Fundamentals of Information Theory

Rate Distortion Theory

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Outline

- Do all information sources need error-free coding?
- System model for rate-distortion coding
- How to evaluate distortion?——distortion function
- Optimization problem for rate-distortion coding
- Rate distortion function
- Shannon’s third theorem: Rate-distortion source coding theorem
- Distortion rate function
- Practical insights

本节学习目标

1. 理解有损压缩的动机与意义
2. 说出率失真信源编码的建模过程
3. 说出失真函数的定义
4. 说出平均失真的定义
5. 说出率失真函数的定义及意义
6. 写出香农第三定理及其意义
7. 理解率失真理论与信道容量的联系

重难点：

- 率失真信源编码的建模
- 率失真函数的定义
- 香农第三定理

01

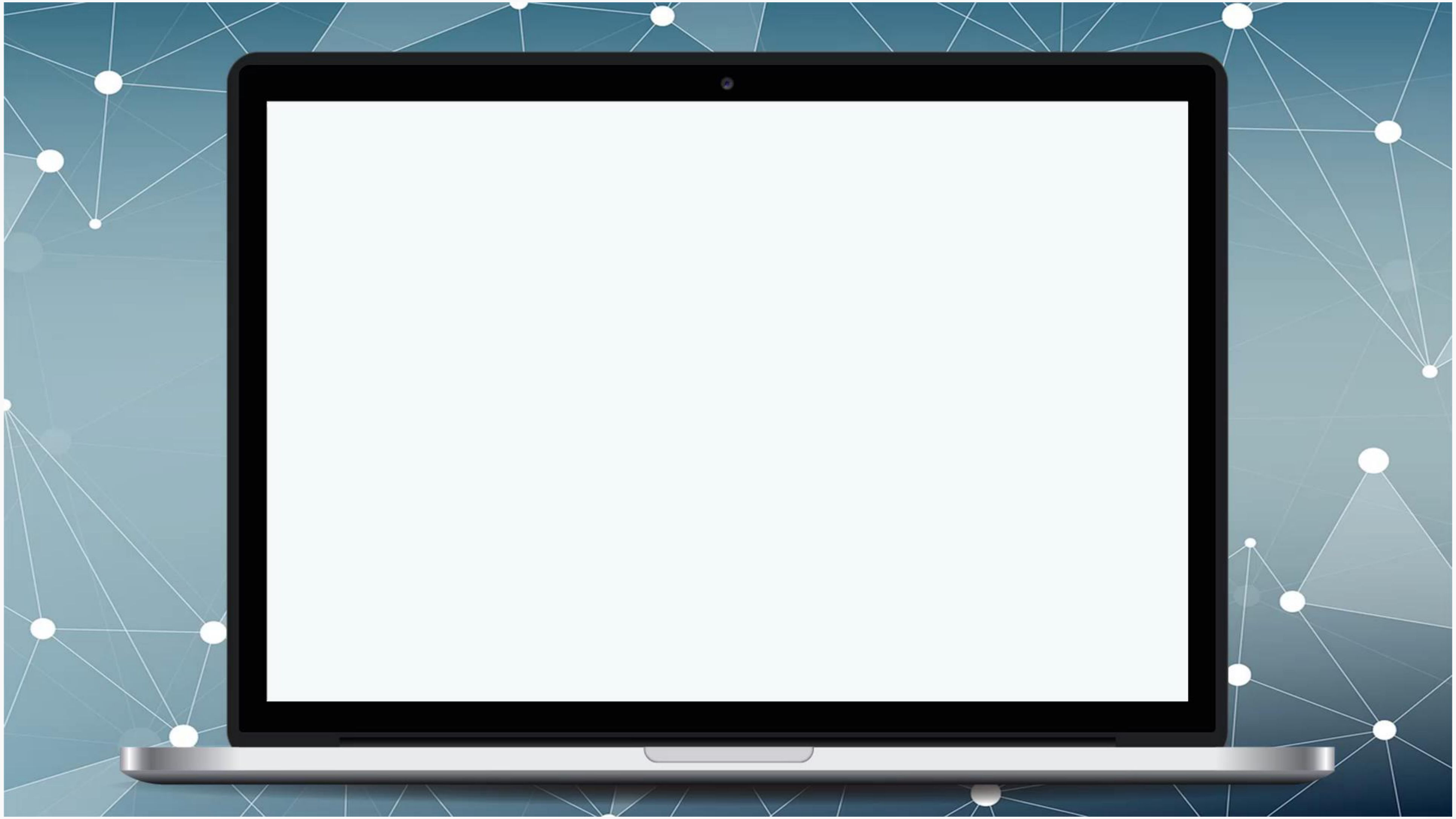
Do all information sources need error-free coding?

Revisiting

- Source coding
  - Eliminate redundancy to compress data and improve efficiency.
  - Represent the source efficiently and **without error**.
- Channel coding
  - Increase redundancy to combat transmission errors.
  - Transmit information reliably over channels **without error**
- **Preserve entropy**
  - To guarantee reliable and error-free transmission.

Do all information sources need error-free coding?

Do all information sources need error-free coding?



Error-free coding are not always needed.

We do not need to reconstruct all the information in continuous sources

Lossy source coding

Fundamental question of rate-distortion theory

Tradeoff between Rate and Distortion

Compression rate

VS

Compression quality

We are all imperfect. But how well can we do?

- Given a requirement on distortion, how small can the source be compressed?
    - What is the minimum description rate?

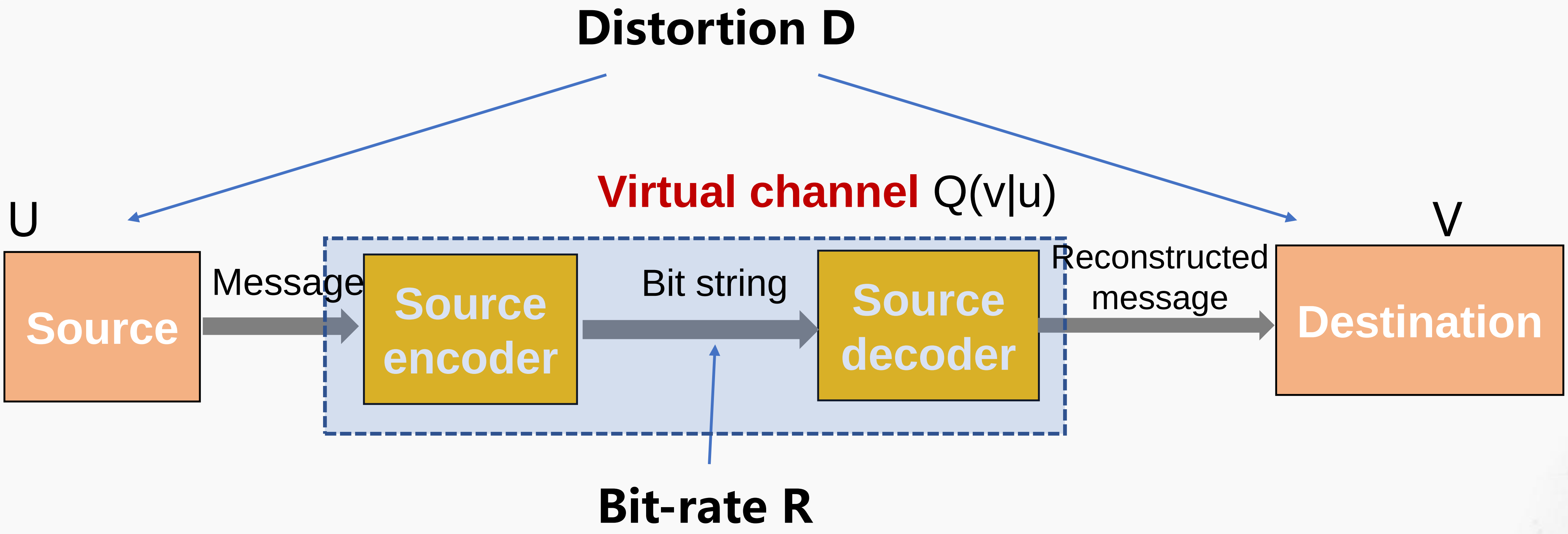
We are all imperfect. But how well can we do?

- Given a specific transmission bit-rate, how high-definition video I can watch?
  - What is the minimum distortion?

02

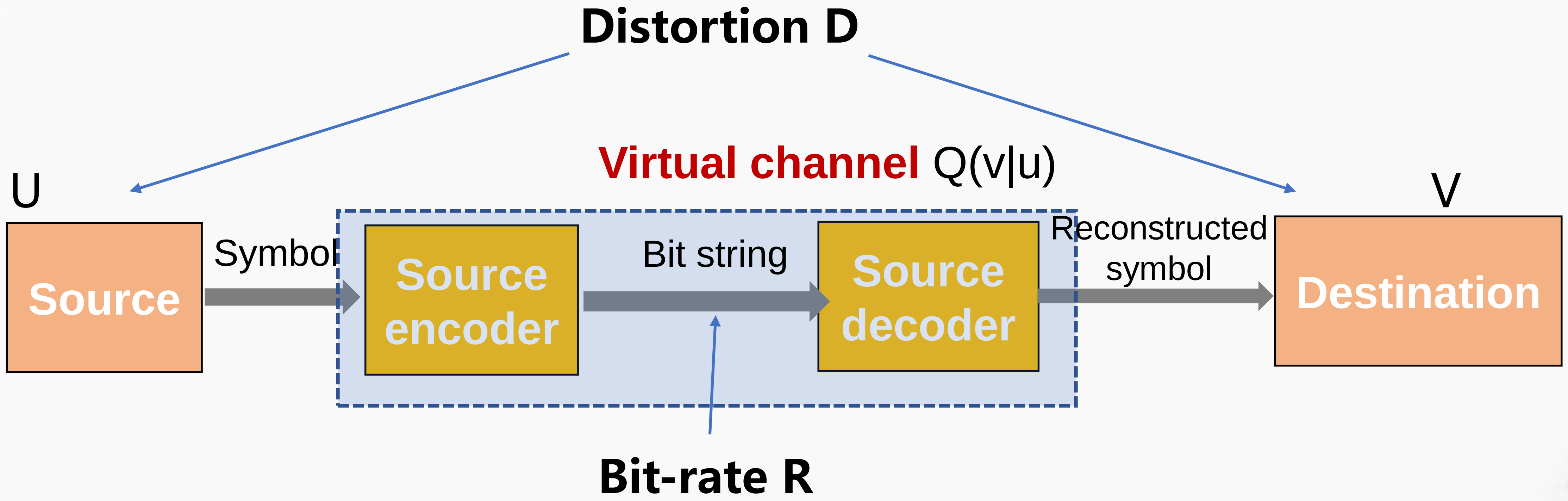
System model
for rate-distortion coding

Rate-distortion source coding: system model



- Objective: Establish functional relationship between source U, destination V, distortion D and information rate R.
- Idea: **Consider the process of rate distortion encoding and decoding as a virtual channel**.

Rate-distortion source coding: system model



conditional probability distribution

Source symbols

$$U = (u_0, u_1, \ldots, u_{M-1})$$

$$\boxed{Q = \{Q(v|u), u \in \mathcal{U}, v \in \mathcal{V}\}}$$

Reconstruction symbols

$$V = (v_0, v_1, \ldots, v_{N-1})$$

Source symbols and reconstruction symbols

- Source symbols are given by the random sequence $\{U_k\}$. Each U_k assumes values in the discrete set $U = (u_0, u_1, \ldots, u_{M-1})$.
  For simplicity, let us assume U_k to be independent and identically distributed (i.i.d.) with the distribution $P(u)$, $u \in \mathcal{U}$.
  Example:
    - For a binary source: $U = (0, 1)$.
    - For a picture: $U = (0, 1, \ldots, 255)$.
- Reconstruction symbols are given by the random sequence $\{V_k\}$ with distribution $P(v)$, $v \in \mathcal{V}$.
  Each V_k assumes values in the discrete set $V = (v_0, v_1, \ldots, v_{N-1})$.
- The sets $\mathcal{U}$ and $\mathcal{V}$ are usually the same.

Coder/Decoder

- The statistical description of the coder/decoder defines the mapping from the source symbols to the reconstruction symbols, via

$$Q = \{Q(v|u), u \in \mathcal{U}, v \in \mathcal{V})\}.$$

- Q is the conditional probability distribution over the letters of the reconstruction alphabet $\mathcal{V}$ given a letter of the source alphabet $\mathcal{U}$.

- The transmission system is described via the joint p.d.f.: $P(u, v)$.

$$P(u, v) = P(u) \cdot Q(v|u) \text{ (Bayes' rule)}$$

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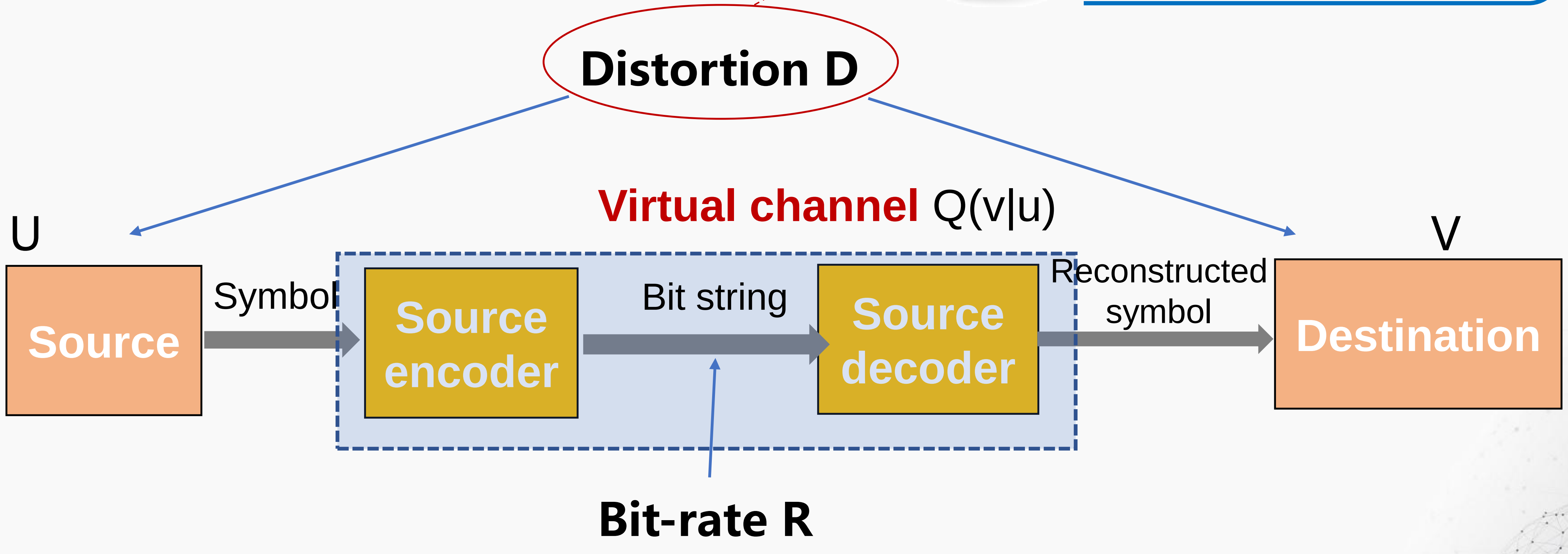
How to characterize distortion?
——distortion function

A natural question...

How to characterize the distortion?



How to characterize distortion?——distortion function

- **Definition**: the distortion between the input symbol u by the output symbol v is measured by a **non-negative** cost function $d(u,v)$.

$$d(u,v) = \begin{cases} 0, & u = v \\ a, & u \neq v \end{cases}$$

- A mapping from the set of source-reconstruction alphabet pairs into the set of nonnegative real numbers.
  - For discrete alphabets, distortion function can be described with **distortion matrix**.

$$d : \mathcal{U} \times \mathcal{V} \to [0, \infty)$$

- **Physical meaning**: the cost of representing u by v

Some common distortion functions

- Hamming distortion

$$d(u,v) = \begin{cases} 0, & \text{for } u = v \\ 1, & \text{for } u \neq v \end{cases}$$

- Hamming distortion matrix

$$\mathcal{D} = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \dots & & & & \dots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$$

- Squared-error distortion

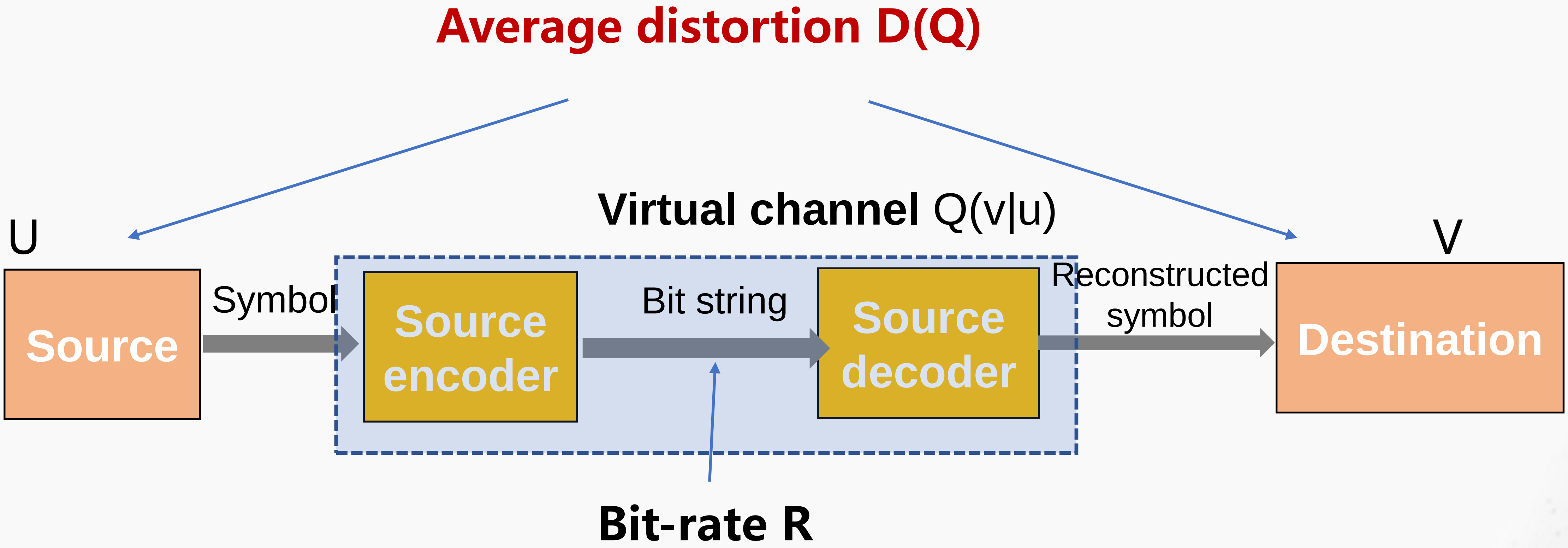
$$d(u,v) = |u - v|^2$$

- Given $U = \{0,1,2\},\ V = \{0,1,2\}$
- Distortion matrix

$$\mathcal{D} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

- Widely adopted for continuous alphabets.

How to measure the overall distortion?



- Average distortion: statistical average of the distortion function

$$D(Q) = E[d(u, v)]$$

Average distortion: consider the source distribution

- Average distortion: statistical average of the distortion function

$$D(Q) = E[d(u,v)]$$

$$D(Q) = \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} P(u,v) \cdot d(u,v)$$
$$= \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} P(u) \cdot Q(v|u) \cdot d(u,v)$$

- Information source: $P(u)$
- Coder/decoder: $Q(v|u)$
- Distortion function: $d(u,v)$

- Given source distribution and the transition probability distribution, the **average measure of distortion** over the channel.

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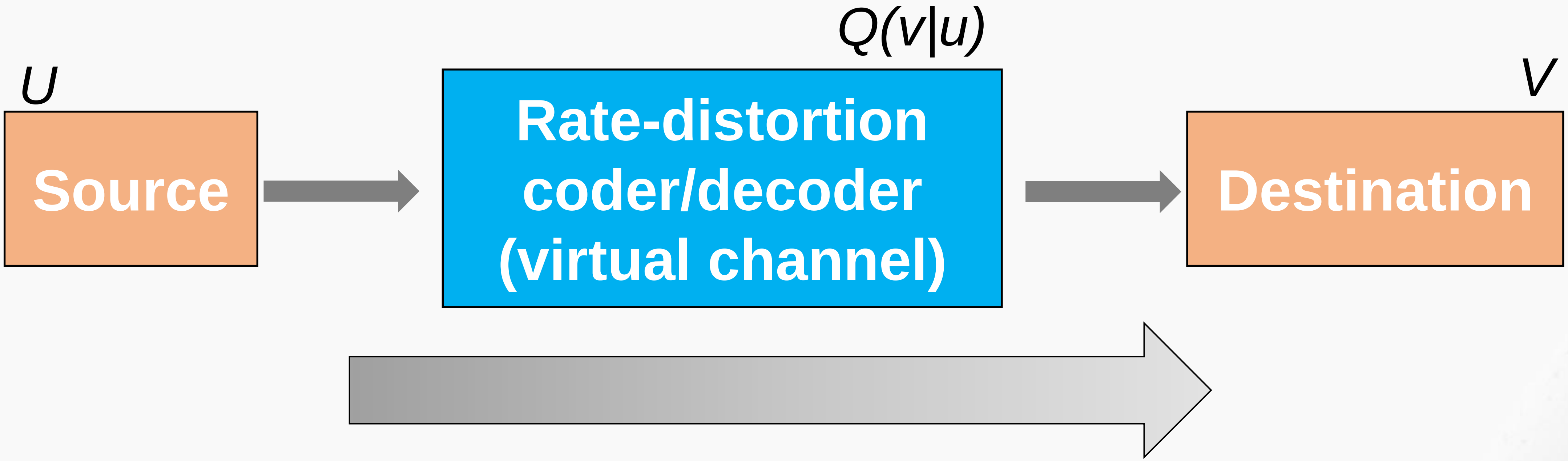
Optimization problem for rate-distortion coding

Question

After compression, how much information of the source are kept in the reconstruction symbols?



Mutual information $I(U;V)$

Information rate

- The Shannon average mutual information is expressed via entropy.

$$I(U; V) = H(U) - H(U|V),$$

where
  - $H(U)$: Source entropy
  - $H(U|V)$: Equivocation (conditional entropy).

- **Equivocation**:
  - The conditional entropy (uncertainty) about the source U given the reconstruction V.
  - A measure for the amount of **missing** [quantized] information in the received signal V.

- I(U; V) denotes **the amount of average information of the source U that contains in the reconstruction one V**.

Optimization problem: objective function

Q: How well can we compress the source?

Less kept information, better compression.

Minimize mutual information

$$\min \ I(U;V)$$

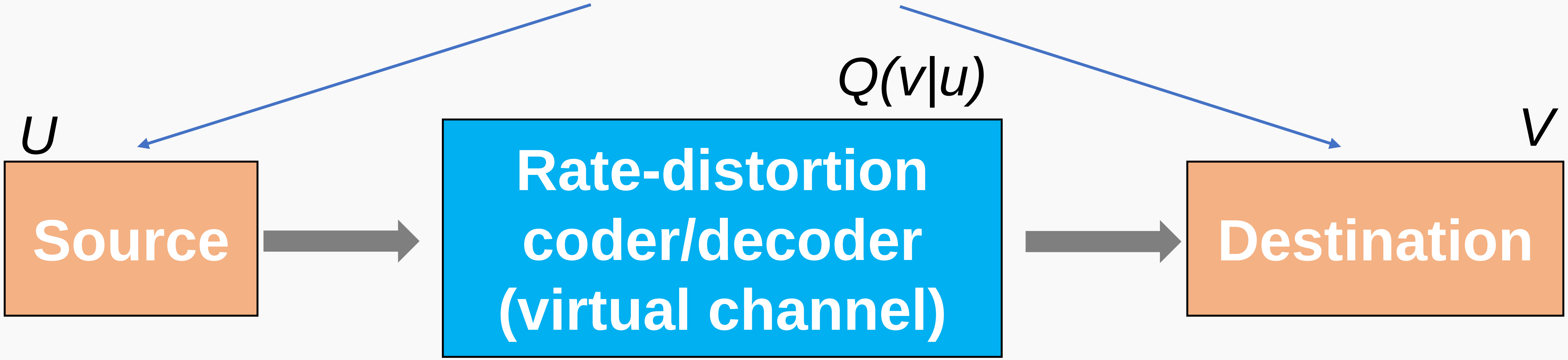
Let V be a constant, then

$$I(U;V) = 0$$

Any problem?

Optimization problem: constraint on average distortion

Average distortion D(Q) $D(Q) = E[d(u,v)]$



- Fidelity criteria: constraint on average distortion

$$D(Q) \le D^*.$$

- D^* : maximum average distortion

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Rate Distortion Function

Rate distortion function: definition

- Definition: For a source *U* and distortion function *d(u, v)*, given the maximum allowable distortion *D**, the **minimum information rate *R*(*D**)**

$$R(D^*) = \min_{\boxed{Q: D(Q) \leq D^*}} \{I(U; V)\}$$

coder/decoder

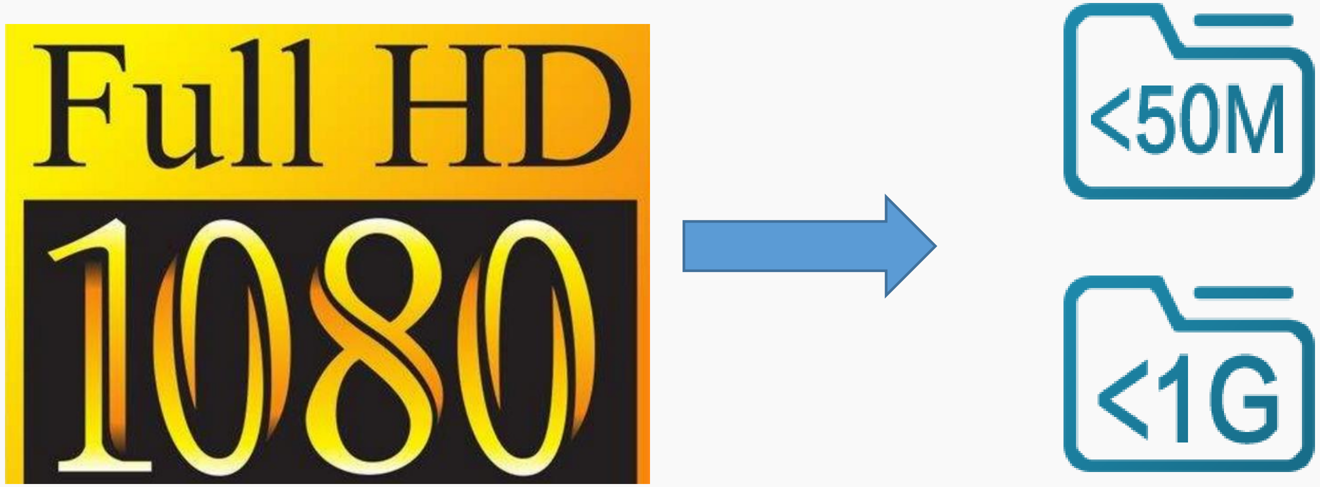
$$\boxed{D(Q) = \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} P(u) \cdot \boxed{Q(v|u)} \cdot d(u, v)}$$

- The minimization is conducted for **all possible mappings Q** that satisfy the average distortion constraint.

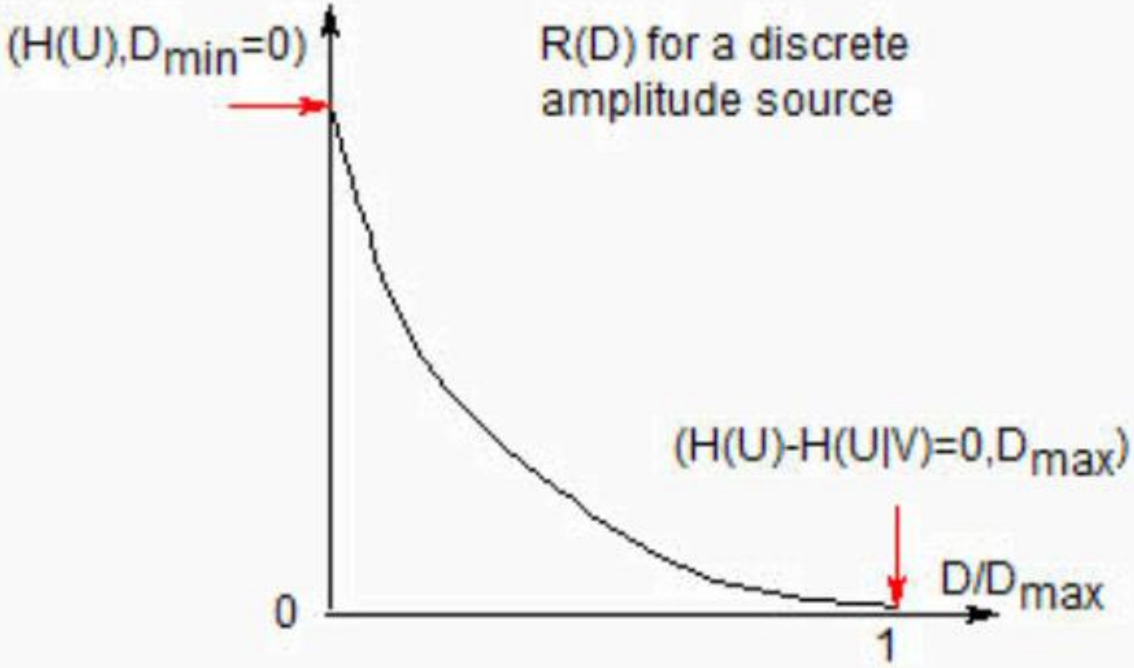
Rate distortion function: Physical meaning

$$\boxed{R(D^*) = \min_{Q:D(Q)\leq D^*} \{I(U;V)\}}$$

- Data compression limit for lossy source coding
- Given a requirement on distortion, how small can the source be compressed? What is the minimum description rate?

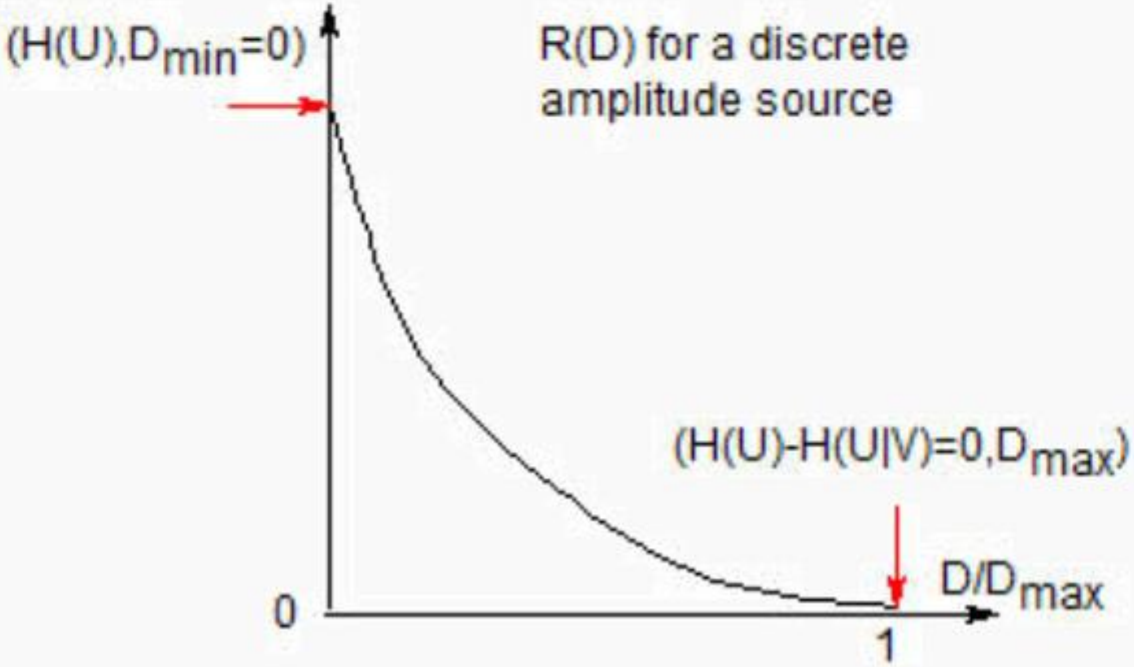


Rate distortion function: properties



- $R(D)$ is well defined for $D \in (D_{\text{min}}, D_{\text{max}})$.
- For discrete amplitude sources, $D_{\text{min}} = 0$.
- $R(D) = H(U)$, if $D = D_{\text{min}} = 0$. (not always true.)
- $R(D) = 0$, if $D \geq D_{\text{max}}$.
- $H(U) > R(D) > 0$, if $0 < D < D_{\text{max}}$.

Rate distortion function: properties



- $R(D)$ is always non-negative.

$$0 \leq I(U;V) \leq H(U)$$

- $R(D)$ is decreasing in the range $(D_{\text{min}}, D_{\text{max}})$.
- $R(D)$ is strictly convex upward in the range $(D_{\text{min}}, D_{\text{max}})$.
- The slope of $R(D)$ is continuous in the range $(D_{\text{min}}, D_{\text{max}})$.

Rate distortion function: discrete source

$$R(D^*) = \min_{Q:D(Q)\leq D*} \{I(U;V)\}$$

- For discrete sources, calculating $R(D^*)$ is to find the local minimum mutual information problem under some constraint conditions.
- Given $p(u)$ and $d(u,v)$, find the minimum $I(U;V)$ under the constraint condition $D(Q) \leq D$. The typical solution applies the Lagrange multiplier method.

Rate distortion function: continuous source

$$\boxed{R(D^*) = \min_{Q:D(Q)\leq D^*} \{I(U;V)\}}$$

$R(D^*)$ for memoryless Gaussian sources.

- Gaussian source, variance σ^2.
- Mean squared error (MSE) $D = E\left\{(u-v)^2\right\}$

$$R(D^*) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D^*}, & 0 \leq D^* \leq \sigma^2 \\ 0, & D^* > \sigma^2 \end{cases}$$

$SNR = 10 \cdot \log_{10}\frac{\sigma^2}{D}$

- Rule of thumb: 6dB $\sim$ 1 bit

- The $R(D^*)$ for non-Gaussian sources with the same variance σ^2 is always below the Gaussian $R(D^*)$ curve.

Rate distortion function: Memoryless Gaussian source

$$R(D) = \min_{f(V|U):E(U-V)^2 \le D} I(U;V).$$

$$\begin{aligned}
I(U;V) &= h(U) - h(U|V) && (h(X+a) = h(X)) \\
&= h(U) - h(U-V|V) && (r.v. U \text{ is Gaussian.}) \\
&\ge \frac{1}{2}\log[(2\pi e)\sigma^2] - h(U-V) && (\text{Conditioning reduce entropy.}) \\
&\ge \frac{1}{2}\log[(2\pi e)\sigma^2] - h\left(\mathcal{N}\left(0, E(U-V)^2\right)\right) && (\text{Gaussian maximum entropy.}) \\
&= \frac{1}{2}\log[(2\pi e)\sigma^2] - \frac{1}{2}\log[(2\pi e)E(U-V)^2] \\
&\ge \frac{1}{2}\log[(2\pi e)\sigma^2] - \frac{1}{2}\log[(2\pi e)D^*] && (\text{Distortion fidelity criteria.}) \\
&= \frac{1}{2}\log\frac{\sigma^2}{D^*}
\end{aligned}$$

Rate distortion function: Example

Let $d(x, \hat{x})$ be a distortion function. We have a source $X \sim p(x)$. Let $R(D)$ be the associated rate distortion function.

(a) Find $\widetilde{R}(D)$ in terms of $R(D)$, where $\widetilde{R}(D)$ is the rate distortion function associated with the distortion $\widetilde{d}(x, \hat{x}) = d(x, \hat{x}) + a$ for some constant $a > 0$. (They are not equal)

(b) Now suppose that $d(x, \hat{x}) \geq 0$ for all $x, \hat{x}$ and define a new distortion function $d^*(x, \hat{x}) = bd(x, \hat{x})$, where b is some number ≥ 0. Find the associated rate distortion function $R^*(D)$ in terms of $R(D)$.

(c) Let $X \sim N(0, \sigma^2)$ and $d(x, \hat{x}) = 5(x - \hat{x})^2 + 3$. What is $R(D)$?

Rate distortion function: Example

(a)

$$\widetilde{R}(D) = \inf_{p(\hat{x}|x):E\left(\widetilde{d}(x,\hat{x})\right)\le D} I(X;\hat{X})$$

$$= \inf_{p(\hat{x}|x):E(d(x,\hat{x}))+a\le D} I(X;\hat{X})$$

$$= \inf_{p(\hat{x}|x):E(d(x,\hat{x}))\le D-a} I(X;\hat{X})$$

$$= R(D-a)$$

Rate distortion function: Example

(b) If $b > 0$,

$$\begin{aligned} R^*(D) &= \inf_{p(\hat{x}|x):E(d^*(x,\hat{x}))\leq D} I(X;\hat{X}) \\ &= \inf_{p(\hat{x}|x):E(bd(x,\hat{x}))\leq D} I(X;\hat{X}) \\ &= \inf_{p(\hat{x}|x):E(d(x,\hat{x}))\leq \frac{D}{b}} I(X;\hat{X}) \\ &= R\left(\frac{D}{b}\right). \end{aligned}$$

If $b = 0$, then $d^* = 0$ and $R^*(D) = 0$.

Rate distortion function: Example

(c) Let $R_{se}(D)$ be the rate distortion function associate with the distortion $d_{se}(x, \hat{x}) = (x - \hat{x})^2$: Then from parts (a) and (b) we have

$$R(D) = R_{se}\left(\frac{D-3}{5}\right).$$

We know that

$$R_{se}(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D}, & 0 \le D \le \sigma^2 \\ 0, & D > \sigma^2 \end{cases}.$$

Therefore, we have

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{5\sigma^2}{D-3}, & 3 \le D \le 5\sigma^2 + 3 \\ 0, & D > 5\sigma^2 + 3 \end{cases}.$$

06

Shannon’s third theorem:

Rate-distortion source coding theorem

Rate-distortion source coding theorem

$R \geq R(D) \Rightarrow$ There exists a coding method C, which satisfies $D(C) \leq D + \varepsilon$ for any given positive D and any minimum ε.

$R < R(D) \Rightarrow$ For any coding method C, $D(C) > D$.

- Known as Shannon's third theorem
- Limits of data compression
  - Zero-error source compression (1^{st} theorem): **H(S)**
  - Distortion source compression (3^{rd} theorem): **R(D)**
  - Given D, normally R(D)<H(S).

07

Distortion Rate Function

Motivation: Another perspective

- Given a requirement of distortion D from the source, what is the minimum transmission bit-rate R?
  - Rate distortion function R(D)

- Given a specific transmission bit-rate R, what is the minimum distortion D?
  - **Distortion rate function D(R)**

- The calculation of the rate distortion function R(D) and the distortion rate function D(R) are called as dual-problems.

Distortion rate function: definition

- **Definition**: For a source U and distortion function d(u, v), given the maximum average rate R*, the **minimum average distortion D(R*)**

$$D(R^*) = \min_{Q:I(U;V)\leq R^*} \{d(Q)\}$$

- We can set R* to the capacity C of the transmission channel and determine the minimum distortion for this ideal communication system.

Distortion rate function: physical meaning

- Given a specific transmission bit-rate, how high-definition video I can watch?
  - What is the minimum distortion?

Distortion rate function: continuous source

$$D(R^*) = \min_{Q:I(U;V)\leq R^*} \{d(Q)\}$$

$D(R^*)$ for memoryless Gaussian sources.

- Gaussian source, variance σ^2.
- Mean squared error (MSE) $D = E\left\{(u-v)^2\right\}$

$$R(D^*) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D^*}, & 0 \leq D^* \leq \sigma^2 \\ 0, & D^* > \sigma^2 \end{cases}$$

$$D(R^*) = \sigma^2 \cdot 2^{-2R^*}, R \geq 0.$$

08

Practical Insights

Rate distortion function vs. Channel capacity

$$R(D^*) = \min_{Q:D(Q)\leq D^*} \{I(U;V)\}$$

$$C = \max_{p(x)} \{I(X;Y)\}$$

- **Rate distortion theory**
  - **minimize** mutual information
  - **Source** is given
  - Search all possible **channels (coder/decoder design)** for the optimal solution
  - **Efficiency** for compression
  - **Decrease redundancy**
  - **Source coding**

- **Channel capacity**
  - **maximize** mutual information
  - **Channel** is given
  - Search all possible **input distributions** for the optimal solution
  - **Reliability** for communication
  - **Increase redundancy**
  - **Channel coding**

Source coding vs. Channel coding

- **Source coding**
  - Core problem: **efficiency**
  - Efficiency: having an average code length that is as small as possible
  - Example: to use shorter code for the English letters which appear frequently, so as to reduce the average code length

- **Channel Coding**
  - Core problem: **reliability**
  - Reliability: to cope with the errors in the transmission
  - Example: to send the same sequence multiple times, so as to recover from the errors in channel

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My Homepage

Thank you!

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