§ 2.11 second-order differential operations on field functions

1.the Laplacian Operator of a scalar field

the Laplacian Operator is symbolically written as ∇ . It is defined as the divergence of a gradient of a scalar function. Simply put, if $f(x, y, z)$ is a continuously differentiable scalar function, the Laplacian of $f(x, y, z)$ is

$$\nabla^2 f(x, y, z) = \nabla \bullet (\nabla f(x, y, z)) \quad \text{or} \quad \nabla^2 f = \nabla \bullet (\nabla f)$$

we can write the divergence of the gradient of a scalar function f in the rectangular coordinate system as

$$\nabla \bullet \left(\nabla f\right) = \left[\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}\right] \bullet \left[\vec{a}_x \frac{\partial f}{\partial x} + \vec{a}_y \frac{\partial f}{\partial y} + \vec{a}_z \frac{\partial f}{\partial z}\right]$$

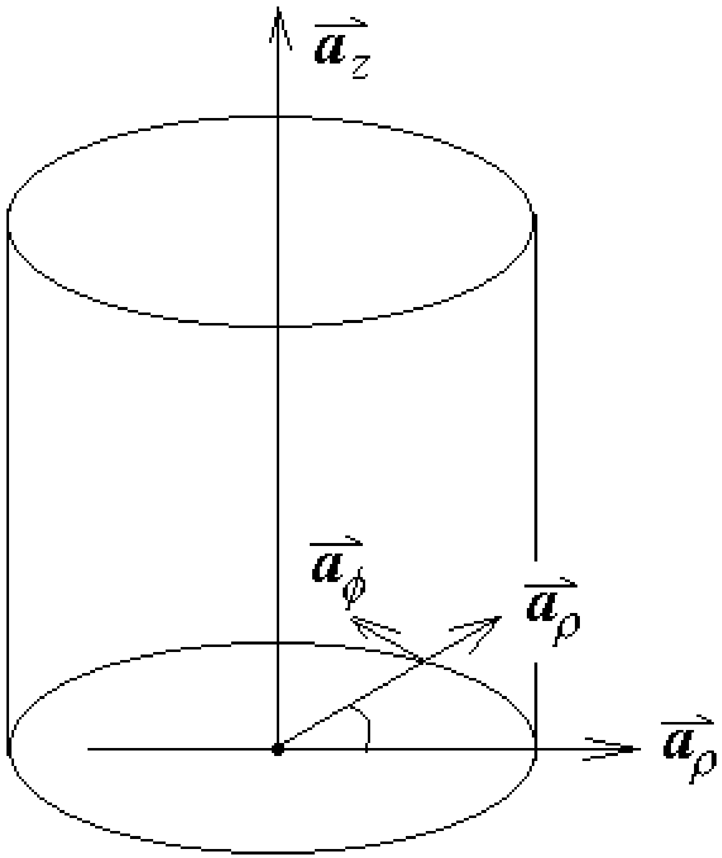
which yields

$$\nabla^2 f = \nabla \bullet \left(\nabla f\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

it is evident that the Laplacian of a scalar function is a scalar and involves second-order partial differentiation of the function.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is a scalar operator.



By simple transformations, we can express the Laplacian of a scalar function in cylindrical coordinates as

$$\nabla^2 f = \nabla \bullet (\nabla f)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

if the Laplacian of a scalar function is zero, that is

$$\nabla^2 f = 0$$

this equation is routinely referred to as Laplacian's equation.

- **2.the Laplacian operator of a vector field**

$$\nabla^2 \vec{F} = \nabla(\nabla \bullet \vec{F}) - \nabla \times \nabla \times \vec{F}$$

it is defined as the gradient of a divergence of a vector field minus the curl of a curl of it.

$$\nabla^2 \vec{F} = \nabla^2(\vec{a}_x F_x + \vec{a}_y F_y + \vec{a}_z F_z)$$
$$= \vec{a}_x \nabla^2 F_x + \vec{a}_y \nabla^2 F_y + \vec{a}_z \nabla^2 F_z$$

- **3.vector identities**
- **（1） $\nabla\times\nabla f=0$**

it is defined as the curl of a gradient of a scalar field, it equals zero. If the curl of a vector field equals zero , the vector field can be represented by the gradient of a scalar field. that is

if $\nabla \times \vec{E} = 0$, $\vec{E} = -\nabla \phi$

Hence, the gradient of a scalar field is a vector field which is an irrotational field.

The irrotational field can be represented by the gradient of a scalar field.

$$(2)\ \nabla \bullet (\nabla \times \vec{A}) = 0$$

it is defined as the divergence of a curl of a vector field, it equals zero. If the divergence of a vector field equals zero, the vector quantity can be represented by the curl of another vector quantity.

If $\nabla \bullet \vec{B} = 0$, $\vec{B} = \nabla \times \vec{A}$

The curl of a vector field is a vector field which is continuous or solenoidal vector field.

A continuous or solenoidal vector field can represented by the curl of a vector field.

4.field classifications

（1）class I fields

we consider a vector field $\vec{F}$ to be a class I field everywhere in a region if

$$\nabla \bullet \vec{F} = 0 \text{ and } \quad \nabla \times \vec{F} = 0$$

However, if the curl of a vector is zero, the vector can be written in terms of a gradient of a scalar function *f*. that is,

$$\vec{F} = -\nabla f \qquad \therefore \nabla^2 f = 0$$

which is Laplace's equation. Therefore, to obtain f ields of class I, we need to solve Laplace's equation subjected to the condition at the boundary of the region. Once we know *f*, we can compute the vector $\vec{F}$ as $\vec{F} = -\nabla f$

- examples:

electrostatic fields in charge-free medium

magnetic fields in current-free medium

(2) class II fields

we consider a vector field $\vec{F}$ to be a class II field in a given region if

$$\nabla \bullet \vec{F} \neq 0 \text{ and } \nabla \times \vec{F} = 0$$

However, if the curl of a vector is zero, the vector can be written in terms of a gradient of a scalar function *f*. that is, $\vec{F} = -\nabla f$

since $\nabla \bullet \vec{F} \neq 0$**, we can let** $\nabla \bullet \vec{F} = \rho$

$$\therefore \nabla^2 f = -\rho$$

which is Poisson's equation. Therefore, to obtain fields of class II, we need to solve Poisson's equation subjected to the constraints of the boundary conditions. Once we know f, we can compute the vector $\vec{F}$ as $\vec{F} = -\nabla f$

- examples:

electrostatic fields in a charged region

- (3) class III fields

$\nabla \bullet \vec{F} = 0$ and $\nabla \times \vec{F} \neq 0$

if the divergence of a vector is zero, then the vector can be expressed in terms of the curl of another vector. For

$\nabla \bullet \vec{F} = 0$, we can express $\vec{F}$ as

$$\vec{F} = \nabla \times \vec{A}$$

where $\vec{A}$ is another vector field.

Because $\nabla \times \vec{F} \neq 0$, we can write it as

$$\nabla \times \vec{F} = \vec{J}$$

where $\vec{J}$ is a known vector field. Substituting

$$\vec{F} = \nabla \times \vec{A}$$

we get

$$\nabla \times \nabla \times \vec{A} = \vec{J}$$

using the vector identity, we can express this equation as

$$\nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A} = \vec{J}$$

if we set an arbitary constraint that $\nabla \bullet \vec{A} = 0$**, we obtain**

$$\nabla^2 \vec{A} = \vec{J}$$

which is called Poisson's vector equation. Therefore, class III fields require a solution of Poisson's vector equation. The vector field $\vec{F}$ **can be computed from** $\vec{A}$ **as**

$$\vec{F} = \nabla \times \vec{A}$$

the constraint $\nabla \bullet \vec{A} = 0$ **is known as Coulomb gauge**

■**Examples:**

The magnetic field within a current-carrying conductor falls into class III.

➤**(4)class IV fields**

for a vector field $\vec{F}$ **to be class IV, neither its divergence nor its curl is zero. However, we can decompose** $\vec{F}$ **into two vector fields** $\vec{G}$ **and** $\vec{H}$ **such that** $\vec{G}$ **satisfies class III and** $\vec{H}$ **satisfies class II requirements. That is,**

$$\vec{F} = \vec{G} + \vec{H}$$

$$\nabla \bullet \vec{G} = 0, \nabla \times \vec{G} \neq 0 \rightarrow \vec{G} = \nabla \times \vec{A}$$

$$\nabla \times \vec{H} = 0, \nabla \bullet \vec{H} \neq 0 \rightarrow \vec{H} = -\nabla f$$

so

$$\vec{F} = \vec{G} + \vec{H} = \nabla \times \vec{A} - \nabla f$$