Chapter 3 Electrostatics

--Field quantities and laws.

contents

3.1 Introduction

- **We study static electric fields(electrostatics), due to charges at rest.**
- **A charge can be either concentrated at a point or distributed in some fashion. The charge is assumed to be constant in time.**

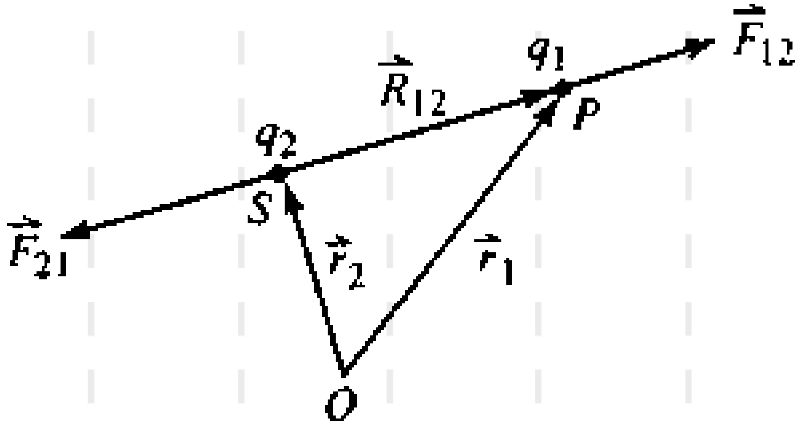
3.2 coulomb's law

Coulomb's law

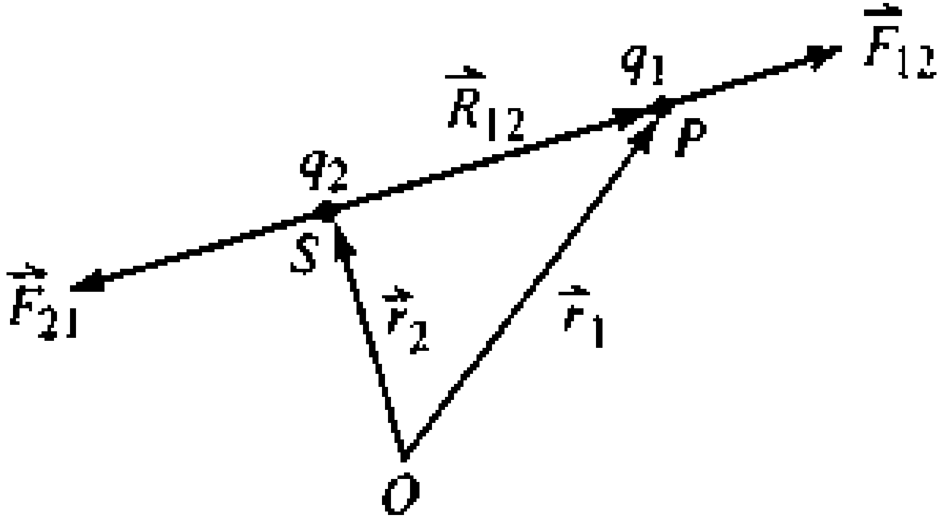
If q_1 and q_2 are two charged particles situated at point $P(x,y,z)$ and $S(x',y',z')$, the electric force acting on q_1 due to q_2 is

- It is the force experienced by q1 due to q2.
- K is the constant of proportionality.

$$\vec{F}_{12} = K \frac{q_1 q_2}{R_{12}^2} \vec{a}_{12} \qquad (3.1)$$



$$\vec{F}_{12} = K \frac{q_1 q_2}{R_{12}^2} \vec{a}_{12}$$



The force is

(a) Directly proportional to the product of their charges.

(b) Inversely proportional to the square of the distance between them.

(c) Directed along the line joining them.

(d) Repulsive (attractive) for like (unlike) charges.

$$\vec{F}_{12} = K\frac{q_1 q_2}{R_{12}^2}\vec{a}_{12}$$

where

(a) $\vec{F}_{12}$ **is the force experienced by** q_1 **due to** q_2

(b) K **is the constant of proportionality, which depend upon the system of units used. In the International System of Units, the constant of proportionality is** $K=1/(4\pi\varepsilon_0)$**.**

(c) R_{12} **is the distance between points P and S**

(d) $\vec{a}_{12}$ **is the unit vector pointing in the direction from point S to point P**

and $\vec{\mathbf{R}}_{12} = R_{12}\vec{\mathbf{a}}_{12}$, $\qquad \vec{\mathbf{a}}_{12} = \dfrac{\vec{\mathbf{R}}_{12}}{R_{12}}$

•coulomb's law states that a charge will always exert a force on another, even when the charges are separated by a large distance.

•In physics, a force acting on one charge due to another is usually referred to as **an action** at a d istance.

We say that there exists **an electric field or electric field intensity** everywhere in space surrounding the charge. When another charge is brought into the electric field, it experiences a force acting on it.

- **It is also clear that the force exerted by q1 on q2 is equal in magnitude but opposite in direction to the force that q2 exerts on q1. that is,** $\vec{F}_{21} = -\vec{F}_{12}$
- **The coulomb's law states that a charge will always exert a force on another charge:**
- **a charge will always create its electric field, the electric field will always exerts the electric force on charges situated in its space.**

That is:

- **There exists an electric field or electric field intensity everywhere in space surrounding the charge.When another charge is brought into this electric field, it experiences a force acting on it.**

Review:

The coulomb's law states that

①there exists an electric field or electric field intensity everywhere in space surrounding the charge.

A charge produces an electric field.

②the basic characteristic of an electric field is that it exerts a force on a charge. The interaction among charges is due to the electric field.

$$\vec{F}_{12} = K\frac{q_1 q_2}{R_{12}^2}\vec{a}_{12}$$

3.3 Electric field intensity

① **Concept:**

the electric field intensity

The force per unit charge exerted on a test charge q_t as the magnitude of $q_t \to 0$, is

$$\vec{E} = \lim_{q_t \to 0} \frac{\vec{F}}{q_t} \qquad (3.7)$$

where $\vec{F}$ is the total force acting on q_t

The electric field intensity $\vec{E} = \lim_{q_t \to 0} \dfrac{\vec{F}}{q_t}$ is a vector.

Its magnitude at a point in space is equal to the magnitude of the force per unit positive charge;

its direction at a point in space is the direction of the force per unit positive charge.

If is the electric field intensity at a point in space, the force acting on a charge q at that point is

$$\vec{F} = \vec{E}q \tag{3.8}$$

from now on we will use (3.3-2) to compute the electrostatic force experienced by a charge when placed in an electric field.

② The electric field created by a point charge

From (3.1) and (3.7), we can express the electric field intensity at any point due to a point charge *q* at *S*. That is

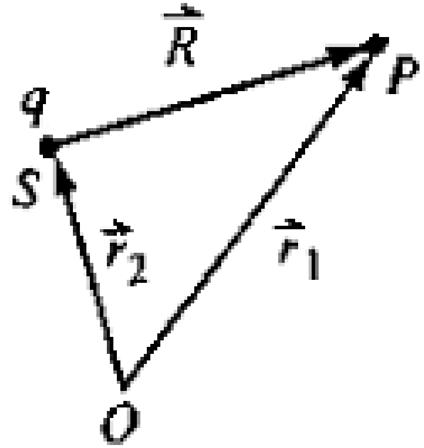
$$\vec{\mathbf{E}} = \frac{q\vec{\mathbf{R}}}{4\pi\varepsilon_0 R^3} = \frac{q\vec{\mathbf{a}}_R}{4\pi\varepsilon_0 R^2} \qquad \textbf{(3.9)}$$

where $\vec{a}_R$ is the unit vector directed from S toward P.

The electric field intensity due to n point charges, from equation (3.9) , is

$$\vec{\mathbf{E}} = \sum_i^n \frac{q_i}{4\pi\varepsilon_0} \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_i}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}_i\right|^3}$$

$$= \sum_i^n \frac{q_i \vec{\mathbf{R}}_i}{4\pi\varepsilon_0 R_i^3} = \sum_i^n \frac{q_i \vec{\mathbf{a}}_{Ri}}{4\pi\varepsilon_0 R_i^2} \qquad (3.10)$$



$\vec{R}_i$ is the distance vector directed from charge q_i to toward the point of .

$\vec{a}_{Ri}$ is the unit vector directed from *the source point* S_i *toward the field point P.*

the electric field intensity has the units of newtons per coulomb(N/C) , newtons per coulomb is dimensionally equivalent to volts per meter.(V/m)

③ electric field intensity due to charge distributions.

(a) Line charge distribution and its electric field intensity

•Line charge density:

when the charge is distributed over a linear element, we define the line charge density, the charge per unit length, as

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l}$$

where Δq is the charge on a length element Δl

the line charges can be calculated by $q = \int_c \rho_l dl'$

•The electric field intensity due to a line charge distribution can be written as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_c \frac{\rho_l(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}dl'$$

(b) Surface charge distribution and its electric field intensity

•Surface charge density:

When the charge is distributed over a surface, we define the surface charge density, the charge per unit area, as where Δq is the charge on a surface element Δs

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s}$$

the surface charges can be calculated by $q = \int_s \rho_s ds'$

•The electric field intensity due to a surface charge distribution can be written as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_s \frac{\rho_s (\vec{r} - \vec{r}\,')}{\left|\vec{r} - \vec{r}\,'\right|^3} ds'$$

(c) Volume charge distribution and its electric field intensity

•**Volume charge density:**

When the charge is confined within a volume, we define the volume charge density, the charge per unit volume, as

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v}$$

the volume charges can be calculated by $q = \int_v \rho_v dv'$

•**The electric field intensity at point P due to a volume charge distribution, is**

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho_v (\vec{r} - \vec{r}\,')}{\left|\vec{r} - \vec{r}\,'\right|^3} dv'$$

Some examples:

- Page 77-80
- Example 3.4
- Example 3.5
- Example 3.6

Read by yourselves.

3.5 The electric potential

- Because

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{q\vec{R}}{4\pi\varepsilon_0 R^3} = \frac{q\vec{a}_R}{4\pi\varepsilon_0 R^2} \quad \textbf{(3.9)}$$

we can obtain

$$\nabla\times\vec{\boldsymbol{E}} = \nabla\times\frac{q\vec{\boldsymbol{R}}}{4\pi\varepsilon_0 R^3} = \frac{q}{4\pi\varepsilon_0}\nabla\times\frac{\vec{\boldsymbol{R}}}{R^3} = 0$$

We know that if the curl of a vector field is zero, the vector field can be represented in terms of the gradient of a scalar field V as

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{q\vec{R}}{4\pi\varepsilon_0 R^3} = -\nabla\left(\frac{q}{4\pi\varepsilon_0 R}\right)$$

$$\vec{E} = -\nabla V \qquad\qquad V_a = \frac{q}{4\pi\varepsilon_0 R} + Const.$$

- **Where V is called the electric potential.**
- **When the external force is pushing the positive charge against the $\vec{E}$ field, the potential energy of the charge is increasing. That is why we have used the negative sign in the above equation.**
- **The work done in moving a positive charge against the electric field is equal to the increase in the potential energy of the charge.**

- **If we set R→∞,then the potential of point P with respect to a point S at infinity is known as the absolute potential. Thus, the absolute potential of point P at r_1=R is**

$$V_a = \frac{q}{4\pi\varepsilon_0 R} + Const. \qquad V = \frac{q}{4\pi\varepsilon_0 R}$$

$$Const = 0, when \qquad R \to \infty$$

•The electric potential due to a line charge distribution can be written as

$$V = \frac{1}{4\pi\varepsilon_0}\int_c \frac{\rho_{l'}}{|\vec{r} - \vec{r}'|} dl'$$

The electric potential can be given as

$$V=\frac{q}{4\pi\varepsilon_0 R}$$

when the electric potential of Point *P* at infinity is zero.

The electric potential due to *n* point charges, from the preceding equation, is

$$V=\frac{1}{4\pi\varepsilon_0}\sum_{i}^{n}\frac{q_i}{R_i}$$

- **The electric potential due to a surface charge distribution can be written as**

$$V = \frac{1}{4\pi\varepsilon_0}\int_s \frac{\rho_{s'}}{|\vec{r}-\vec{r}\,'|}ds'$$

- **The electric potential due to a volume charge distribution can be written as**

$$V = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho_{v'}}{|\vec{r}-\vec{r}\,'|}dv'$$

- **Example 3.11**

Read by yourselves.

- **The potential difference, therefore, is the change in potential energy per unit charge in the limit q→0. that is**

$$V_{ab} = \lim_{q \to 0} \frac{W_{ab}}{q} = -\int_b^a \vec{E} \bullet d\vec{l}$$

- **Example 3.10**
- **Determine the potential difference between two points due to a point charge *q* at the origin.**

Solution:

$$V_{ab} = -\int_b^a \vec{E} \bullet d\vec{l} = -\int_b^a \frac{q}{4\pi\varepsilon_0 r^2} \vec{a}_r \bullet d\vec{l}$$

$$= -\int_b^a \frac{q}{4\pi\varepsilon_0 r^2} \vec{a}_r \bullet \vec{a}_r dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

3.4 Electric flux and electric flux density

- (1) the electric flux density

•electric dipole---------------------- 3.6 page 89

We define an electric dipole as a pair of equal charges of opposite sign that are very close together,

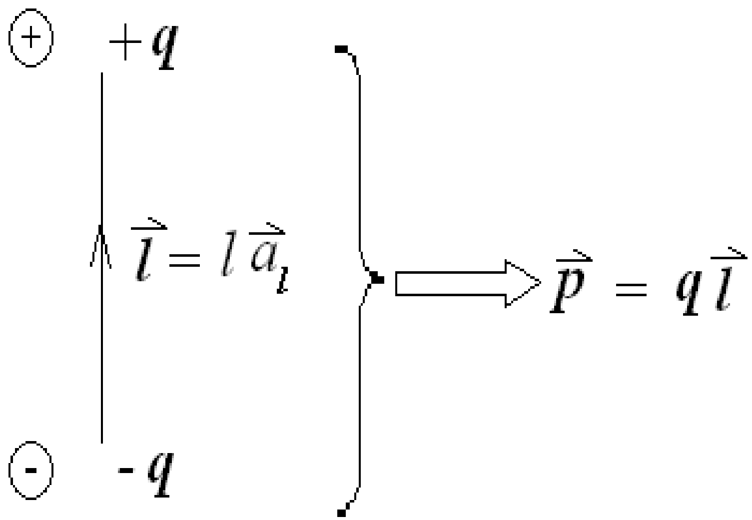
two charges of equal strength but of opposite polarity but separated by a small distance. a positive

charge and a negative constitute an electric dipole.

Electric dipole moment vector: $\vec{p} = q\vec{l}$

Where *q* is the magnitude of each charge and is the distance vector from the negative charge to the positive charge.

A dipole moment vector is with magnitude *p=ql* and direction along the line from the negative to the positive charge.



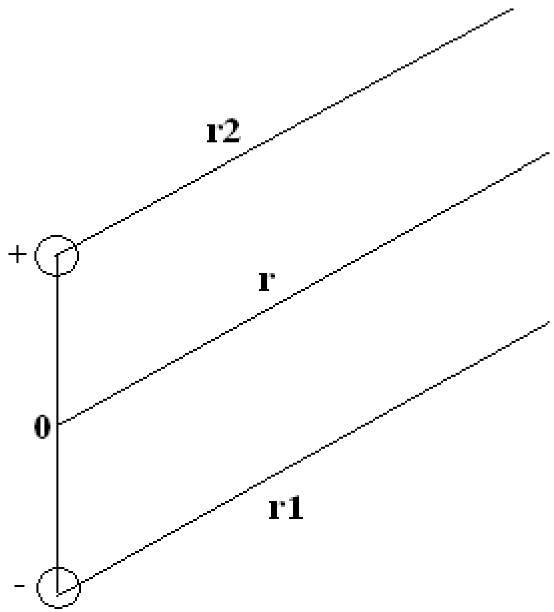
- **Example: determine the potential at any point *P*(*x*,*y*,*z*) in space established by the dipole.**
- **Solutions:**

Assuming that the separation between the charges is very small compared with the distance to the point of observation. The total potential at point P is

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{r_1 - r_2}{r_1 r_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{l\cos\theta}{r^2}\right) = \frac{q\,l\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{q\,l\;\vec{a}_l \bullet \vec{a}_r}{4\pi\varepsilon_0 r^2}$$

$$= \frac{\vec{P} \bullet \vec{a}_r}{4\pi\varepsilon_0 r^2} = \frac{\vec{P} \bullet \vec{r}}{4\pi\varepsilon_0 r^3}$$



- **Charges: free charges and bound charges**
  - **(1)charges**

    (a) class

free charges:

free electrons in a conductor;

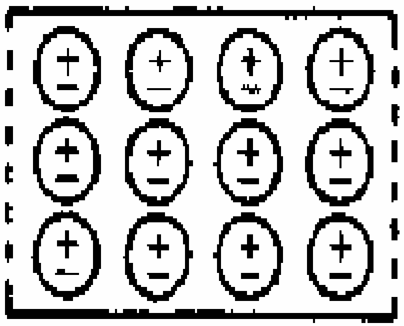
positively charged ions and negatively charged

ions in a liquor, that is free ions in a liquor;

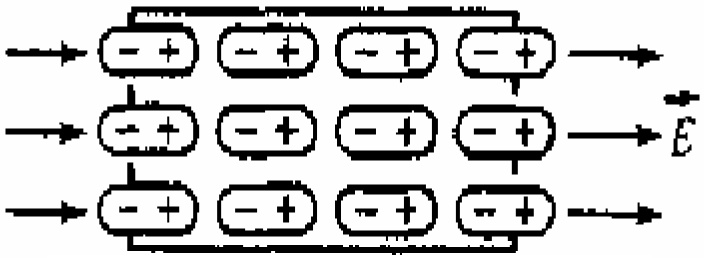
charged particles and electron beam

bound charges: polar charges

Under the influence of an electric field,the molecules of a material experience distortion in the sense that the center of a positive charge of a molecule no longe r coincides with the center of a negative charge. The material is polarized. In its polarized state the material contain a large of number of dipoles



Random orientation
the center of a positive charge coincides with that of a negative charge.



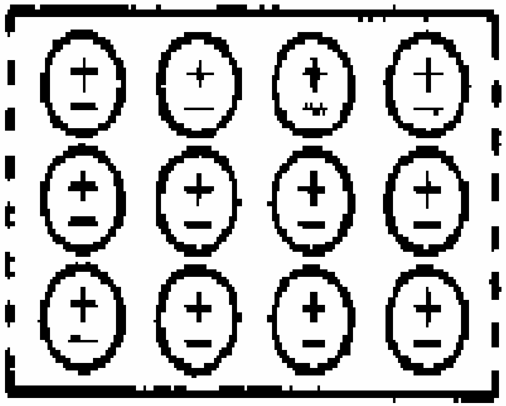
A polarized dielectric showing the separation between charge pairs electric dipoles are arrayed along extern $\vec{E}$

Polar molecules:

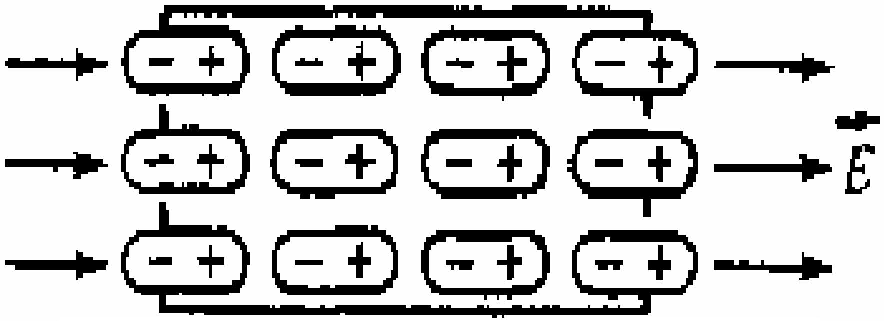
The center of a positive charge of a molecule does not coincide with the center of a negative charge. But all molecules are in a ruleless movement state. The mater ial does not appear polarity. Under the influence of an electric field, all molecules(positive /negative charges) will be arrayed along the direction of the electric field, The material appear polarity.

Non-polar molecules: the center of a positive charge of a molecule coincides with the center of a negative charge. Under the influence of an electric field, the molecules of a material experience distortion in the sense that the center of a positive charge of a molecu le no longer coincides with the center of a negative charge. all molecules(positive /negative charges) will be arrayed along the direction of the electric field, The material appear polarity.

The direction of the dipole moment coincides with the direction of the electric field .



Random orientation
the center of a positive charge coincides with that of a negative charge.



A polarized dielectric showing the separation between charge pairs electric dipoles are arrayed along extern $\vec{E}$

Under the influence of an electric field, The material is polarized. In its polarized state the material contain a large of number of dipoles . An electric dipole is arrayed along the direction of the electric field.

A polarization vector: the number of dipole moments per unit volume. That is,

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\Delta \vec{p}}{\Delta V} \qquad (c/m^2)$$

where $\Delta\vec{p}$ is the dipole moment of volume ΔV

In the limit $\Delta V \to 0$.

The dipole moment $\Delta\vec{p}$ or polarization vector is induced by an external field .

A material is said to be linear if

$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

Because The dipole moment $\Delta\vec{p}$ or polarization vector also creates its own electric field and distorts the external field, the electric field for any medium includes the external electric field and the effect of the polarization. That is,

$$\varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} = \vec{D}$$

where χ_e is called the electric susceptibility(电极化率)

$(1+\chi_e)=\varepsilon_r$ is called the relative permittivity or dielectric constant of the medium(相对电容率或相对介电常数)

ε is called the permittivity or dielectric constant of the medium(电容率或介电常数)。

$\vec{D}$ is called the electric flux density. For free space,

$$\vec{D} = \varepsilon_0 \vec{E}$$

because $\chi_e = 0$, $\vec{P} = 0$ or $(1+\chi_e) = \varepsilon_r = 1$

（a）if the electromagnetic properties of the dielectric (are specified by electromagnetic parameters: permittivity $\varepsilon_r, \varepsilon$, permeability μ_r, μ, conductivity σ) do not vary with the magnitude of the electromagnetic field, dielectric is said to be linear, otherwise non-linear.

(b) **if the electromagnetic properties of the dielectric (are specified by electromagnetic parameters: permittivity ε_r,ε, permeability μ_r, μ, conductivity σ) do not vary with the direction of the electromagnetic field (are independent of the direction), the dielectric is said to be isotropic, *otherwise anisotropic.***

(c) **electromagnetic parameters are constant for the all portions of the dielectric (the all portions of the dielectric are identical), the dielectric is said to be homogeneous, otherwise unhomogeneous.**

(2) Electric flux

- **Definition of electric flux**

The electric flux ψ in terms of electric flux density D as

$$\Psi = \int_s \vec{D} \bullet d\vec{s}$$

The flux passing through s is maximum if $\vec{D}$ and *ds* are in the same direction.

(3) Gauss's Law

Gauss's Law states that the net outward flux passing through a closed surface is equal to the total charge enclosed by that surface. That is

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\oint_s \vec{D} \bullet d\vec{s} = Q$$

$$\vec{\mathbf{E}} = \frac{Q\,\vec{\mathbf{R}}}{4\pi\varepsilon_0 R^3} = \frac{Q\,\vec{\mathbf{a}}_R}{4\pi\varepsilon_0 R^2}$$

Prove:

$$\oint_s \frac{Q}{4\pi R^2}\vec{a}_R \bullet d\vec{s} = \oint_s \frac{Q}{4\pi R^2}\vec{\boldsymbol{a}}_{\boldsymbol{R}} \bullet \vec{\boldsymbol{a}}_{\boldsymbol{R}} ds = \frac{Q}{4\pi}\oint_s \frac{ds}{R^2}$$

$$= \frac{Q}{4\pi}\frac{1}{R^2}\oint_s ds = \frac{Q}{4\pi}\frac{1}{R^2}4\pi R^2 = Q$$

The total electric flux emanating from a closed surface is numerically equal to the net positive charge inside the closed surface.

- **Gauss's Law can also be expressed in terms of electric field intensity in free space as**

$$\oint_s \vec{E} \bullet d\vec{s} = Q / \varepsilon_0$$

- **If the charges are distributed in a volume bounded by a surface**

$$\oint_s \vec{D} \bullet d\vec{s} = Q = \int_v \rho_v dv$$

- **For distributions of surface charge density,**

$$\oint_s \vec{D} \bullet d\vec{s} = Q = \int_s \rho_s ds$$

- **For distributions of line charge density,**

$$\oint_s \vec{D} \bullet d\vec{s} = Q = \int_l \rho_l dl$$

- **By applying the divergence theorem, equation**

$$\oint_s \vec{D} \bullet d\vec{s} = Q = \int_v \rho_v dv$$

can be written as

$$\oint_s \vec{D} \bullet d\vec{s} = Q = \int_v \rho_v dv$$

$$\int_v (\nabla \bullet \vec{D}) dv = \int_v \rho_v dv$$

Because the volume is arbitrary, we can obtain

$$\nabla \bullet \vec{D} = \rho_v$$

$$\nabla \bullet \vec{D} = \rho_v$$

This equation is called the point or the differential form of Gauss's Law.

- **It can be stated in words as follows:**
- **Lines of electric flux emanate from any point in space at which there exists a positive charge density.**
- **If the charge density is negative, the lines of electric flux converge toward the point.**

- **Considering the electric field intensity and the electric flux, we have**

$$\nabla \bullet \vec{E} = \rho_v / \varepsilon$$

review

$$\nabla\times\vec{E}=\nabla\times\frac{q\vec{\boldsymbol{R}}}{4\pi\varepsilon_0 R^3}=\frac{q}{4\pi\varepsilon_0}\nabla\times\frac{\vec{\boldsymbol{R}}}{R^3}=0$$

$$\oint_l \vec{\boldsymbol{E}}\bullet d\vec{\boldsymbol{l}}=\mathbf{0}$$

$$\nabla\bullet\vec{D}=\rho_v$$

$$\nabla\bullet\vec{E}=\rho_v/\varepsilon$$

$$\oint_s \vec{D}\bullet d\vec{s}=Q=\int_v \rho_v dv$$

3.7 The electric field in Materials

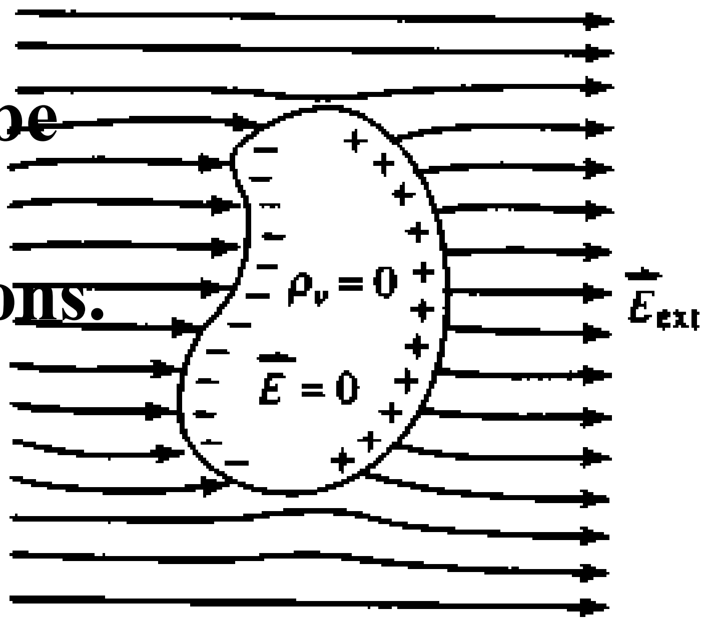
- 1. the electric field in conductors
- The net electric field inside the conductor is zero when the steady state is reached. Thus,

$$\vec{E} = 0$$

Under steady-state (equilibrium) conditions, the net volume charge density within the conductor is zero, that is

$$\rho_v = 0$$

Neither volume charge density nor electric field intensity can be maintained within an isolated conductor under static conditions. Each conductor forms an equipotential region of space.



- **2. dielectrics in an electric field**
- **An ideal dielectric is a material in which positive and negative charges are so sternly bound that they are inseparable, It has zero conductivity.**

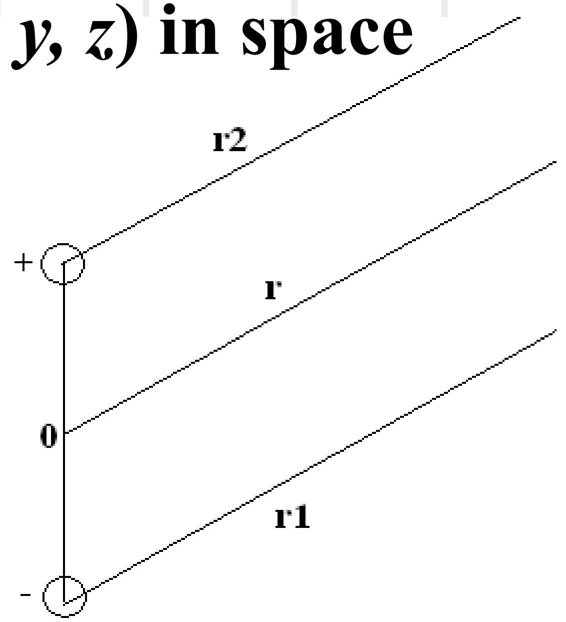
- **For the polarization vector (the number of dipole moments per unit volume)**

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\Delta \vec{p}}{\Delta V} \qquad \text{(c/m}_2\text{)}$$

where $\Delta\vec{p}$ is the dipole moment of volume ΔV

- **since the potential at any point $P\,(x, y, z)$ in space established by the dipole.**

$$V = \frac{\vec{P} \bullet \vec{r}}{4\pi\varepsilon_0 r^3}$$



For the volume *dv'* We can expressed electric dipoles as

$$d\vec{p} = \vec{P}dv'$$

The potential at point *P* due to $\overrightarrow{dp}$ is given by

$$dV = \frac{\vec{P} \bullet \vec{a}_R}{4\pi\varepsilon_0 R^2}dv'$$

Because

$$\nabla'\left(\frac{1}{R}\right) = \frac{\vec{a}_R}{R^2}$$

We obtain

$$dV = \frac{\vec{P} \bullet \nabla'\left(\frac{1}{R}\right)dv'}{4\pi\varepsilon_0}$$

- **Now using the vector identity**

$$\vec{\boldsymbol{P}} \bullet \nabla'\left(\frac{1}{R}\right) = \nabla' \bullet \left(\frac{\vec{\boldsymbol{P}}}{R}\right) - (\nabla' \bullet \vec{\boldsymbol{P}})\frac{1}{R}$$

We have

$$dV = \frac{\vec{P} \bullet \nabla'\left(\frac{1}{R}\right) dv'}{4\pi\varepsilon_0}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\nabla' \bullet \left(\frac{\vec{\boldsymbol{P}}}{R}\right) - (\nabla' \bullet \vec{\boldsymbol{P}})\frac{1}{R}\right] dv'$$

- **Integrating over volume *v'* of the polarized dielectric, we obtain the potential at point P as**

$$V=\frac{1}{4\pi\varepsilon_0}\int_{v'}\left[\nabla'\bullet\left(\frac{\vec{P}}{R}\right)-\frac{\nabla'\bullet\vec{P}}{R}\right]dv'$$

$$=\frac{1}{4\pi\varepsilon_0}\left(\int_{v'}\nabla'\bullet\left(\frac{\vec{P}}{R}\right)dv'+\int_{v'}\frac{-\nabla'\bullet\vec{P}}{R}dv'\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\left(\int_{s'}\left(\frac{\vec{P}}{R}\right)\bullet\vec{\boldsymbol{a}}_n ds'+\int_{v'}\frac{-\nabla'\bullet\vec{P}}{R}dv'\right)$$

- **If we define** $\rho_{sb}=\vec{\boldsymbol{P}}\bullet\vec{\boldsymbol{a}}_n$, $\rho_{vb}=-\nabla'\bullet\vec{P}$

as the bound surface charge density and the bound volume charge density, respectively.

- **The potential created by a polarized materials can be expressed by**

$$V = \frac{1}{4\pi\varepsilon_0}\left(\int_{s'}\left(\frac{\vec{P}}{R}\right)\bullet\vec{a}_n ds' + \int_{v'}\frac{-\nabla'\bullet\vec{P}}{R}dv'\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\oint_{s'}\frac{\rho_{sb}}{R}ds' + \int_{v'}\frac{\rho_{vb}}{R}dv'\right)$$

- **The potential is created by a bound surface charge density and the bound volume charge density. The polarization of a dielectric material results in bound distributions.**

- **If a dielectric region contains the free charge density ρ_v in addition to the bound charge density ρ_{vb}, the contribution due to the free charge density ρ_v and the bound charge density ρ_{vb} must be considered to determine the $\vec{\mathrm{E}}$ field, that is**

$$\nabla \bullet \vec{E} = \frac{\rho_v}{\varepsilon_0} + \frac{\rho_{vb}}{\varepsilon_0} = \frac{\rho_v}{\varepsilon_0} + \frac{-\nabla \bullet \vec{\boldsymbol{P}}}{\varepsilon_0}$$

or

$$\nabla \bullet (\varepsilon_0 \vec{E} + \vec{\boldsymbol{P}}) = \nabla \bullet \vec{\boldsymbol{D}}$$

- **Now we can generalize our definition of electric flux density for any medium as**

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

- **But we should note that the divergence of $\vec{\mathbf{D}}$ will always represent the free charge density in any medium, namely,** $\nabla \bullet \vec{D} = \rho_v$
- **Example 3.14**
- **A point charge *q* is enclosed in a linear, isotropic, and homogeneous dielectric medium of infinite extent, calculate the electric field intensity ,the electric flux density, the polarization vector, the bound surface charge density and the bound volume charge density.**

-----------Read by yourselves.

3.9 Boundary conditions

In this section, we investigate the conditions that govern the behavior of electromagnetic fields at the boundary (interface) between two media. The interface may be between a dielectric and a conduct or between different dielectrics. **(Page104.)**

Concept:

The equations governing the behavior of electro-magnetic fields on either side of an interface are known as the boundary conditions.

Since electromagnetic parameters（ε、μ、σ）change suddenly from one side of an interface to another, the fields at the interface will also vary.

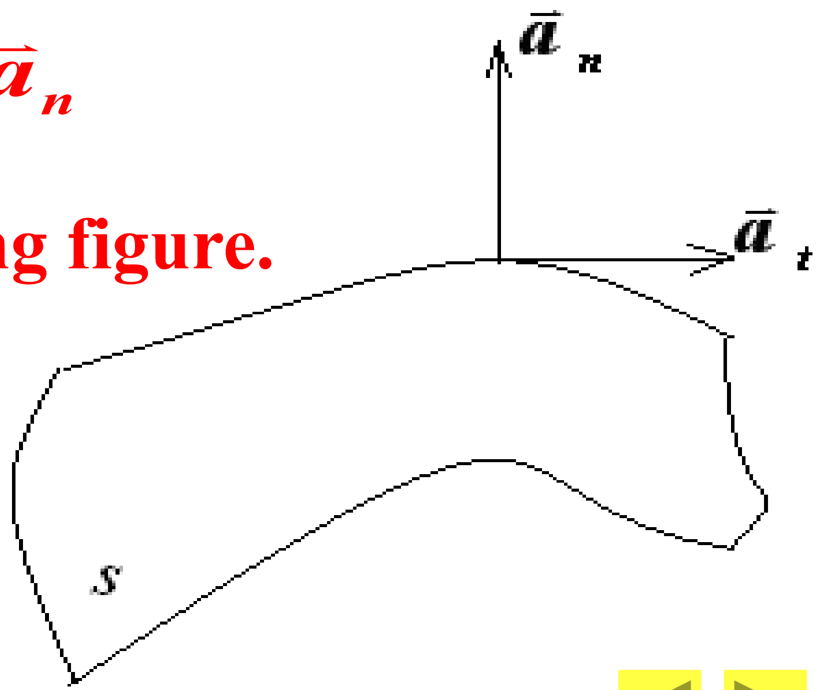
Therefore, Maxell's equations in the point (differential) form are invalid at the interface. We can only use Maxwell's equations in integral form at the interface for obtaining the behavior of electromagnetic fields at the interface.

❖**Introduction**

Since the electromagnetic fields are vector fields, we shall consider the magnitudes and directions of them at an interface. At a point on the surface *S*, the tangential direction $\vec{a}_t$

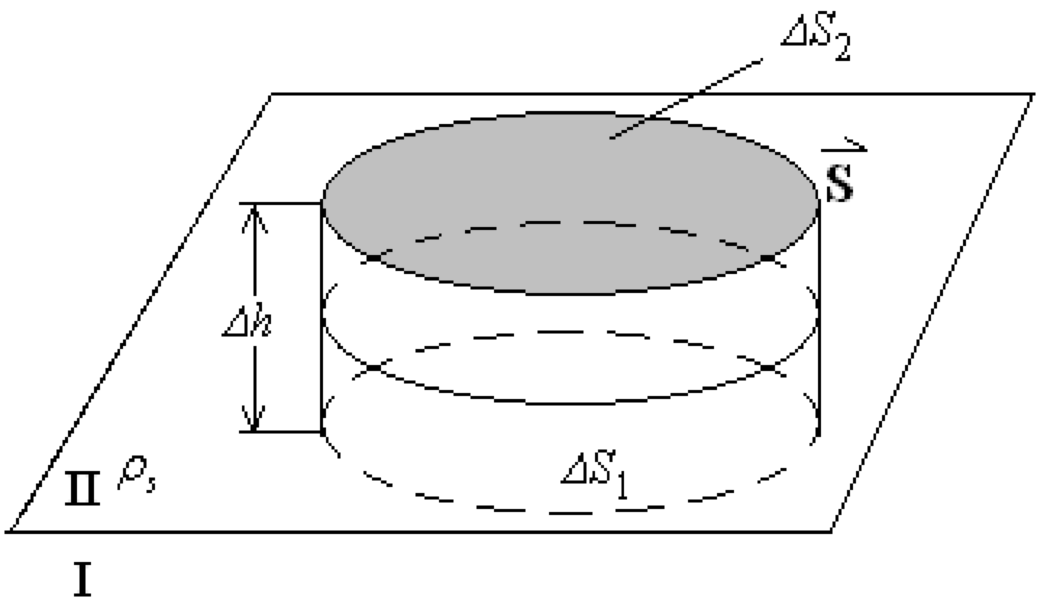
and the normal direction $\vec{a}_n$

as exhibited in the following figure.

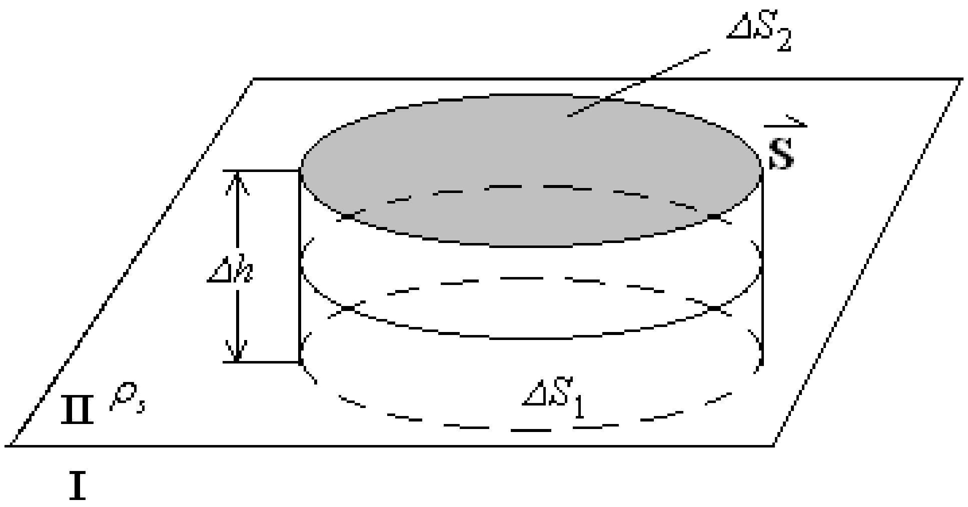


1. the normal component of $\vec{D}$ (page104).

Let us apply Gauss's law to find the boundary condition pertaining to the normal component of the electric flux density at an interface. We have constructed a Gaussian surface $\vec{S}$ in the form of a pillbox, with half in medium 1, and the other half in medium 2, as exhibited in the following figure.



Gaussian surface $\vec{S}$: a closed surface

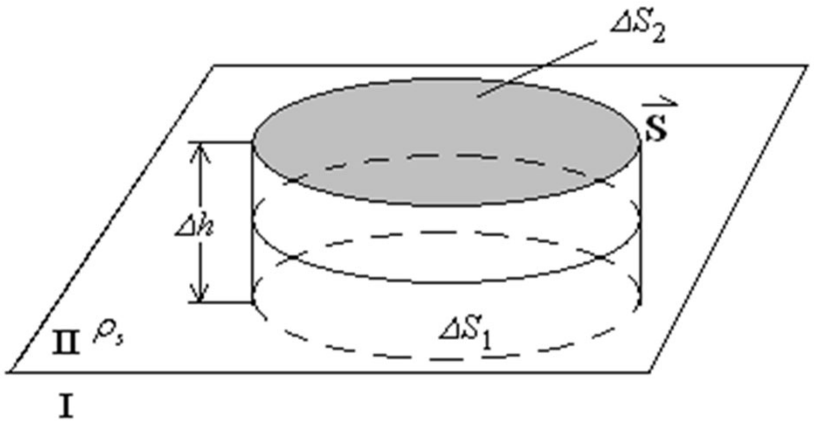


The Gaussian surface $\vec{S}$

includes

- **the surface area ΔS_1,**
- **the surface area ΔS_2 and**
- **The cylinder with its height being Δh**

the Gaussian surface $\vec{S}$



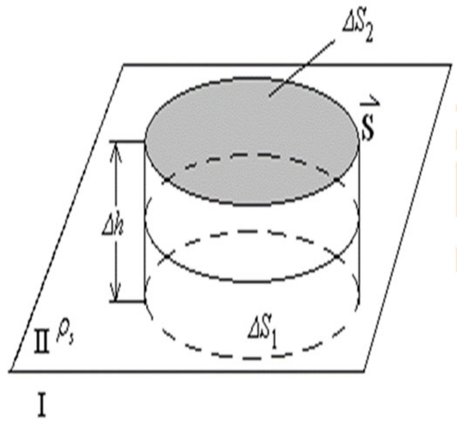
we assume that

- $\Delta S = \Delta S_1 = \Delta S_2$
- a free surface charge density ρ_s at the interface.
- $\vec{a}_n$ is the unit vector normal to the interface pointing from medium 1 to medium 2.

Applying Gauss's law

$$\oint_s \vec{\mathbf{D}} \bullet d\vec{s} = \int_V \rho_v dv$$

we get



$$\oint_s \vec{\mathbf{D}} \bullet d\vec{s} = \int_{\Delta S_1} \vec{\mathbf{D}} \bullet d\vec{s} + \int_{\Delta S_2} \vec{\mathbf{D}} \bullet d\vec{s} + \int_{cylinder} \vec{\mathbf{D}} \bullet d\vec{s}$$

$$= \int_{\Delta S_1} \vec{\mathbf{D}}_1 \bullet d\vec{\mathbf{s}} + \int_{\Delta S_2} \vec{\mathbf{D}}_2 \bullet d\vec{\mathbf{s}} + \int_{cylinder} \vec{\mathbf{D}} \bullet d\vec{\mathbf{s}}$$

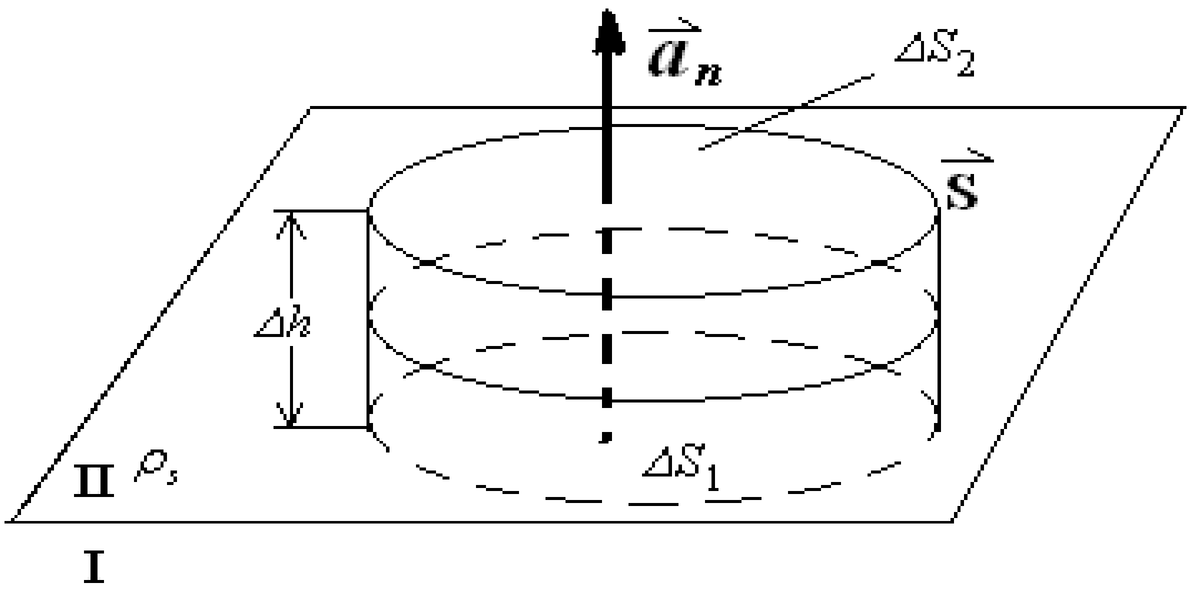
Since fields at a point between medium 1 and medium 2 are studied, we can let

① $\Delta S = \Delta S1 = \Delta S2$ is so small that the electric flux density ($\vec{D}_1$ $\vec{D}_2$) in each medium is essentially constant over the surface in that medium.

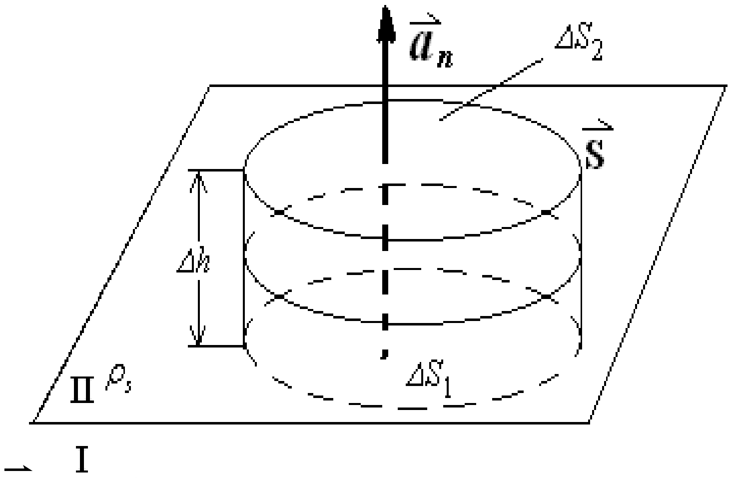
$$\int_{\Delta S_1} \vec{\mathbf{D}}_1 \bullet d\vec{\mathbf{s}} + \int_{\Delta S_2} \vec{\mathbf{D}}_2 \bullet d\vec{s}$$

$$= \vec{\mathbf{D}}_1 \bullet \Delta \vec{s}_1 + \vec{\mathbf{D}}_2 \bullet \Delta \vec{s}_2$$

$$= \vec{\mathbf{D}}_1 \bullet (-\vec{\mathbf{a}}_n \Delta s_1) + \vec{\mathbf{D}}_2 \bullet \vec{\mathbf{a}}_n \Delta s_2$$



② $\Delta h \to 0$, we assume that the area of the curved surface is negligibly small as the height of the pillbox Δh shrinks to zero.



$$\lim_{\Delta h \to 0} \int_{cylinder} \vec{\mathbf{D}} \bullet d\,\vec{\mathbf{s}} = 0$$

From ① and ②, we get

$$\oint_s \vec{\mathbf{D}} \bullet d\vec{s} = \int_{\Delta S_1} \vec{\mathbf{D}} \bullet d\vec{s} + \int_{\Delta S_2} \vec{\mathbf{D}} \bullet d\vec{s} + \int_{cylinder} \vec{\mathbf{D}} \bullet d\vec{s}$$

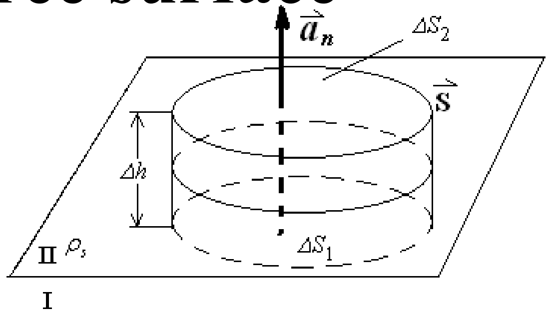
$$= \int_{\Delta S_1} \vec{\mathbf{D}}_1 \bullet d\vec{s} + \int_{\Delta S_2} \vec{\mathbf{D}}_2 \bullet d\vec{s}$$

$$= \vec{\mathbf{D}}_1 \bullet (-\vec{\mathbf{a}}_{\mathbf{n}} \Delta s) + \vec{\mathbf{D}}_2 \bullet \vec{\mathbf{a}}_{\mathbf{n}} \Delta s \qquad (3.7.2)$$

③ **Since the height of the pillbox Δh shrinks to zero, the total charge q enclosed by the Gaussian surface $\vec{S}$**

$$\oint_s \vec{\mathbf{D}} \bullet d\vec{\mathbf{s}} = \int_V \rho_v dv = q$$

only includes the free charges at the interface: we shall also assume that there exists a free surface charge density ρ_s at the interface.



$$\lim_{\Delta h \to 0} \int_V \rho_v dv = \int_{\Delta S} \rho_s ds$$

because ΔS is so small that the surface charge density is constant, the above equation can be rewritten as

$$\lim_{\Delta h \to 0} \int_V \rho_v \, dv = \int_{\Delta S} \rho_s \, ds = \rho_s \Delta s$$

from (3.7.1), (3.7.2) and (3.7.3), we have

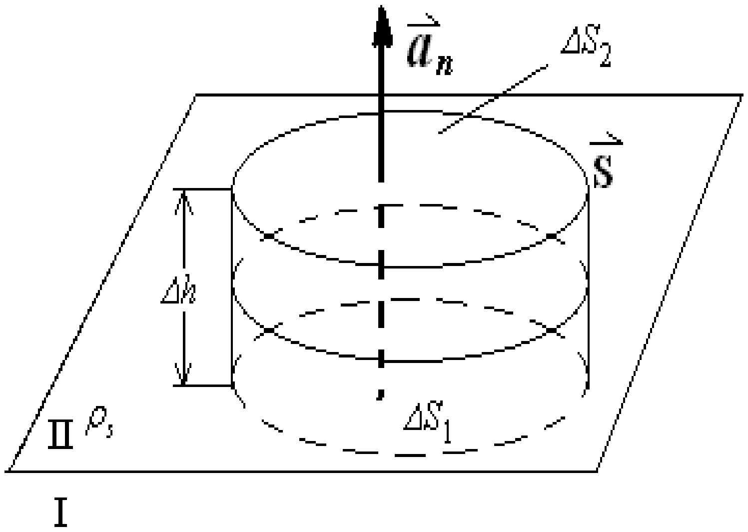
$$\vec{D}_1 \bullet (-\vec{a}_n \Delta s) + \vec{D}_2 \bullet \vec{a}_n \Delta s = \rho_s \Delta s$$

or

$$\vec{a}_n \bullet (\vec{D}_2 - \vec{D}_1) = \rho_s$$

or

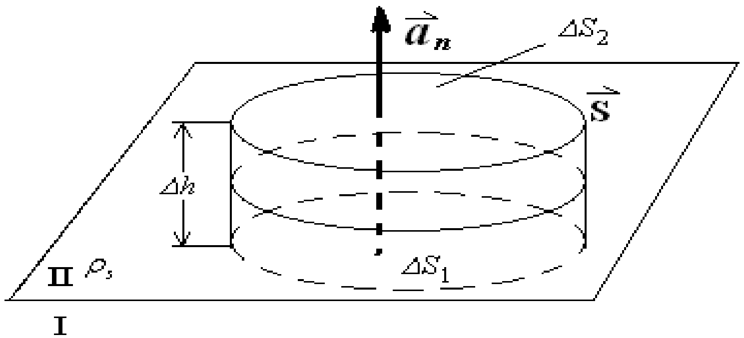
$$D_{2n} - D_{1n} = \rho_s$$



Where D_{2n}, D_{1n} are the components of the $\vec{D}$ field normal to the interface in medium 1 and medium 2. Note that $\vec{a}_n$

is the unit vector normal to the interface

pointing from medium 1 to medium 2. If $\vec{a}_n$ is the unit vector normal to the interface pointing from medium 2 to medium 1, we shall get



$$\vec{a}_n \bullet (\vec{D}_1 - \vec{D}_2) = \rho_s \qquad \textbf{(3.70a)}$$

$$D_{1n} - D_{2n} = \rho_s \qquad \textbf{(3.70b)}$$

Equation (3.70a) states that the normal components of the electric flux density $\vec{D}$ are discontinuous if a free surface charge density exists at the interface.

❖**Discussion:**

❑ **i)if the free charge density ρ_s does not exist at the interface, that is $\rho_s = 0$, we get**

$$\vec{a}_n \bullet (\vec{D}_2 - \vec{D}_1) = 0$$

or

$$D_{2n} - D_{1n} = 0$$

The normal components of the electric flux density are continuous across a dielectric boundary.

- **ii) since $\vec{D} = \varepsilon \vec{E}$ we can obtain**

$$\vec{\mathbf{a}}_n \bullet (\varepsilon_2 \vec{\mathbf{E}}_2 - \varepsilon_1 \vec{\mathbf{E}}_1) = \rho_s \qquad (2.7.4d)$$

or

$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_s$ $\quad \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = 0$, **Obviously, if $\varepsilon_2 \neq \varepsilon_1$, we have $E_{2n} \neq E_{1n}$**

- **iii) if medium 1 is a conductor, the electric flux density $\vec{D}_1$ equals zero. From equation(3.70),**

We get

$$\vec{a}_n \bullet \vec{D}_2 = \rho_s \quad , \; D_{2n}=\rho_s$$

or

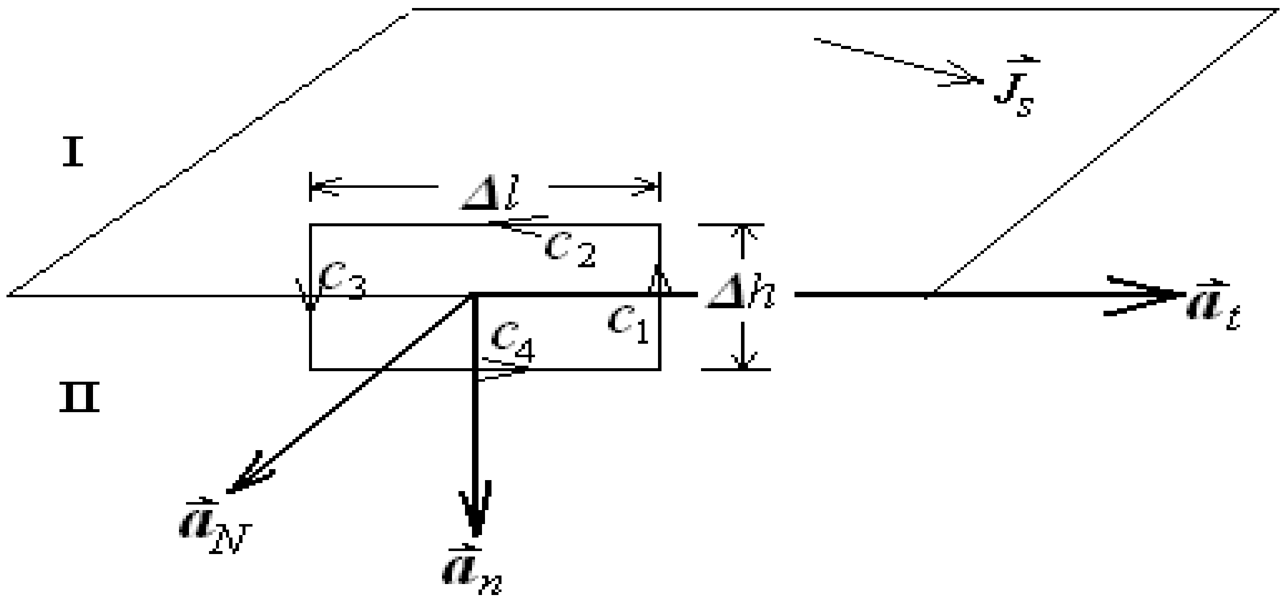
$$\vec{a}_n \bullet \varepsilon_2 \vec{E}_2 = \rho_s \quad , \; \varepsilon_2 E_{2n}=\rho_s$$

obviously, ρ_s is distributed over the surface of the conductor it states that the normal component of the electric flux density in a dielectric medium just above the surface of a conductor is equal to the surface charge density on the conductor.

2. Boundary conditions for tangential components of $\vec{E}$ Field

page106

To obtain the Boundary conditions for Tangential components of $\vec{E}$ field, consider a closed path shown in the following figure.



We already know that the electric field is conservative in nature and, accordingly,

$$\oint_l \vec{E} \bullet d\vec{l} \quad = 0$$

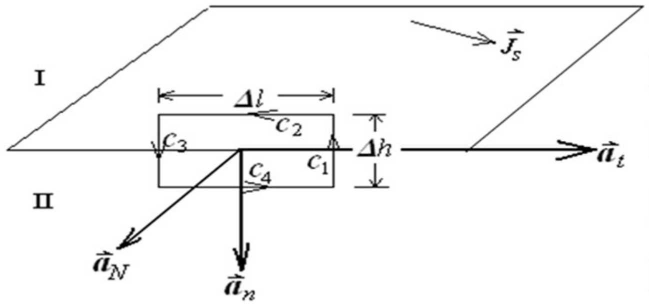
we have

$$\oint_l \vec{E} \bullet d\vec{l} = \int_{c1} \vec{E} \bullet d\vec{l} + \int_{c2} \vec{E} \bullet d\vec{l} + \int_{c3} \vec{E} \bullet d\vec{l} + \int_{c4} \vec{E} \bullet d\vec{l}$$

$$= 0$$

Similarly, consider $\Delta h \to 0$, and also $\Delta S = \Delta h \cdot \Delta l \to 0$, we can obtain

$$\int_{c1} \vec{E} \bullet d\vec{l} = 0, \qquad \int_{c3} \vec{E} \bullet d\vec{l} = 0$$



therefore, $\int_{c2} \vec{E}_1 \bullet d\vec{l} + \int_{c4} \vec{E}_2 \bullet d\vec{l} = 0$

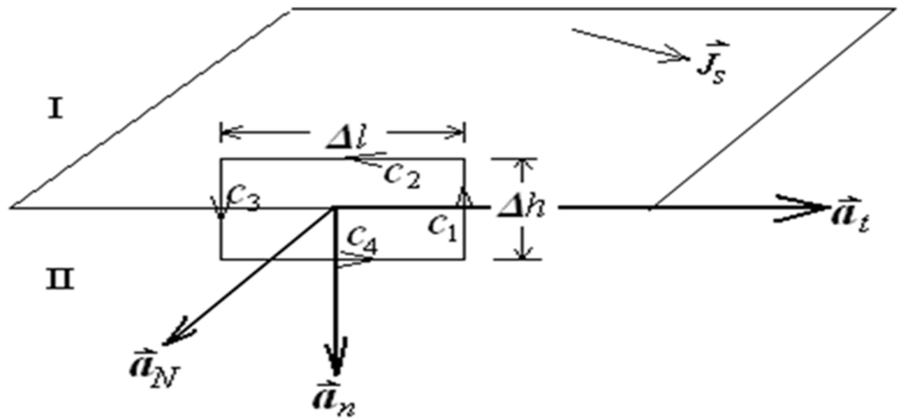
$$\vec{E}_1 \bullet \Delta\vec{l}_2 + \vec{E}_2 \bullet \Delta\vec{l}_4 = 0$$

$$\vec{E}_1 \bullet (-\vec{a}_t)\Delta l + \vec{E}_2 \bullet \vec{a}_t \Delta l = 0$$

$$\vec{a}_t \bullet (\vec{E}_2 - \vec{E}_1) = 0 \quad \Rightarrow \quad \vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

or

$$E_{2t} = E_{1t}$$



If $\vec{D} = \varepsilon\vec{E}$ **,we have**

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

when $\varepsilon_1 \neq \varepsilon_2$, ***we have*** $D_{2t} \neq D_{1t}$

❖**Summary:**

■**Normal components of electric fields**

$$\vec{a}_n \bullet (\vec{D}_2 - \vec{D}_1) = \rho_s \qquad D_{2n} - D_{1n} = \rho_s$$

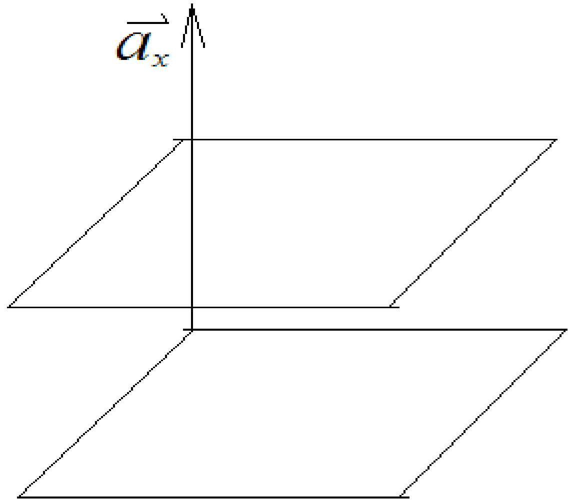
■**Tangential components of electric fields**

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0 \qquad E_{2t} = E_{1t}$$

homework:

$$\overrightarrow{E} = \overrightarrow{a}_x E \cos(\omega t - \beta z)$$

Find: ρ_s

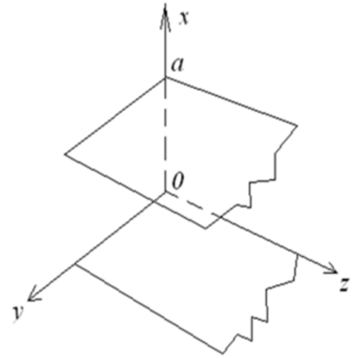


(1)the surface charge density everywhere in these two conductors.

❖**Solution:**

Since σ→∞, charges and currents are distributed over the surfaces of these two conducting plates.

The charge distribution on the upper plate is



$$\rho_{upper} = \vec{\mathbf{a}}_n \bullet (\vec{\mathbf{D}}_{air} - \vec{\mathbf{D}}_{con}) = \vec{\mathbf{a}}_n \bullet (\vec{\mathbf{D}}_{air} - 0)$$

$$= -\vec{a}_x \bullet \varepsilon_0 \vec{\mathbf{E}} \quad = -\varepsilon_0 E \cos(\omega t - \beta z)$$

The charge distribution on the lower plate is

$$\rho_{lower} = \vec{\mathbf{a}}_n \bullet (\vec{\mathbf{D}}_{air} - \vec{\mathbf{D}}_{con}) = \vec{\mathbf{a}}_n \bullet (\vec{\mathbf{D}}_{air} - 0)$$

$$= \vec{a}_x \bullet \varepsilon_0 \vec{\mathbf{E}} = \varepsilon_0 E \cos(\omega t - \beta z)$$

The current distribution on the lower plate is

$$\vec{J}_{lower} = \vec{\mathbf{a}}_n \times (\vec{\mathbf{H}}_{air} - \vec{\mathbf{H}}_{con}) = \vec{\mathbf{a}}_n \times (\vec{\mathbf{H}}_{air} - 0)$$

$$= \vec{a}_x \times \vec{a}_y \frac{E}{\eta_0} \cos(\omega t - \beta z)$$

$$= \vec{a}_z \frac{E}{\eta_0} \cos(\omega t - \beta z)$$

where $\vec{a}_n$ is unit vector pointing from the lower conducting plate to the air.

3.12 Method of image

- Read by yourselves
- Read 3.6,3.7,3.8 from page117 to page 130.

Excises :

T 3.2,T3.6(1),T3.12,T3.18,T3.28,

T3.29,T3.32