetic induction

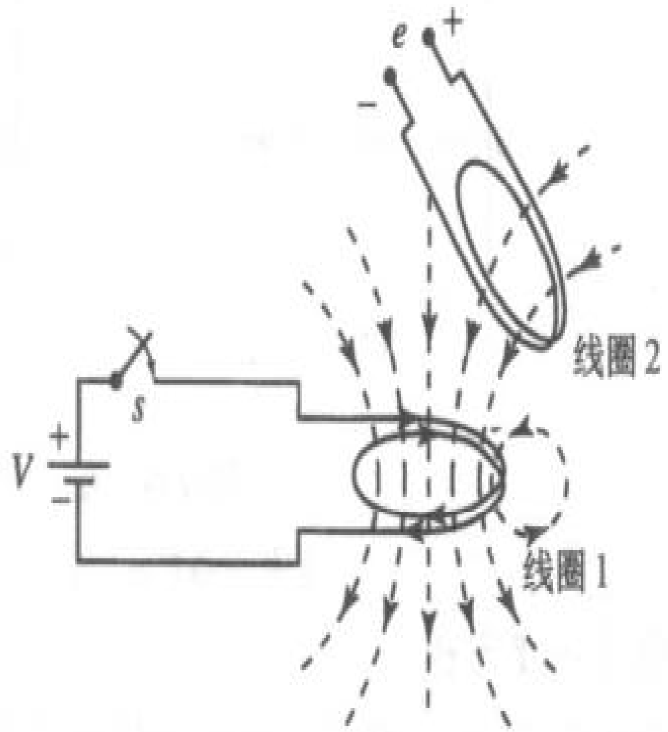
introduction

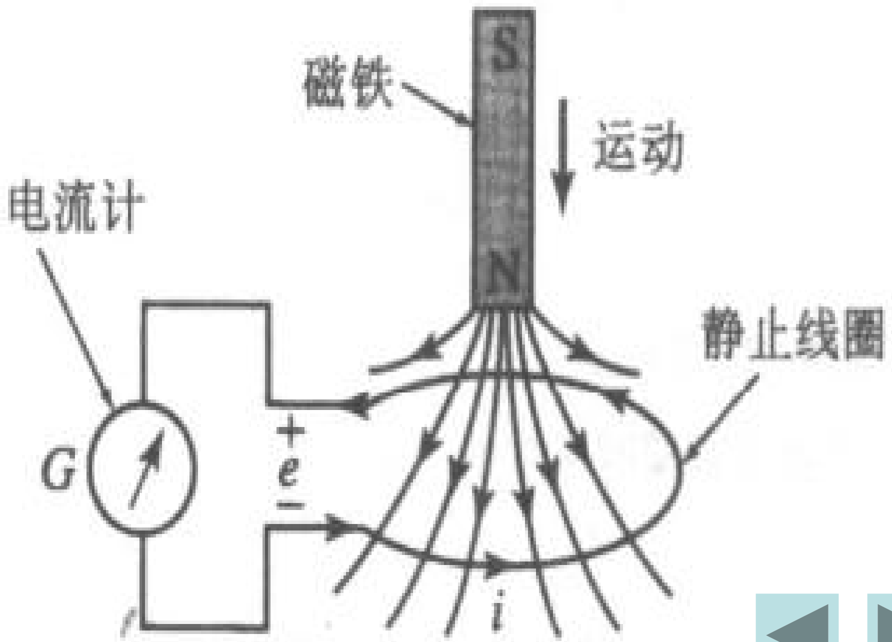
static electric fields are created by charges at rest; static magnetic fields are producted by charges in motion or steady currents.

⇒the static electric field can exist even when there is no the static magnetic field and vice versa. that is, the static electric field is independent of the static magnetic field.

After conducting a series of experiments with stationary coils, Michael[`maikl] Faraday discovered that

①an electromotive force(emf) is induced in a coil when the magnetic flux passing through (linking) a stationary coil varies as a function of time:





eg. induced emf and current in a coil due to increase in the magnetic flux by moving a magnet in the vicinity of the coil

induced emf in a coil at the time of opening or closing a switch in another coil.

②The induced emf around a closed path is equal to the rate of change of the magnetic flux passing through the area enclosed by the path. That is,

$$\varepsilon_{in} = -d\phi/dt$$

③the negative sign of the equation was introduced by Lenz:

the induced emf created the induced current, **the current induced** in a closed conducting loop by a change in magnetic flux through the loop **is in a direction** such that the flux generated by the induced current tends to **counterbalance** the change in the original magnetic flux.

- **The induced emf$\Rightarrow$the induced current $\Rightarrow$ The magnetic flux $\Rightarrow$ counterbalance the change of the original magnetic flux.**

- **The induced emf$\Rightarrow$the induced current $\Leftarrow$ the induced electric field intensity $\vec{E}$**

In fact, the induced emf in a conductor (coil) can defined in terms of the induced electric field intensity created by a time-varying magnetic field.

$$\varepsilon_{in} = \oint_c \vec{\boldsymbol{E}} \bullet d\vec{\boldsymbol{l}}$$

we have

$$\oint_c \vec{\mathbf{E}} \bullet d\,\vec{l} = -\frac{d}{dt}\int_s \vec{\mathbf{B}} \bullet d\vec{\mathbf{s}}$$

the equation is also a definition of Faraday's law of induction in the integral form as applied to a stationary loop immersed in a time-varying magnetic field.

in terms of stokes' theorem, (to convert a line integral into an equivalent surface integral and vice sersa), the above equation can be rewritten as

$$\oint_c \vec{\mathbf{E}} \bullet d\vec{l} = -\frac{d}{dt}\int_s \vec{\mathbf{B}} \bullet d\vec{\mathbf{s}}$$

$$\int_s \left(\nabla \times \vec{\mathbf{E}}\right) \bullet d\vec{\mathbf{s}} = -\frac{d}{dt}\int_s \vec{\mathbf{B}} \bullet d\vec{\mathbf{s}}$$

$$\int_s \left(\nabla \times \vec{\mathbf{E}}\right) \bullet d\vec{\mathbf{s}} = -\int_s \frac{\partial \vec{\mathbf{B}}}{\partial t} \bullet d\vec{\mathbf{s}}$$

$$\int_s \left(\nabla \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t}\right) \bullet d\vec{\mathbf{s}} = 0$$

since the surface S under consideration is arbitrary,we can obtain the integrand to vanish at each point.

$$\nabla \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \quad or \quad \nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

the equation is also a statement of Faraday's law of induction for a fixed point of observation in a stationary medium. It is also one of the four well-known Maxwell's equation.

- The electric field is the induced electric field.

the curl of the induced electric field intensity is non zero.⇒the time-varying magnetic field is its source. It is different from the coulomb's electric field, or the static field since

$$\nabla \times \vec{E} = 0$$

●the time-varying magnetic field can create the electric field.

The electric field is dependent of the time-varying magnetic field.

2. Gauss's theorem

the electric field induced by the time-varying magnetic field.

The induced electric field $\vec{E}_2$($\vec{\mathbf{D}}_2$)is rotational, the electric field lines are always continuous. the flux penetrating a closed surface is equal to the flux lea-ving the closed surface. Hence, the flux of $\vec{\mathbf{D}}_2$

is

$$\oint_s \vec{D}_2 \bullet d\vec{s} = 0$$

- in general, the electric field includes the static electric field $\vec{\mathbf{D}}_1$ and the induced electric field $\vec{\mathbf{D}}_2$, that is,

$$\vec{D} = \vec{D}_1 + \vec{D}_2$$

the flux of $\vec{D}$ is

$$\oint_s \vec{\mathbf{D}} \bullet d\vec{\mathbf{s}} = \oint_s \left(\vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2\right) \bullet d\vec{\mathbf{s}}$$

$$= \oint_s \vec{\mathbf{D}}_1 \bullet d\vec{\mathbf{s}} + \oint_s \vec{\mathbf{D}}_2 \bullet d\vec{\mathbf{s}}$$

$$= \int_v \rho_v \, dv$$

or

$$\oint_s \vec{\mathbf{D}} \bullet d\,\vec{\mathbf{s}} = \int_v \rho_v dv = q$$

$$\oint_s \vec{\boldsymbol{D}} \bullet d\vec{\mathbf{s}} = \int_v \nabla \bullet \vec{\boldsymbol{D}} dv$$

the flux of $\vec{\mathbf{D}}$ is equal to the total free charge present at any time t within the volume *v* bounded by the cl-osed surface *s*

meanwhile, we can obtain a mathematical statement for Gauss's law in the point (differential) form as

$$\nabla \bullet \vec{\boldsymbol{D}} = \rho_v$$

* $\vec{D}$ and ρ_v are now time-dependent field quantities. They are functions of time and space.

* the divergence of $\vec{D}$ is equal to the free charge volume density at the point.

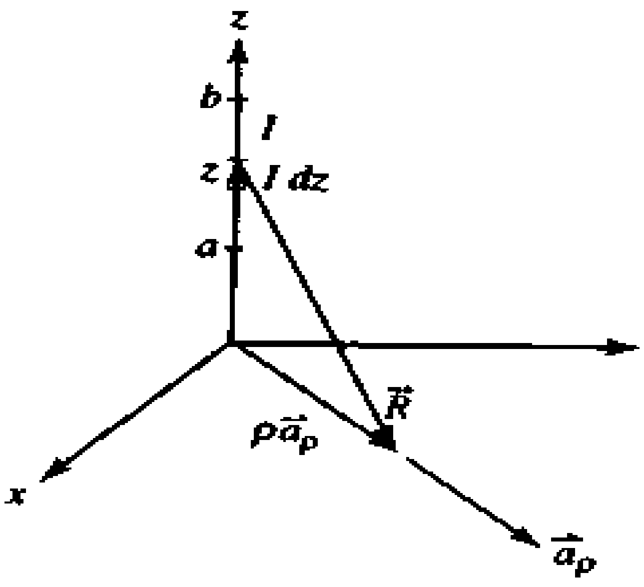
* the electric field can be created by the electric charge. The charge is its source.

- 3. law of continuity of the total current

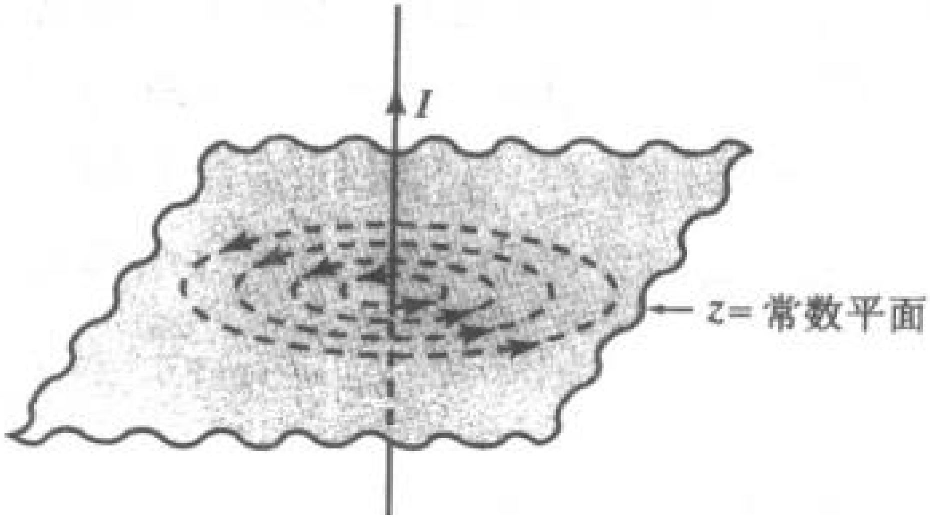
  - (1)example5.1 page 58.

(1)example5.1 page 58.

a filamentary wire of finite length extends from *z=a* to *z=b*, as shown in fig.5.5a, determine the magnetic flux density at a point P in the xy plane. What is the magnetic flux density at P if *a*→-∞ and *b*→∞ ?



(a)



$$\vec{B} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{l} \times \vec{R}}{R^3} = \frac{\mu_0}{4\pi}\int_c \frac{I d\vec{l} \times \vec{a}_R}{R^2}$$

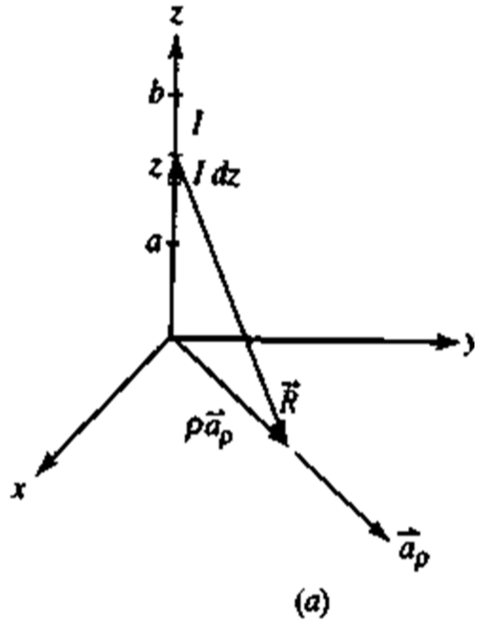
solution

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\int_c \frac{Id\vec{\mathbf{l}} \times \vec{\mathbf{R}}}{R^3}$$

$$= \frac{\mu_0}{4\pi}\int_c \frac{Idz\vec{\mathbf{a}}_{\mathbf{z}} \times (\rho\vec{\mathbf{a}}_{\boldsymbol{\rho}} - z\vec{\mathbf{a}}_{\mathbf{z}})}{[\rho^2 + z^2]^{3/2}}$$

$$= \frac{\mu_0}{4\pi}\int_c \frac{Idz\vec{\mathbf{a}}_{\mathbf{z}} \times \rho\vec{\mathbf{a}}_{\boldsymbol{\rho}}}{[\rho^2 + z^2]^{3/2}}$$

$$= \frac{\mu_0}{4\pi}\int_c \frac{I\rho dz\vec{\mathbf{a}}_{\phi}}{[\rho^2 + z^2]^{3/2}}$$



$$\vec{B} = \vec{\mathbf{a}}_{\phi} \frac{\mu_0 I \rho}{4\pi} \int_c \frac{dz}{[\rho^2 + z^2]^{3/2}}$$

$$= \vec{\mathbf{a}}_{\phi} \frac{\mu_0 I \rho}{4\pi} \int_a^b \frac{dz}{[\rho^2 + z^2]^{3/2}}$$

$$= \vec{\mathbf{a}}_{\phi} \frac{\mu_0 I}{4\pi\rho} \left[\frac{b}{\sqrt{\rho^2 + b^2}} - \frac{a}{\sqrt{\rho^2 + a^2}} \right]$$

By setting *a*= –∞ and *b*=∞ in the preceding expression, we obtain the magnetic field $\vec{B}$ produced at a point by a wire of infinite extent as

$$\vec{B} = \vec{a}_{\phi} \frac{\mu_0 I}{4\pi\rho} \left[1 - (-1) \right]$$

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi\rho}\vec{\mathbf{a}}_\phi$$

Since we can define the magnetic flux density in a medium in terms of the current(Biot-Savart law) as

$$\vec{\mathbf{B}} = \frac{\mu}{4\pi}\int_c \frac{Id\vec{l} \times \vec{\boldsymbol{R}}}{R^3}$$

we obtain the magnetic field $\vec{\boldsymbol{B}}$ produced at a point in a medium μ by a wire of infinite extent as

$$\vec{\mathbf{B}} = \frac{\mu I}{2\pi\rho}\vec{\mathbf{a}}_\phi$$

(2) Ampère's Circuital Law page304

it states that the line integral of the magnetic field intensity around a closed path equals the current enclosed. That is

$$\oint_c \vec{H} \bullet d\vec{l} = \oint_c \frac{\vec{\mathbf{B}}}{\mu} \bullet d\vec{l} = \oint_c \frac{\vec{B}}{\mu} \bullet \vec{a}_l \, dl$$

$$= \int_0^{2\pi} \frac{\vec{\mathbf{B}}}{\mu} \bullet \vec{\mathbf{a}}_\phi \, \rho \, d\phi$$

$$= I$$

where *I* is the uniform current enclosed by contour c.

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi\rho} \vec{\mathbf{a}}_\phi$$

by describing I in terms of the volume current density $\vec{J}_v$ over surface S bounded by closed contour C as

$$I = \int_s \vec{J}_v \bullet d\vec{s}$$

we obtain

$$\oint_c \vec{H} \bullet d\vec{l} = \int_s \vec{J}_v \bullet d\vec{s}$$

the equation can rewritten in point (differential) form such that

$$\nabla \times \vec{H} = \vec{J}_v \qquad (7.59)$$

because $\quad \nabla \bullet (\nabla \times \vec{H}) = \nabla \bullet \vec{J}_v = 0$

In fact, the equation of continuity states that

$$\nabla \bullet \vec{J}_v = -\frac{\partial \rho_v}{\partial t}$$

for static fields, $\nabla \bullet \vec{J}_v = -\dfrac{\partial \rho_v}{\partial t}$ is zero and equation(7.59) is valid.

however, for time-varying fields, $\nabla \bullet \vec{J}_v = -\dfrac{\partial \rho_v}{\partial t}$ is not necessarily zero, and equation (7.59) is invalid.

Thereforce, we should replace $\vec{J}_v$ in (7.59) with

$$\vec{J}_v + \frac{\partial \vec{D}}{\partial t}$$

that is,

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_v + \frac{\partial \vec{\mathbf{D}}}{\partial t} \qquad (7.66)$$

Because

$$\nabla \bullet \left(\vec{\mathbf{J}}_v + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right)$$

$$= \nabla \bullet \vec{\mathbf{J}}_v + \frac{\partial}{\partial t} \nabla \bullet \vec{\mathbf{D}}$$

$$= \nabla \bullet \vec{\mathbf{J}}_v + \frac{\partial}{\partial t} \rho_v$$

$$= \nabla \bullet (\nabla \times \vec{\mathbf{H}}) = 0$$

the term $\frac{\partial \vec{D}}{\partial t}$ is called as the displacement current density.

$\vec{J}_v + \frac{\partial \vec{D}}{\partial t}$ **=total current density. It is continuous, namely, the divergence of it is zero.**

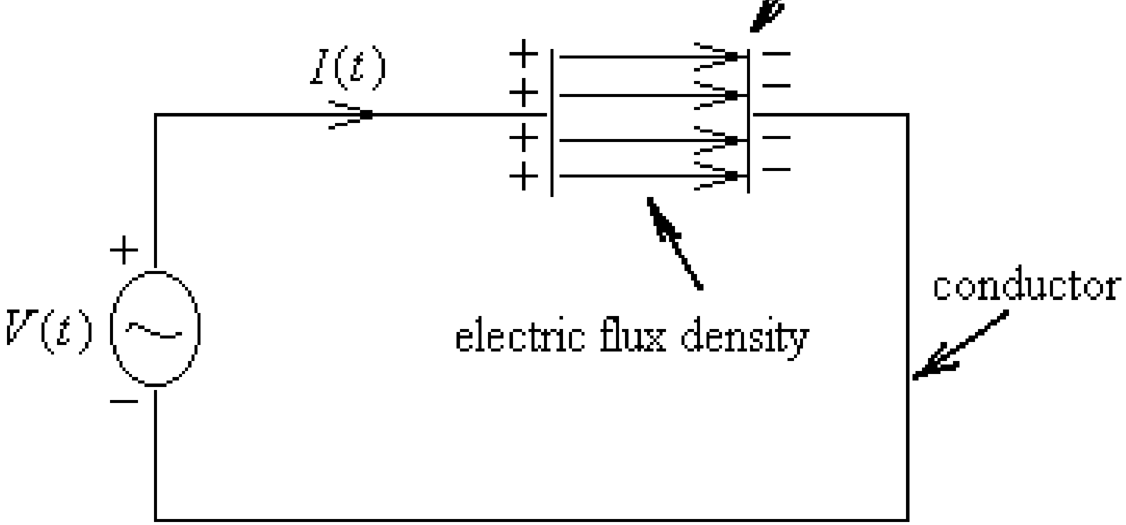
It includes the conduction current density $\vec{\mathbf{J}}_v$ and the displacement current density $\frac{\partial \vec{\boldsymbol{D}}}{\partial t}$

For any arbitrary open surface s bounded by a closed contour *c* we can rewrite(7.66) in the integral form as

$$\oint_c \vec{\boldsymbol{H}} \bullet d\vec{\boldsymbol{l}} = \int_s \vec{\boldsymbol{J}}_v \bullet d\vec{\boldsymbol{s}} + \int_s \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \bullet d\vec{\boldsymbol{s}} \qquad (7.68)$$

equation (7.66) or (7.68) can be understood by imagining a capacitor connected to a time -varying voltage source as shown in following figure.

$$\oint_c \vec{H} \bullet d\vec{l} = \int_s \vec{J}_v \bullet d\vec{s} + \int_s \frac{\partial \vec{D}}{\partial t} \bullet d\vec{s}$$



the displacement current in a capacitor establishes the continuity of the conduction current in the conductor.

For the capacitor circuit,

$$\oint_c \vec{H} \bullet d\vec{l} = \int_s \vec{J}_v \bullet d\vec{s} + \int_s \frac{\partial \vec{D}}{\partial t} \bullet d\vec{s}$$

- **the displacement current through the capacitor must be equal to the conduction current in the conductor**

- **the current is continuous in the capacitor.**

- **The time-varying current $I(t)$ or $\vec{J}_v(t)$ creates a time-varying magnetic field**
- **The time-varying electric field $\frac{\partial \vec{D}}{\partial t}$ creates a time-varying magnetic field.**

$\frac{\partial \vec{D}}{\partial t}$ is a source for the magnetic field.

●The time-varying electric and magnetic fields are interdependent.

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_v + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$