Chapter - 3 = ELECTROSTATICS =

Exercise 3.1 $P(2, \pi/2, -3) \Rightarrow P(0, 2, -3)$, $Q(5, \pi, 0) \Rightarrow Q(-5, 0, 0)$

$\vec{R} = \vec{PQ} = -5\vec{a}_x - 2\vec{a}_y + 3\vec{a}_z \qquad R = \sqrt{38} \qquad \frac{1}{4\pi\epsilon_0} = 9\times10^9$

$$\vec{F}_{-10nC} = -\frac{5\times10^{-9}\times 10\times10^{-9}}{38^{3/2}} \cdot 9\times10^9 [-5\vec{a}_x - 2\vec{a}_y + 3\vec{a}_z] = +9.61\,\vec{a}_x + 3.84\,\vec{a}_y - 5.76\,\vec{a}_z \text{ nN}$$

$$|\vec{F}_{-10nC}| = 11.84 \text{ nN}$$

Exercise 3.2 $P(2, \pi/2, \pi/4) \Rightarrow P(\sqrt{2}, \sqrt{2}, 0)$, $Q(1, \pi, \pi/2) \Rightarrow Q(0, 0, -1)$

$S(5, \frac{\pi}{3}, \frac{2\pi}{8}) \Rightarrow S(-2.17, 3.75, 2.5)$

$\vec{R}_1 = \vec{QP} = 1.414\,\vec{a}_x + 1.414\,\vec{a}_y + \vec{a}_z \qquad R_1 = \sqrt{5}$

$\vec{R}_2 = \vec{SP} = 3.584\,\vec{a}_x - 2.336\,\vec{a}_y - 2.5\,\vec{a}_z \qquad R_2 = 4.955$

$$\vec{F}_{2,-0.5} = \frac{2\times10^{-9}\times -5\times10^{-9}\times 9\times10^9}{5^{3/2}}[\vec{R}_1] = -11.38\,\vec{a}_x - 11.38\,\vec{a}_y - 8.05\,\vec{a}_z \text{ nN}$$

$$\vec{F}_{2,0.2} = \frac{2\times10^{-9}\times 0.2\times10^{-9}\times 9\times10^9}{4.955^3}\vec{R}_2 = 0.106\,\vec{a}_x - 0.07\,\vec{a}_y - 0.074\,\vec{a}_z \text{ nN}$$

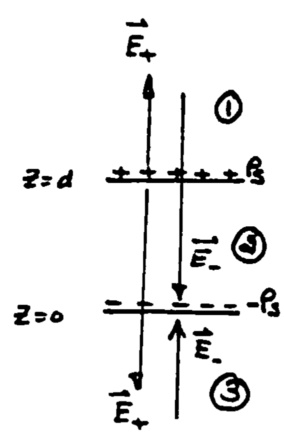
$$\vec{F}_2 = \vec{F}_{2,-0.5} + \vec{F}_{2,0.2} = -[11.274\,\vec{a}_x + 11.45\,\vec{a}_y + 8.124\,\vec{a}_z]$$

$F_2 = 18$ nN Force of attraction

Exercise 3.3 $dq = 2\pi\rho'd\rho'\rho_s \Rightarrow d\vec{E} = \dfrac{2\pi\rho'd\rho' z\rho_s}{4\pi\epsilon_0[z^2+\rho'^2]^{3/2}}\vec{a}_z$

Hence $$E_z = \frac{z\rho_s}{2\epsilon_0}\int_a^b \frac{\rho'd\rho'}{(\rho'^2+z^2)^{3/2}} = \frac{\rho_s z}{2\epsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]$$

Exercise 3.4 $|\vec{E}_+| = |\vec{E}_-| = \dfrac{\rho_s}{2\epsilon_0}$



Region - 1: $\vec{E} = 0$

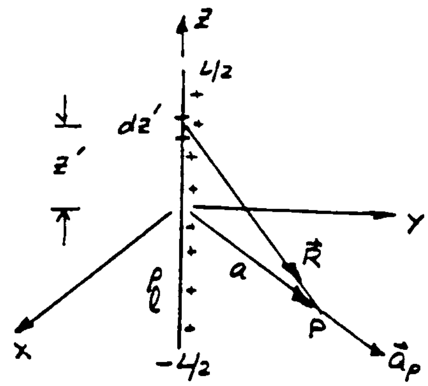
Region - 3: $\vec{E} = 0$

Region - 2:

$$\vec{E}_2 = -\left[\frac{\rho_s}{2\epsilon_0} + \frac{\rho_s}{2\epsilon_0}\right]\vec{a}_z$$

$$= -\frac{\rho_s}{\epsilon_0}\vec{a}_z \quad \text{V/m}$$

Exercise 3.5 $L = 10\,m$ $a = 5\,m$ $\rho_\ell = 10\,\mu c/m$

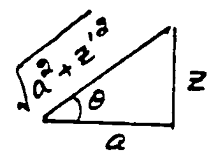


$$\vec{E} = \frac{\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{(a\,\vec{a}_\rho - z'\vec{a}_z)\,dz'}{(a^2+z'^2)^{3/2}} \Rightarrow$$

$$E_\rho = \frac{a\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{dz'}{(a^2+z'^2)^{3/2}}$$

Let $z' = a\tan\theta$, $dz' = a\sec^2\theta\, d\theta$, then

$$\int \frac{dz'}{(a^2+z'^2)^{3/2}} = \frac{1}{a}\int \cos\theta\, d\theta = \frac{1}{a}\sin\theta = \frac{1}{a}\frac{z'}{\sqrt{a^2+z'^2}}$$

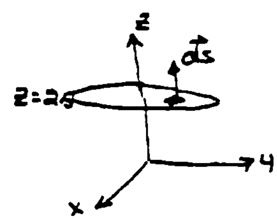


$$E_\rho = \frac{\rho_\ell}{4\pi\epsilon_0 a}\left[\frac{L}{\sqrt{a^2+(L/2)^2}}\right]$$

$$E_z = -\frac{\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{z'dz'}{(a^2+z'^2)^{3/2}} = \frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{a^2+z'^2}}\right]_{-\frac{L}{2}}^{\frac{L}{2}} = 0$$

Hence, $\vec{E} = E_\rho\,\vec{a}_\rho$. Substitute values and obtain $\vec{E} = 25.46\,\vec{a}_\rho$ kV/m

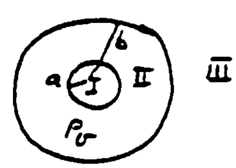
Exercise 3.6 $\vec{D} = 10\sin\phi\,\vec{a}_\rho + 12z\cos(\phi/4)\,\vec{a}_z$ C/m^2



$\vec{ds} = \rho\, d\rho\, d\phi\;\vec{a}_z$ at $z = 2.5$

$$\Psi = \int_s \vec{D}\cdot\vec{ds} = 12\times 2.5\int_0^{0.5}\rho\, d\rho\int_0^{2\pi}\cos(\phi/4)\,d\phi = 15C$$

Exercise 3.7 $\vec{E}_I = 0$



$\oint \vec{E}\cdot\vec{ds} = 4\pi r^2 E_r$ $a \le r \le b$

$$Q = \int_a^r \frac{k}{r} r^2 dr\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi = 2\pi k\,(r^2 - a^2)$$

Region - II $\vec{E}_{II} = \frac{k}{2\epsilon_0}\left(\frac{r^2-a^2}{r^2}\right)\vec{a}_r$

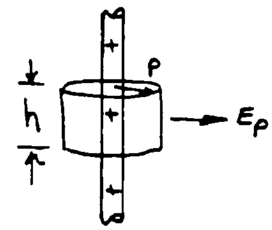
Region - III $Q = 2\pi k\,(b^2-a^2) \Rightarrow \vec{E}_{III} = \frac{k}{2\epsilon_0}\left[\frac{b^2-a^2}{r^2}\right]\vec{a}_r$

Exercise 3.8 $\quad a \le \rho \le \infty$

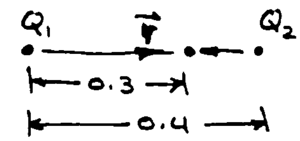
$$Q = \int_s \rho_s \, ds = 2\pi a h \rho_s$$

$$\oint_s \vec{D}\cdot\vec{ds} = 2\pi\rho h D_\rho \Rightarrow D_\rho = \frac{a\rho_s}{\rho} \Rightarrow E_\rho = \frac{a\rho_s}{\rho\epsilon_0}$$

$$\psi = \int_s \vec{D}\cdot\vec{ds} = a\rho_s \int_0^{2\pi} \frac{1}{\rho}\rho\, d\phi \int_0^{\ell} dz = 2\pi a \ell \rho_s$$



Exercise 3.9 $\quad Q_1 = 120\,nC,\ Q_2 = 800\,nC$



$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\vec{a}_r$$

$$W = -Q_2 \int_c \vec{E}\cdot\vec{dl} = -\frac{Q_1 Q_2}{4\pi\epsilon_0}\int_{0.4}^{0.3} \frac{1}{r^2}\,dr = \frac{Q_1 Q_2}{4\pi\epsilon_0}\left[\frac{1}{0.3} - \frac{1}{0.4}\right] = 720\,\mu J$$

Since $W > 0$, the external force is doing the work.

Exercise 3.10 $\quad \vec{E} = \frac{Q}{4\pi\epsilon_0}\cdot\frac{\vec{a}_r}{r^2}$ but $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{a}_r}{r^2}$

a) Thus, $\vec{E} = -\frac{Q}{4\pi\epsilon_0}\nabla(1/r) = -\nabla\left(\frac{Q}{4\pi\epsilon_0 r}\right) = -\nabla V$

b)

$$\nabla\times\vec{E} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ E_r & 0 & 0 \end{vmatrix} = 0$$

Exercise 3.11 $\quad E_\rho = \frac{\rho_\ell}{2\pi\epsilon_0\rho}$

$$V = -\int_{\rho_1}^{\rho_2} \vec{E}\cdot\vec{dl} = -\frac{\rho_\ell}{2\pi\epsilon_0}\int_{\rho_1}^{\rho_2}\frac{1}{\rho}\,d\rho = \frac{\rho_\ell}{2\pi\epsilon_0}\ln(\rho_1/\rho_2)$$

If we establish a reference point at ρ_2 = constant, then at ρ_1 = constant, V = constant. Thus, equipotential surfaces are concentric cylinders.

Exercise 3.12 $\quad V = \dfrac{p\cos\theta}{4\pi\epsilon_0 r^2}$ where $p = qd$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\right] = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right]$$

Exercise 3.13 $\quad$ Let $d\vec{l} = c\vec{E}$ where c is a constant

Since $d\vec{l} = dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi$

$= cE_r\,\vec{a}_r + cE_\theta\,\vec{a}_\theta$,

$$dr = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}c, \quad r\,d\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}c \Rightarrow \frac{dr}{r} = \frac{2\cos\theta\,d\theta}{\sin\theta} \text{ or}$$

$2\ln(\sin\theta) = \ln(r) + \ln(k)$ where k is a constant of integration

or $\sin^2\theta = rk \Rightarrow r = d\sin^2\theta$ where $d = \frac{1}{k}$

Thus, $r = d\sin^2\theta$ yields the lines of force of a dipole. These are the curves shown in Figure 3.26.

Exercise 3.14 $\quad E_r = \dfrac{p}{4\pi\epsilon_0 r^3}2\cos\theta, \quad E_\theta = \dfrac{p}{4\pi\epsilon_0 r^3}\sin\theta$

$$E = \sqrt{E_r^2 + E_\theta^2} = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{1 + 3\cos^2\theta}$$

Exercise 3.15

$$\nabla\times\vec{E} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} & \frac{p\sin\theta}{4\pi\epsilon_0 r^3} & 0 \end{vmatrix}$$

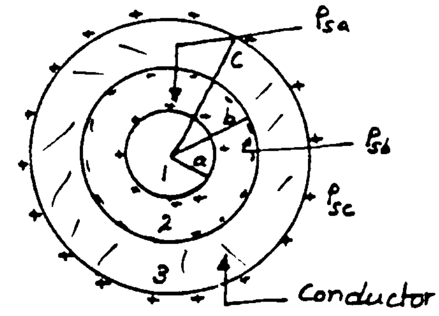
$$= \frac{1}{r}\vec{a}_\phi\left[-\frac{2p\sin\theta}{4\pi\epsilon_0 r^3} + \frac{2p\sin\theta}{4\pi\epsilon_0 r^3}\right] = 0$$

Since $\nabla\times\vec{E} = 0$, $\vec{E}$ represents a conservative field.

Exercise 3.16

Region-1 : $\vec{E}_1 = 0$

Region-2: $\vec{E}_2 = \dfrac{Q}{2\pi\epsilon_0\rho}\vec{a}_\rho \qquad a \le \rho \le b$



Charge per unit length: $Q = 2\pi a \rho_{sa}$

at $\rho = b$, $\quad \rho_{sb} = -\dfrac{Q}{2\pi b} = -\dfrac{a}{b}\rho_{sa}$

at $\rho = c$, $\quad \rho_{sc} = \dfrac{a}{c}\rho_{sa}$

Region-3: $\vec{E}_3 = 0$ conductor, Region-4: $\vec{E}_4 = \dfrac{Q}{2\pi\epsilon_0\rho}\vec{a}_\rho$, $\rho \ge c$

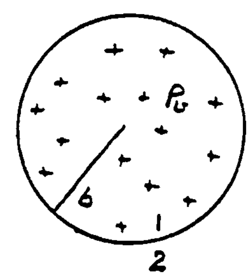
When the outer conductor is grounded, $\rho_{sc} \to 0$ and $\vec{E}_4 \to 0$.

Exercise 3.17

$$V_{ab} = -\int_b^a \vec{E}_2 \cdot \vec{dl} = -\frac{Q}{2\pi\epsilon_0}\int_b^a \frac{1}{\rho}d\rho = \frac{a\rho_{sa}}{\epsilon_0}\ln(b/a)$$

Potential at a is higher than at b.

Exercise 3.18 $\quad \rho_v = b^2 - r^2 \ C/m^3 \qquad \oint_S \vec{E}\cdot\vec{ds} = 4\pi r^2 E_r$



Region-1 : $r \le b$

$$Q_{enc} = \int_0^r (b^2 - r^2)r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{4\pi r^3}{15}(5b^2 - 3r^2)$$

Thus, $\quad \vec{E}_1 = \dfrac{(5b^2 - 3r^2)r}{15\epsilon_0}\vec{a}_r \qquad r \le b$

Region-2: $r \ge b$: $\quad Q_{enc} = \dfrac{8\pi b^5}{15} \Rightarrow \vec{E}_2 = \dfrac{2b^5}{15\epsilon_0 r^2}\vec{a}_r$

Potential at any point at $r \ge b$:

$$V_r = -\int_\infty^r \frac{2b^5}{15\epsilon_0 r^2}dr = \frac{2b^5}{15\epsilon_0}\cdot\frac{1}{r}$$

at $r = \infty$ $\quad V_\infty \to 0$, Reference point

at $r = b$, $\quad V_b = \dfrac{2b^4}{15\epsilon_0}$

Potential inside the sphere:

$$V_r' = -\int \vec{E}_1 \cdot \vec{dl} = -\frac{5b^2}{15\epsilon_0}\left(\frac{r^2}{2}\right) + \frac{3r^4}{60\epsilon_0} + k$$

where k is an integration constant

at $r = b$, $V_r' = V_b \Rightarrow k = \dfrac{b^4}{4\epsilon_0}$

Hence

$$V_r' = \frac{b^4}{4\epsilon_0} - \frac{5b^2r^2}{30\epsilon_0} + \frac{3r^4}{60\epsilon_0}$$

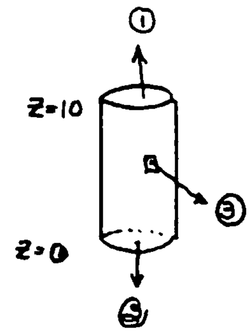
Exercise 3.19 $\vec{R} = (x-x')\,\vec{a}_x + (y-y')\,\vec{a}_y + (z-z')\,\vec{a}_z$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \qquad \frac{\partial}{\partial x'}\left(\frac{1}{R}\right) = \frac{x-x'}{R^3}$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\partial}{\partial x'}\left(\frac{1}{R}\right)\vec{a}_x + \frac{\partial}{\partial y'}\left(\frac{1}{R}\right)\vec{a}_y + \frac{\partial}{\partial z'}\left(\frac{1}{R}\right)\vec{a}_z$$

$$= \frac{(x-x')}{R^3}\vec{a}_x + \frac{y-y'}{R^3}\vec{a}_y + \frac{z-z'}{R^3}\vec{a}_z = \frac{\vec{R}}{R^3} = \frac{\vec{a}_R}{R^2}$$

Exercise 3.20 $A = \pi a^2 = \pi(10\times10^{-3})^2 = 314.16\times10^{-6}\ m^2$



$\vec{P} = 2z^2+10\,\vec{a}_z \qquad \rho_{vb} = -\nabla\cdot\vec{P} = -4z$

Surface-1: $\rho_{sb1} = \vec{P}\cdot\vec{a}_z\Big|_{z=10} = 2z^2+10\Big|_{z=10} = 210\ C/m^2$

Surface-2: $\rho_{sb2} = -\vec{P}\cdot\vec{a}_z\Big|_{z=0} = -10\ C/m^2$

Surface-3: $\rho_{sb3} = \vec{P}\cdot\vec{a}_r = 0$

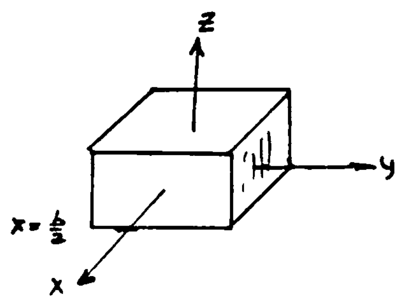
bound charge on surface-1: $210\pi a^2$

bound charge on surface 2: $-10\pi a^2$

bound volume charge: $\int_v \rho_{vb}\,dv = \pi a^2\int_0^{10} -4z\,dz = -200\pi a^2$

Total bound charge: $210\pi a^2 - 10\pi a^2 - 200\pi a^2 = 0$ (As expected)

Exercise 3.21 $\vec{P} = x\,\vec{a}_x + y\,\vec{a}_y + z\,\vec{a}_z$



$\nabla\cdot\vec{P} = 3 \;\Rightarrow\; \rho_{sv} = -\nabla\cdot\vec{P} = -3\ C/m^3$

Surface at	bound ρ_{sb}	Total Charge (bound)
$x = b/2$	$b/2$	$b^3/2$
$x = -b/2$	$b/2$	$b^3/2$
$y = b/2$	$b/2$	$b^3/2$
$y = -b/2$	$b/2$	$b^3/2$
$z = b/2$	$b/2$	$b^3/2$
$z = -b/2$	$b/2$	$b^3/2$

Total bound surface charge = $3b^3$

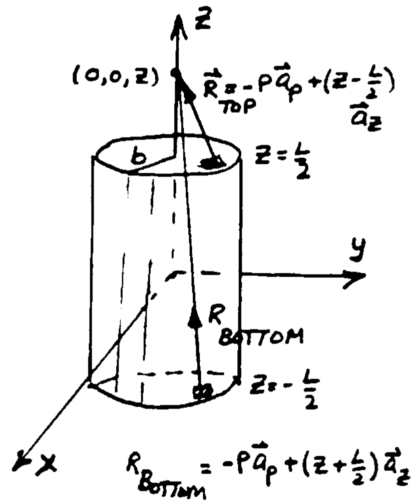
Total bound volume charge: $-3b^3$

Hence, net bound charge = 0

Exercise 3.22 $\quad \vec{P} = P\,\vec{a}_z$

$$\nabla\cdot\vec{P} = 0 \;\Rightarrow\; \rho_{vb} = 0$$

$$\rho_{sb}\Big|_{z=\frac{L}{2}} = \vec{P}\cdot\vec{a}_z = P\,, \quad \rho_{sb}\Big|_{z=-\frac{L}{2}} = -P$$



$$\vec{E}_{TOP} = \frac{P}{4\pi\epsilon_0}\int_0^{b}\int_0^{2\pi}\frac{\rho\, d\rho\, d\phi\,(z-\frac{L}{2})\,\vec{a}_z}{\left[\rho^2 + (z-\frac{L}{2})^2\right]^{3/2}}$$

$$= -\frac{P}{2\epsilon_0}\left[\frac{z-\frac{L}{2}}{\sqrt{b^2+(z-\frac{L}{2})^2}} - 1\right]\vec{a}_z$$

$$\vec{E}_{BOTTOM} = \frac{P}{2\epsilon_0}\left[\frac{z+\frac{L}{2}}{\sqrt{b^2+(z+\frac{L}{2})^2}} - 1\right]\vec{a}_z$$

$$\vec{E} = \vec{E}_{TOP} + \vec{E}_{BOTTOM} = \frac{P}{2\epsilon_0}\left[\frac{z+\frac{L}{2}}{\sqrt{b^2+(z+\frac{L}{2})^2}} + \frac{z-\frac{L}{2}}{\sqrt{b^2+(z-\frac{L}{2})^2}}\right]\vec{a}_z$$

Exercise 3.23 $\quad \oint_s \vec{D}\cdot d\vec{s} = Q_{enc} \;\Rightarrow\; 2\pi\rho L\, D_\rho = \rho_\ell L \quad \text{or} \quad D_\rho = \dfrac{\rho_\ell}{2\pi\rho}$

and $E_\rho = \dfrac{D_\rho}{\epsilon} = \dfrac{\rho_\ell}{2\pi\epsilon\rho} \qquad \epsilon = \epsilon_0\epsilon_r$

Since $D_\rho = \epsilon_0 E_\rho + P_\rho \;\Rightarrow\; P_\rho = \dfrac{\rho_\ell}{2\pi\rho}\left(\dfrac{\epsilon_r - 1}{\epsilon_r}\right)$ or $\vec{P} = \dfrac{\rho_\ell}{2\pi\rho}\left(\dfrac{\epsilon_r-1}{\epsilon_r}\right)\vec{a}_\rho$

$$\nabla\cdot\vec{P} = 0 \;\Rightarrow\; \rho_{vb} = 0$$

For a to be the radius of the line, $\rho_{sb} = -\vec{P}\cdot\vec{a}_\rho\Big|_{r=a} = -\left(\dfrac{\epsilon_r-1}{\epsilon_r}\right)\dfrac{\rho_\ell}{2\pi a}$

$Q_{sb}\Big|$ unit length (bound) $= 2\pi a\,\rho_{sb} = -\left(\dfrac{\epsilon_r-1}{\epsilon_r}\right)\rho_\ell$

Hence, the effective line charge density

$$\rho_e = \rho_\ell - \frac{\epsilon_r - 1}{\epsilon_r}\rho_\ell = \frac{\rho_\ell}{\epsilon_r}$$

The line charge has decreased by a factor of ϵ_r. Note that ϵ_r has no effect on the $\vec{D}$ field.

Exercise 3.24 From Gauss' Law: $D_r = \frac{\rho_v}{3} r \quad r \le a$

$$= \frac{\rho_v}{3}\frac{a^3}{r^2} \quad r \ge a$$

for $r > a$ $$V_o = -\int_{-\infty}^{r} \vec{E}\cdot\vec{dl} = -\frac{\rho_v}{3\epsilon_0} a^3 \int_{-\infty}^{r} \frac{1}{r^2}dr = \frac{\rho_v}{3\epsilon_0}\frac{a^3}{r} \Rightarrow V_a = \frac{\rho_v}{3\epsilon_0}a^2 \text{ at } r = a$$

for $r \le a$ $$V_r - V_a = -\int_a^r \frac{\rho_v}{3\epsilon_0} r\,dr = -\frac{\rho_v}{3\epsilon_0}\left[\frac{r^2}{2}\right]_a^r = -\frac{\rho_v}{6\epsilon_0}\left[r^2 - a^2\right]$$

Hence, inside the sphere, $$V_r = \frac{\rho_v}{3\epsilon_0}a^2 - \frac{\rho_v}{6\epsilon_0}(r^2 - a^2) = \frac{\rho_v a^2}{2\epsilon_0} - \frac{\rho_v}{6\epsilon_0}r^2$$

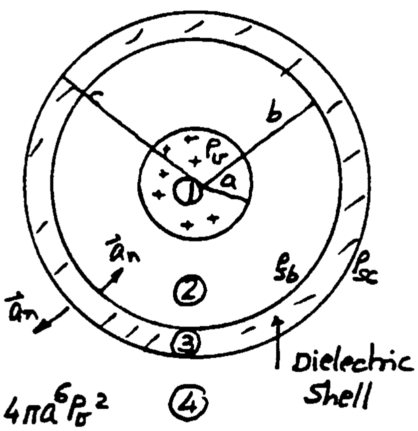
$$W = \frac{1}{2}\int_v \rho_v V_r\, dv = \frac{1}{2}\rho_v^2 \int_0^a \left(\frac{a^2}{2\epsilon_0} - \frac{r^2}{6\epsilon_0}\right) r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{4\pi a^5}{15\epsilon_0}\rho_v^2$$

Exercise 3.25 $$W = \frac{1}{2}\int_v \vec{D}\cdot\vec{E}\, dv = \frac{\rho_v^2}{2}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^a \epsilon_0\left(\frac{r}{3\epsilon_0}\right)^2 r^2 dr$$

$$+ \frac{\rho_v^2}{2}\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \int_a^{\infty} \epsilon_0 \left(\frac{a^3}{3\epsilon_0 r^2}\right)^2 r^2 dr = \frac{4\pi a^5 \rho_v^2}{15\epsilon_0}$$

Exercise 3.26 $$D_r = \frac{\rho_v}{3} r \quad r \le a$$

$$= \frac{\rho_v}{3}\frac{a^3}{r^2} \quad r \ge a$$



Region-1: $r \le a$
$$W_1 = \frac{1}{2\epsilon_0}\int_0^a \left(\frac{\rho_v}{3}r\right)^2 r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{4\pi}{90\epsilon_0}a^5\rho_v^2$$

Region-2: $a \le r \le b$: $$W_2 = \frac{4\pi}{2\epsilon_0}\int_a^b \left(\frac{\rho_v a^3}{3r^2}\right)^2 r^2 dr = \frac{4\pi a^5 \rho_v^2}{18\epsilon_0} - \frac{4\pi a^6 \rho_v^2}{18 b\epsilon_0}$$

Region-3 $b \le r \le c$: $$W_3 = \frac{4\pi}{2\epsilon}\int_b^c \left(\frac{\rho_v a^3}{3r^2}\right)^2 r^2 dr = \frac{4\pi a^6 \rho_v^2}{18 b\epsilon} - \frac{4\pi a^6 \rho_v^2}{18\epsilon c}$$

Region-4, $c \le r \le \infty$ $$W_4 = \frac{4\pi}{2\epsilon_0}\int_c^{\infty}\left(\frac{\rho_v a^3}{3r^2}\right)^2 r^2 dr = \frac{4\pi a^6 \rho_v^2}{18\epsilon_0 c}$$

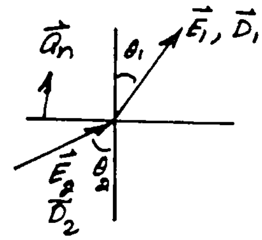
$$W = W_1 + W_2 + W_3 + W_4 = \frac{4\pi a^5}{18\epsilon_0}\rho_v^2\left[\frac{6}{5} + \frac{\epsilon_r - 1}{\epsilon_r}\left(\frac{a}{c} - \frac{a}{b}\right)\right]$$

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = \frac{\rho_v}{3}\frac{a^3}{r^2}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\vec{a}_r \qquad \nabla\cdot\vec{P} = 0 \Rightarrow \rho_{vb} = 0$$

At $r = b$ surface: $$\rho_{sb}\big|_{r=b} = -\vec{P}\cdot\vec{a}_r = -\frac{\rho_v}{3}\frac{a^3}{b^2}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)$$

At $r = c$ surface: $$\rho_{sc}\big|_{r=c} = \frac{\rho_v}{3}\frac{a^3}{c^2}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)$$

Exercise 3.27 Dielectric Media $\Rightarrow \rho_s = 0$

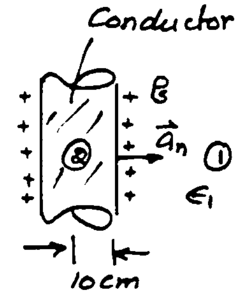


$E_{t1} = E_{t2} \Rightarrow E_1 \sin\theta_1 = E_2 \sin\theta_2$ ①

$D_{n1} = D_{n2} \Rightarrow \epsilon_1 E_1 \cos\theta_1 = \epsilon_2 E_2 \cos\theta_2$ ②

From ① and ②: $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\epsilon_1}{\epsilon_2} = \dfrac{\epsilon_{r1}}{\epsilon_{r2}}$.

Exercise 3.28 $\rho_s = 200\ \mu c/m^2 \quad \vec{a}_n = \vec{a}_\rho$



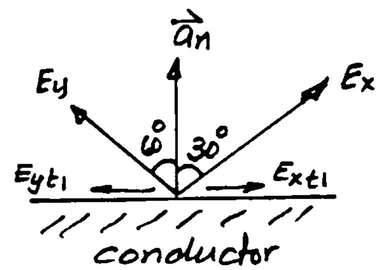
$\vec{D}_2 = 0$ and $\vec{E}_2 = 0$ Conducting region. Thus.

$D_{n1} = D_{\rho 1} = \rho_s = 200\ \mu c/m^2$

$E_{\rho 1} = \dfrac{D_{\rho 1}}{\epsilon_1} = \dfrac{200 \times 10^{-6}}{5 \times 10^{-9}} 36\pi = 4.52\ MV/m$

$\rho_{\rho 1} = D_{\rho 1} - \epsilon_0 E_{\rho 1} = 160\ \mu c/m^2 \Rightarrow \rho_{sb}\Big|_{\rho = 10cm} = -160\ \mu c/m^2$

Exercise 3.29 $E_x = 10\ V/m$



$E_{xt1} = 10 \sin 30^\circ = 5\ V/m$

$E_{yt1} = E_y \sin 60^\circ = 0.866 E_y\ V/m$.

At the boundary, $E_{xt1} = E_{yt1} \Rightarrow E_y = 5.77\ V/m$

Exercise 3.30 $C = \dfrac{\epsilon A}{d} = \dfrac{6 \times 10^{-9}}{36\pi} \cdot \dfrac{40 \times 10^{-4}}{2 \times 10^{-3}} = 106.1\ PF$

$\vec{E} = -\dfrac{V}{d} \vec{a}_z = -\dfrac{1500}{2 \times 10^{-3}} \vec{a}_z = -750\ \vec{a}_z\ kV/m$, $\vec{D} = \epsilon \vec{E} = -\dfrac{6 \times 10^{-6}}{36\pi} \cdot 750 \times 10^{3}\ \vec{a}_z$.

$= -39.79\ \vec{a}_z\ \mu c/m^2$

$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \vec{D}\left(\dfrac{\epsilon_r - 1}{\epsilon_r}\right) = -33.16\ \vec{a}_z\ \mu c/m^2$

$\rho_{s+} = 39.79\ \mu c/m^2$ (Top-plate), $\rho_{s-} = -39.79\ \mu c/m^2$ (bottom plate)

Bound charges: $\rho_{sb\ Top} = -33.16\ \mu c/m^2$, $\rho_{sb\ BOTTOM} = 33.16\ \mu c/m^2$

Energy density: $w = \frac{1}{2} \epsilon E^2 = \frac{1}{2} \times \dfrac{6 \times 10^{-9}}{36\pi} \cdot (750 \times 10^{3})^2 = 14.92\ J/m^3$

Total Energy : $W = \int w\, dv = 14.92 \times 40 \times 10^{-4} \times 2 \times 10^{-3} = 119.37\ \mu J$

Exercise 3.31 $D_\rho = \frac{a}{\rho}\rho_s \Rightarrow E_\rho = \frac{a\rho_s}{\rho\epsilon}$. $V_{ab} = -\int_b^a \frac{a\rho_s}{\epsilon\rho} d\rho = \frac{a\rho_s}{\epsilon}\ln(b/a)$

$$C = \frac{Q}{V_{ab}} = \frac{(2\pi a L \rho_s)\epsilon}{a\rho_s \ln(b/a)} = \frac{2\pi\epsilon L}{\ln(b/a)}$$

Exercise 3.32 $\frac{1}{C} = \int_a^b \frac{d\rho}{2\pi\rho L\epsilon} = \frac{1}{2\pi\epsilon L}\ln(b/a) \Rightarrow C = \frac{2\pi\epsilon L}{\ln(b/a)}$

Exercise 3.33 $V_{ab} = 1000\,V \quad a = 0.1\,m \quad b = 0.12\,m \quad \epsilon_r = 2.5$

$$V_{ab} = \frac{Q}{4\pi\epsilon}\left[\frac{1}{a} - \frac{1}{b}\right] \Rightarrow Q = 1000 \times 4\pi \times \frac{2.5\times10^{-9}}{36\pi}\,\frac{(0.1)(0.12)}{0.12-0.1} = 166.67\ nC$$

$$E_r = \frac{Q}{4\pi\epsilon r^2} = \frac{166.67\times10^{-9}}{r^2}\cdot\frac{9\times10^{9}}{2.5} = \frac{600}{r^2}\ V/m$$

$$D_r = \epsilon E_r = \frac{600}{r^2}\cdot\frac{2.5\times10^{-9}}{36\pi} = \frac{13.26}{r^2}\ nC/m^2 \Rightarrow P_r = D_r - \epsilon_0 E_r = \frac{7.96}{r^2}\ nC/m^2$$

Free charge densities: $\rho_{sa} = \frac{Q}{4\pi a^2} = 1.326\,\mu C/m^2, \quad \rho_{sb} = -\frac{Q}{4\pi b^2} = -921.1\,nC/m^2$

bound charge densities: $\rho_{vb} = -\nabla\cdot\vec{P} = 0 \qquad \rho_{sb} = \vec{P}\cdot\vec{a}_n \Rightarrow$

at $r = a$, $\rho_{sb}\big|_{r=a} = -\frac{7.96\times10^{-9}}{100\times10^{-4}} = -796\ nC/m^2, \quad \rho_{sb}\big|_{r=b} = +\frac{7.96\times10^{-9}}{144\times10^{-4}} = 552.78\ nC/m^2$

$$C = \frac{Q}{V_{ab}} = 166.67\ PF$$

Exercise 3.34 $A = 40\times10^{-4}\ m^2 \qquad d = 2\times10^{-3}\ m$

$\vec{E} = -750\,\vec{a}_z\ kV/m \quad \vec{D} = \epsilon_0\vec{E} = -6.63\,\vec{a}_z\ \mu C/m^2 \quad C = \frac{\epsilon_0 A}{d} = 17.68\ PF, \quad \vec{P} = 0$

$\rho_{s\,TOP} = 6.63\,\mu C/m^2, \quad \rho_{s\,BOTTOM} = -6.63\,\mu C/m^2, \quad w = \frac{1}{2}\vec{D}\cdot\vec{E} = 2.487\ J/m^3$

$$W = \int w\,dv = 2.487\times40\times10^{-4}\times2\times10^{-3} = 19.89\ \mu J$$

Exercise 3.35 $Q = \frac{V_{ab}(4\pi\epsilon_0)ab}{b-a} = \frac{1000\times0.1\times0.12}{9\times10^{9}(0.12-0.1)} = 66.67\ nC$

$E_r = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{600}{r^2}\ V/m, \quad D_r = \epsilon_0 E_r = \frac{5.31}{r^2}\ nC/m^2$

$\rho_s\big|_{r=a} = \frac{66.67\times10^{-9}}{4\pi\times100\times10^{-4}} = 530.54\ nC/m^2 \qquad C = \frac{Q}{V_{ab}} = \frac{66.67\times10^{-9}}{1000}$

$\rho_s\big|_{r=b} = -\frac{66.67\times10^{-9}}{4\pi\times144\times10^{-4}} = -368.43\ nC/m^2 \qquad = 66.67\ PF$

$C_{dielectric}/C_{air} = 2.5$

Exercise 3.36

$\epsilon = \epsilon_1 + \frac{\epsilon_2 - \epsilon_1}{d} z \qquad \epsilon = \epsilon_1$ at $z=0$, $\epsilon = \epsilon_2$ at $z = d$

$$\frac{1}{C} = \int_0^d \frac{dz}{(\epsilon_1 + \frac{\epsilon_2-\epsilon_1}{d} z)A} = \frac{d}{A(\epsilon_2 - \epsilon_1)} \ln(\epsilon_2/\epsilon_1) \qquad A = \text{Area}$$

Hence $C = \frac{A(\epsilon_2 - \epsilon_1)}{d \ln(\epsilon_2/\epsilon_1)}$. ①

When $\epsilon_2 \to \epsilon_1$, $e^{\ln(\epsilon_2/\epsilon_1)} \cong 1 + \ln(\frac{\epsilon_2}{\epsilon_1}) \Rightarrow \ln(\frac{\epsilon_2}{\epsilon_1}) = \frac{\epsilon_2}{\epsilon_1} - 1 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_1}$

Hence from ① $\quad C = \frac{A\epsilon_1}{d}$

Exercise 3.37

$\nabla^2 V = 0 \Rightarrow \frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) = 0 \Rightarrow V = -\frac{C_1}{r} + C_2$

at $r = b$, $V = 0 \Rightarrow C_2 = \frac{C_1}{b}$. Thus, $V = -C_1[\frac{1}{r} - \frac{1}{b}]$

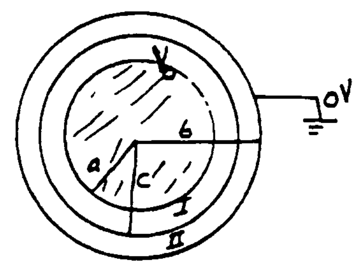
at $r = a$, $V = V_0 \Rightarrow C_1 = \frac{V_0}{\frac{1}{b} - \frac{1}{a}}$. Finally, $V = \frac{V_0}{\frac{1}{a} - \frac{1}{b}}\{\frac{1}{r} - \frac{1}{b}\}$

$\vec{E} = -\nabla V = -\frac{C_1}{r^2}\vec{a}_r = \frac{ab V_0}{(b-a) r^2}\vec{a}_r \qquad D_r = \frac{\epsilon V_0 ab}{(b-a) r^2} \qquad \rho_{sa} = \frac{\epsilon V_0 ab}{(b-a)a^2}, \rho_{sb} = \frac{-\epsilon V_0 ab}{(b-a) b^2}$

$Q_a = \frac{\epsilon V_0 ab}{(b-a)a^2} \cdot 4\pi a^2 = \frac{4\pi\epsilon V_0 ab}{(b-a)}, \quad C = \frac{Q_a}{V_0} = \frac{4\pi\epsilon ab}{b-a}, \quad W = \frac{1}{2}CV_0^2 = \frac{2\pi\epsilon ab}{b-a}V_0^2$

Exercise 3.38

L = Length



$\nabla^2 V = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial V}{\partial r}) = 0$

Region-1: $V_1 = k_1 \ln\rho + k_2$

at $\rho = a$, $V_1 = V_0 \Rightarrow k_2 = V_0 - k_1 \ln a$

$V_1 = V_0 + k_1 \ln(\rho/a)$

$E_{1\rho} = -\frac{\partial V_1}{\partial \rho} = -\frac{k_1}{\rho}, \quad D_{1\rho} = -\frac{\epsilon_1 k_1}{\rho}$

at $\rho = c$, $V_1 = V_2$ and $D_{1\rho} = D_{2\rho}$

Thus, $V_0 + k_1 \ln(\frac{c}{a}) = k_3 \ln(c/b)$

and $\epsilon_1 k_1 = \epsilon_2 k_3$

$k_1 = \frac{-\epsilon_2 V_0}{M}, \quad k_3 = \frac{-\epsilon_1 V_0}{M}$

where $M = \epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)$

$\rho_s|_{\rho=a} = D_{1\rho}|_{\rho=a} = -\frac{\epsilon_1 k_1}{a} = \frac{\epsilon_1\epsilon_2 V_0}{a M}$

$Q_a = 2\pi a L \,\rho_s|_{\rho=a} = \frac{2\pi\epsilon_1\epsilon_2 V_0 L}{M}$

Region-2: $V_2 = k_3 \ln\rho + k_4$

at $\rho = b$, $V_2 = 0 \Rightarrow k_4 = -k_3 \ln b$

$V_2 = k_3 \ln(\rho/b)$

$E_{2\rho} = -\frac{\partial V_2}{\partial \rho} = -\frac{k_3}{\rho}, \quad D_{2\rho} = -\frac{\epsilon_2 k_3}{\rho}$

$C = \frac{Q_a}{V_0} = \frac{2\pi\epsilon_1\epsilon_2 L}{\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)}$

$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ where

$C_1 = \frac{2\pi\epsilon_1 L}{\ln(c/a)}, \quad C_2 = \frac{2\pi\epsilon_2 L}{\ln(b/a)}$

C_1 and C_2 are connected in series.

Exercise 3.39 $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$$\frac{\partial V}{\partial x}: \quad \frac{\partial^2}{\partial x^2}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial V}{\partial x}\right) \Rightarrow \frac{\partial}{\partial x}(\nabla^2 V) = 0$$

Similarly $\frac{\partial^2 V}{\partial x^2}$ is a solution because $\frac{\partial^2}{\partial x^2}(\nabla^2 V) = 0$

Finally $\frac{\partial^2}{\partial x \partial y}(\nabla^2 V) = 0 \Rightarrow \frac{\partial^2 V}{\partial x \partial y}$ is also a solution

Exercise 3.40 For $0 \le z \le d$, $\nabla \cdot \vec{D} = 0 \Rightarrow \nabla \cdot \epsilon \vec{E} = 0 \Rightarrow \epsilon \nabla \cdot \vec{E} + \vec{E} \cdot \nabla \epsilon = 0$

$\epsilon = \epsilon_0(1 + mz) \Rightarrow \nabla \epsilon = m \epsilon_0 \vec{a}_z$. Thus, $\epsilon \frac{\partial E_z}{\partial z} + m \epsilon_0 E_z = 0$

Or $\frac{dE_z}{E_z} = -\frac{m}{1+mz} dz \Rightarrow \ln(E_z) = -\ln(1+mz) + \ln A$, $\ln A$ = Integration constant

Thus, $E_z = \frac{A}{1+mz} \Rightarrow -\frac{dV}{dz} = \frac{A}{1+mz} \Rightarrow V = -\frac{A}{m}\ln(1+mz) + B$, B = constant of Integration

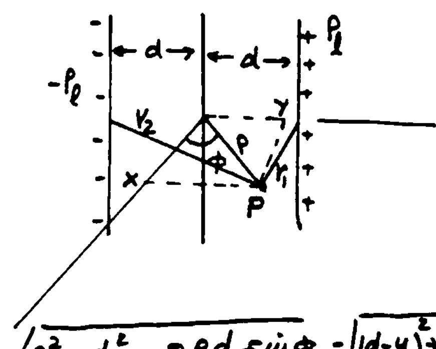
at $z=0$ $V=0 \Rightarrow B=0$ and at $z=d$, $V=V_0 \Rightarrow A = -\frac{V_0 m}{\ln(1+md)}$

Hence, $V = V_0 \frac{\ln(1+mz)}{\ln(1+md)}$ and $E_z = -\frac{V_0 m}{(1+mz)\ln(1+md)}$

$$D_z = -\frac{\epsilon V_0 m}{(1+mz)\ln(1+md)} = -\frac{V_0 m \epsilon_0}{\ln(1+md)} \qquad \rho_s\Big|_{z=d} = \frac{V_0 m \epsilon_0}{\ln(1+md)}$$

$Q\Big|_{z=d} = \frac{V_0 m \epsilon_0 S}{\ln(1+md)}$ and $C = \frac{m \epsilon_0 S}{\ln(1+md)}$ where S = Area of each plate

Exercise 3.41



Let the potential at $\rho = b$ is zero. Then

$V_+ = \frac{\rho_\ell}{2\pi\epsilon}\ln(b/r_1) \qquad V_- = -\frac{\rho_\ell}{2\pi\epsilon}\ln(b/r_2)$

$V_P = V_+ + V_- = \frac{\rho_\ell}{2\pi\epsilon}\ln(r_2/r_1)$

$r_1 = \sqrt{\rho^2 + d^2 - 2\rho d \sin\phi} = \sqrt{(d-y)^2 + x^2}$

$r_2 = \sqrt{\rho^2 + d^2 + 2\rho d \sin\phi} = \sqrt{(d+y)^2 + x^2}$

Equipotential surfaces at

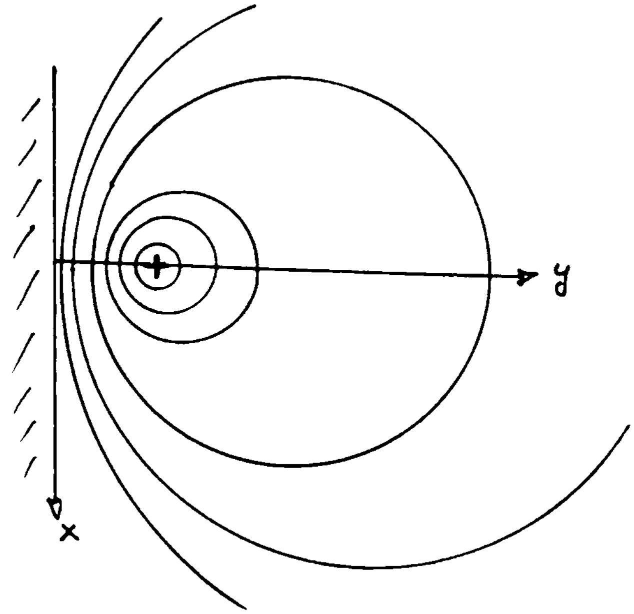
$\frac{r_2}{r_1} = k$ where k is a constant

or $(d+y)^2 + x^2 = k^2[(d-y)^2 + x^2] \Rightarrow x^2 + y^2 - 2yd\frac{k^2+1}{k^2-1} + d^2 = 0$

Add $d^2\left[\frac{k^2+1}{k^2-1}\right]^2$ to both sides to complete squares,

$$x^2 + \left(y - d\,\frac{k^2+1}{k^2-1}\right)^2 = \left(\frac{2kd}{k^2-1}\right)^2$$

This equation describes a series of circles (as shown) in the xy plane.



To determine surface charge density, let us compute normal component ($\vec{a}_y$) of $\vec{D}$-field.

$$E_y = -\frac{\partial V}{\partial y} = -\frac{\rho_\ell}{2\pi\epsilon}\left[\frac{1}{r_2}\frac{\partial r_2}{\partial y} - \frac{1}{r_1}\frac{\partial r_1}{\partial y}\right]$$

$$= -\frac{\rho_\ell}{2\pi\epsilon}\left[\frac{d+y}{r_2^2} + \frac{d-y}{r_1^2}\right] \Rightarrow D_y = -\frac{\rho_\ell}{2\pi}\left[\frac{d+y}{r_2^2} + \frac{d-y}{r_1^2}\right]$$

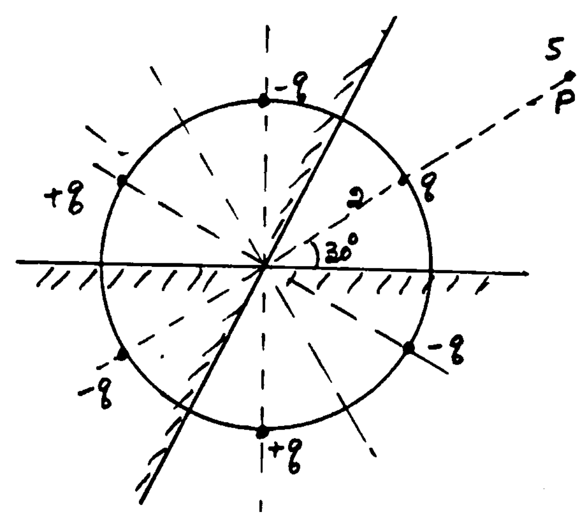
$$\rho_s\Big|_{y=0} = -\frac{\rho_\ell d}{\pi(d^2+x^2)}.$$

Thus, the charge upon the infinite conductor per meter of length along the line is

$$Q = -\frac{\rho_\ell d}{\pi}\int_{-\infty}^{\infty}\frac{dx}{d^2+x^2}\int_0^1 dy = -\frac{\rho_\ell}{\pi}\left[\tan^{-1}\frac{x}{d}\right]_{-\infty}^{\infty} = -\rho_\ell \;\; \text{C/m}.$$

<u>Exercise 3.42</u>

The two intersecting planes can be replaced by image charges as shown.



$$V_P = \frac{q}{4\pi\epsilon}\left[\frac{1}{3} - \frac{1}{7} - \frac{1}{4.36} + \frac{1}{6.25} + \frac{1}{6.25} - \frac{1}{4.36}\right]$$

$$= \frac{0.52q}{4\pi\epsilon}.$$

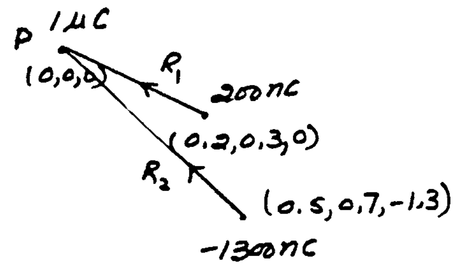
Problem 3.1 $\vec{R} = \vec{QP} = -3\vec{a}_x + 4\vec{a}_y, \quad R = 5$

$$\vec{F}_P = \vec{F}_{2\mu C} = \frac{2\times10^{-6}\times10\times10^{-6}}{5^3}\cdot 10^9\times9[-3\vec{a}_x+4\vec{a}_y] = -4.32\,\vec{a}_x + 5.76\,\vec{a}_y \text{ mN}$$

$$\vec{F}_Q = 4.32\vec{a}_x - 5.76\,\vec{a}_y \text{ mN}, \qquad |\vec{F}_P| = |FQ| = 7.2 \text{ mN}$$

Problem 3.2 $\vec{R}_1 = -0.2\,\vec{a}_x - 0.3\vec{a}_y \Rightarrow R_1 = 0.36$

$\vec{R}_2 = -0.5\vec{a}_x - 0.7\vec{a}_y + 1.3\vec{a}_z, \; R_2 = 1.56$



$$\vec{F}_{P1} = \frac{1\times10^{-6}\times200\times10^{-9}}{0.36^3}\cdot9\times10^9[\vec{R}_1]$$

$$= -7.72\vec{a}_x - 11.57\,\vec{a}_y \text{ mN}$$

$$\vec{F}_{P2} = -\frac{1\times10^{-6}\times1300\times10^{-9}}{1.56^3}\cdot9\times10^9\,\vec{R}_2$$

$$= 1.54\,\vec{a}_x + 2.16\vec{a}_y - 4.01\,\vec{a}_z \text{ mN}$$

$$\vec{F}_P = \vec{F}_{P1} + \vec{F}_{P2} = -6.18\,\vec{a}_x - 9.41\vec{a}_y - 4.01\,\vec{a}_z \text{ mN}, \; |\vec{F}_P| = 11.95 \text{ mN}$$

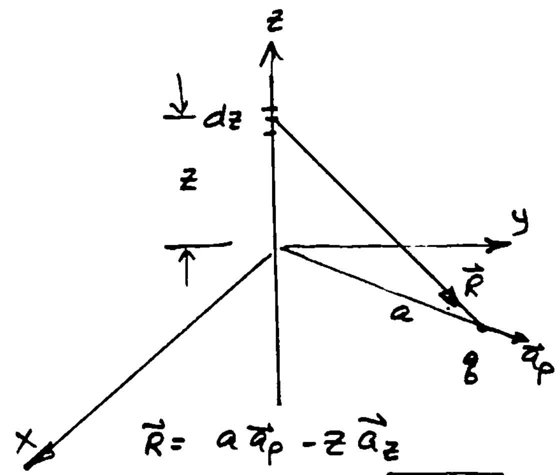
Problem 3.3

$$\vec{F} = \frac{q\rho_\ell}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{(a\,\vec{a}_\rho - z\,\vec{a}_z)\,dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{\rho_\ell\, q}{2\pi\epsilon_0 a}$$

$$= \frac{500\times10^{-9}\times100\times10^{-9}\times36\pi\times10^9}{2\pi\times5}$$

$$= 180\ \mu N$$



$\vec{R} = a\,\vec{a}_\rho - z\,\vec{a}_z$

$a = 5 \qquad R = \sqrt{a^2+z^2}$

Problem 3.4 $\vec{E} = \dfrac{\rho_\ell\,\vec{a}_\rho}{2\pi\epsilon_0 d} \Rightarrow \vec{F} = -\dfrac{\rho_\ell^2}{2\pi\epsilon_0 d}\,\vec{a}_\rho$

$$\vec{F} = -\frac{(100\times10^{-9})^2\times36\pi\times10^9}{2\pi\times1\times10^{-3}}\vec{a}_\rho = -0.18\,\vec{a}_\rho$$

This is a force of attraction.

Problem 3.5

$$|F_e| = \left|\frac{(-1.6\times10^{-19})(1.6\times10^{-19})\ 9\times10^{9}}{(0.05\times10^{-9})^2}\right| = 92.16\times10^{-9}N$$

$$|F_g| = \frac{GmM}{r^2} = \frac{6.67\times10^{-11}\times9.1\times10^{-31}\times1.672\times10^{-27}}{(0.05\times10^{-9})^2} = 40.59\times10^{-48}N$$

$$\frac{|F_e|}{|F_g|} = 2.27\times10^{39}.$$

Electric force perdominates.

Problem 3.6

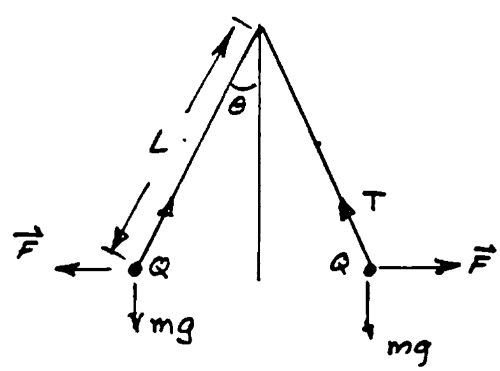
$$|F_e| = \left|\frac{-1.6\times10^{-19}\times1.6\times10^{-19}\times9\times10^{9}}{(0.05\times10^{-9})^2}\right| = 92.16\times10^{-9}N$$

Since $|F_e| = \frac{mv^2}{r}$ $\quad v = r\omega \Rightarrow \omega = \sqrt{\frac{F}{mr}} = \sqrt{\frac{92.16\times10^{-9}}{9.1\times10^{-31}\times0.05\times10^{-9}}} = 45\times10^{15}$ rad/s

$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} = 139.6\times10^{-18}s$

Problem 3.7

$$F = \frac{Q^2}{4\pi\epsilon_0\, 4L^2\sin^2\theta} \quad ①$$



$F = T\sin\theta \qquad mg = T\cos\theta$

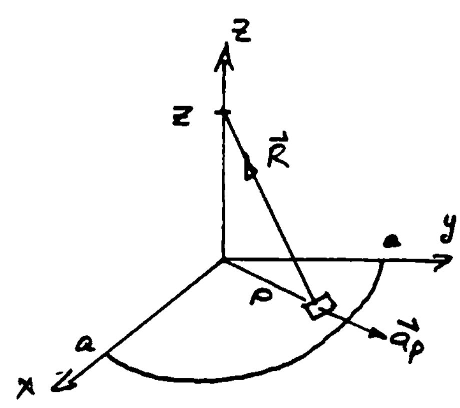
Thus, $F = mg\tan\theta$ ②

From ① and ②

$$\frac{\sin^3\theta}{\cos\theta} = \frac{Q^2}{16\pi\epsilon_0L^2mg}$$

or $$\frac{\tan^3\theta}{1+\tan^2\theta} = \frac{Q^2}{16\pi\epsilon_0L^2mg}$$

Problem 3.8



$$\vec{E} = \frac{k}{4\pi\epsilon_0}\int_0^a\int_0^{\pi/2}\frac{\cos\phi\ \rho\, d\rho\, d\phi\,[z\vec{a}_z - \rho\vec{a}_\rho]}{(\rho^2+z^2)^{3/2}}$$

$\vec{a}_\rho = \vec{a}_x\cos\phi + \vec{a}_y\sin\phi$ and $z = h$

$$E_x = -\frac{k}{4\pi\epsilon_0}\int_0^a\frac{\rho^2 d\rho}{(\rho^2+h^2)^{3/2}}\int_0^{\pi/2}\cos^2\phi\, d\phi$$

$\vec{R} = z\vec{a}_z - \rho\vec{a}_\rho$

$R = \sqrt{\rho^2+z^2}$

$$E_y = -\frac{k}{4\pi\epsilon_0}\int_0^a \frac{\rho^2 d\rho}{(\rho^2+h^2)^{3/2}}\int_0^{\pi/2} \cos\phi \sin\phi \, d\phi$$

$$E_z = \frac{kh}{4\pi\epsilon_0}\int_0^a \frac{\rho \, d\rho}{(\rho^2+h^2)^{3/2}}\int_0^{\pi/2} \cos\phi \, d\phi$$

Evaluating the integrals, we get

$$E_x = \frac{k}{16\epsilon_0}\left[\frac{a}{\sqrt{a^2+h^2}} - \ln\left(\frac{a+\sqrt{a^2+h^2}}{h}\right)\right]$$

$$E_y = \frac{k}{8\pi\epsilon_0}\left[\frac{a}{\sqrt{a^2+h^2}} - \ln\left(\frac{a+\sqrt{a^2+h^2}}{h}\right)\right]$$

$$E_z = \frac{kh}{4\pi\epsilon_0}\left[\frac{1}{h} - \frac{1}{\sqrt{a^2+h^2}}\right]$$

$$\int_0^{\pi/2} \cos\phi \, d\phi = 1$$

$$\int_0^{\pi/2} \sin\phi \cos\phi \, d\phi = \frac{1}{2}$$

$$\int_0^{\pi/2} \cos^2\phi \, d\phi = \frac{\pi}{4}$$

$$\int_0^a \frac{\rho \, d\rho}{(\rho^2+h^2)^{3/2}} = -\frac{1}{\sqrt{\rho^2+h^2}}\Bigg|_0^a = -\frac{1}{\sqrt{a^2+h^2}} + \frac{1}{h}$$

$$\int_0^a \frac{\rho^2 d\rho}{(\rho^2+h^2)^{3/2}} = \left[-\frac{\rho}{\sqrt{\rho^2+h^2}} + \ln\left(\rho+\sqrt{\rho^2+h^2}\right)\right]_0^a = -\frac{a}{\sqrt{a^2+h^2}} + \ln\frac{a+\sqrt{a^2+h^2}}{h}$$

Problem 3.9

$$\vec{E} = \frac{kb}{4\pi\epsilon_0}\int_0^{\pi} \frac{\sin\phi\left[h\vec{a}_z - b\vec{a}_x\cos\phi - b\vec{a}_y\sin\phi\right]d\phi}{(b^2+h^2)^{3/2}}$$

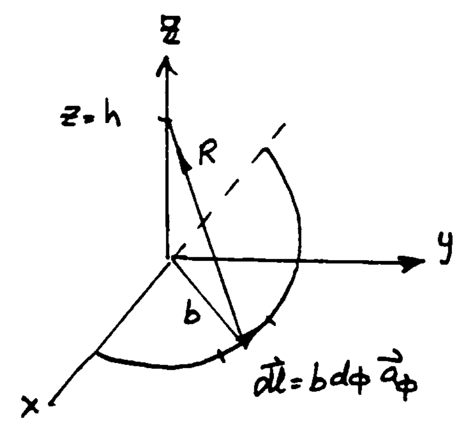
$$E_x = \frac{b^2 k}{4\pi\epsilon_0(b^2+h^2)^{3/2}}\int_0^{\pi} \cancel{-\sin\phi\cos\phi}^{\,0} \, d\phi = 0$$

$$E_y = -\frac{kb^2}{4\pi\epsilon_0(b^2+h^2)^{3/2}}\int_0^{\pi} \sin^2\phi \, d\phi$$

$$E_z = \frac{bkh}{4\pi\epsilon_0(b^2+h^2)^{3/2}}\int_0^{\pi} \sin\phi \, d\phi$$

Evaluate integrals.

$$\vec{E} = \frac{bk}{4\pi\epsilon_0(b^2+h^2)^{3/2}}\left[-\frac{b}{2}\pi\vec{a}_y + 2h\vec{a}_z\right]$$



$$\vec{R} = h\vec{a}_z - b\vec{a}_\rho$$

$$R = \sqrt{h^2+b^2}$$

since $\vec{a}_\rho = \vec{a}_x\cos\phi + \vec{a}_y\sin\phi$

$$\vec{R} = -(\vec{a}_x\cos\phi + \vec{a}_y\sin\phi)b + h\vec{a}_z$$

$$\int_0^{\pi} \sin\phi \, d\phi = 2$$

$$\int_0^{\pi} \sin^2\phi \, d\phi = \pi/2$$

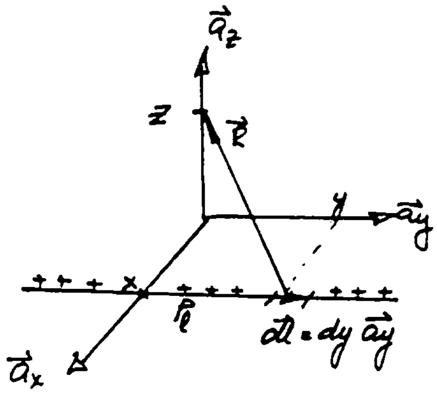
Problem 3.10 The charge distribution ρ_{sa} yields the same $\bar{E}$-field as was obtained in Problem 3.8. Follow the same method to obtain $\bar{E}$-field due to ρ_{sb}. The x- and y-components will be exactly the same but the z-component will be in opposite direction.

Thus,

$$E_x = \frac{A}{8\epsilon_0}\left[\frac{b}{\sqrt{b^2+h^2}} - \ln\left(\frac{b+\sqrt{b^2+h^2}}{h}\right)\right]$$

$$E_y = \frac{A}{4\pi\epsilon_0}\left[\frac{b}{\sqrt{b^2+h^2}} - \ln\left(\frac{b+\sqrt{b^2+h^2}}{h}\right)\right], \quad E_z = 0$$

Problem 3.11 $R = -x\,\vec{a}_x - y\,\vec{a}_y + z\,\vec{a}_z$



$$\vec{E} = \frac{\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{(-x\,\vec{a}_x - y\,\vec{a}_y + z\,\vec{a}_z)\,dy}{(x^2+y^2+z^2)^{3/2}}$$

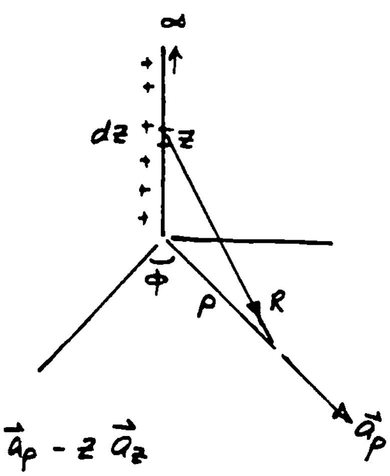
$$E_x = -\frac{\rho_\ell x}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{dy}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{\rho_\ell x}{4\pi\epsilon_0}\left[\frac{1}{x^2+z^2}\,\frac{y}{\sqrt{x^2+y^2+z^2}}\right]_{-L/2}^{L/2} = -\frac{\rho_\ell x L}{4\pi\epsilon_0(x^2+z^2)}\cdot\frac{1}{\sqrt{x^2+z^2+\frac{L^2}{4}}}$$

$E_y = 0$

$$E_z = \frac{\rho_\ell z L}{4\pi\epsilon_0(x^2+z^2)}\cdot\frac{1}{\sqrt{x^2+z^2+L^2/4}}.$$

Problem 3.12



$$\vec{E} = \frac{\rho_\ell}{4\pi\epsilon_0}\int_0^\infty\frac{(\rho\,\vec{a}_\rho - z\,\vec{a}_z)\,dz}{(\rho^2+z^2)^{3/2}}$$

$\vec{R} = \rho\,\vec{a}_\rho - z\,\vec{a}_z$

$$E_\rho = \frac{\rho_\ell\,\rho}{4\pi\epsilon_0}\int_0^\infty\frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{\rho_\ell}{4\pi\epsilon_0\rho}$$

$$E_z = -\frac{\rho_\ell}{4\pi\epsilon_0}\int_0^\infty\frac{z\,dz}{(\rho^2+z^2)^{3/2}} = -\frac{\rho_\ell}{4\pi\epsilon_0\rho}$$

$$\int_0^\infty\frac{z\,dz}{(\rho^2+z^2)^{3/2}} = -\frac{1}{\sqrt{\rho^2+z^2}}\Big|_0^\infty = \frac{1}{\rho}$$

$$\vec{E} = \frac{\rho_\ell}{4\pi\epsilon_0\rho}\left[\vec{a}_\rho - \vec{a}_z\right]$$

$$\rho^2\int_0^\infty\frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{z}{\sqrt{\rho^2+z^2}}\Big|_0^\infty = 1$$

Problem 3.13

$\rho_+ = 1\mu C/m$, $\rho_- = -1\,\mu C/m$

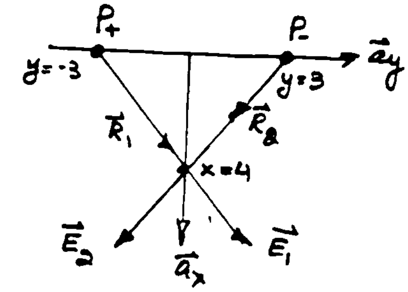
$\vec{R}_1 = 4\vec{a}_x + 3\vec{a}_y$, $R_1 = 5$

$R_2 = 4\vec{a}_x - 3\vec{a}_y$, $R_2 = 5$

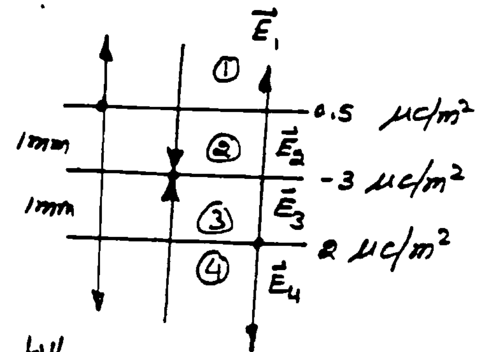
$\vec{E}_1 = \frac{\rho_+}{2\pi\epsilon_0 R_1}\vec{a}_{R_1}$, $\vec{E}_2 = \frac{\rho_-}{2\pi\epsilon_0 R_2}\vec{a}_{R_2}$

$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{1\times 10^{-6}}{50\pi\epsilon_0}\left[4\vec{a}_x + 3\vec{a}_y - 4\vec{a}_x + 3\vec{a}_y\right]$

$= 4320\,\vec{a}_y$ V/m



Problem 3.14



Region-1: $\vec{E}_1 = \frac{1}{2\epsilon_0}\left[0.5 - 3 + 2\right]10^{-6}\vec{a}_z$

$= -28.274\,\vec{a}_z$ kV/m

Region-2: $\vec{E}_2 = \frac{1}{2\epsilon_0}\left[-0.5 - 3 + 2\right]10^{-6}\vec{a}_z = -84.823\,\vec{a}_z$ kV/m

Region-3: $\vec{E}_3 = \frac{1}{2\epsilon_0}\left[-0.5 + 3 + 2\right]10^{-6}\vec{a}_z = 254.47\,\vec{a}_z$ kV/m

Region 4: $\vec{E}_4 = \frac{1}{2\epsilon_0}\left[-0.5 + 3 - 2\right]10^{-6}\vec{a}_z = 28.274\,\vec{a}_z$ kV/m

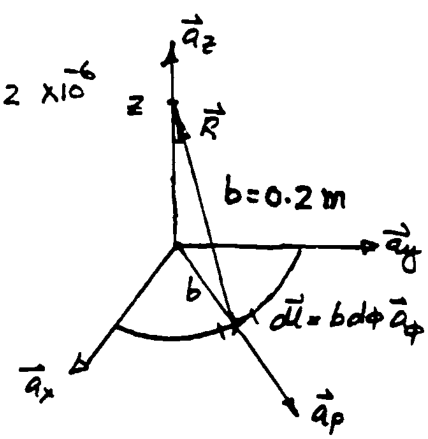
Problem 3.15

$\vec{R} = z\,\vec{a}_z - b\,\vec{a}_\rho$ $\qquad \rho_\ell = \rho_{\ell 0}\sin\phi\cos\phi$

$\rho_{\ell 0} = 1.2\times 10^{-6}$

$= z\,\vec{a}_z - b\cos\phi\,\vec{a}_x - b\sin\phi\,\vec{a}_y$

$$\vec{E} = \frac{\rho_{\ell 0} b}{4\pi\epsilon_0}\left[\int_0^{\pi/2} -b\sin\phi\cos^2\phi\,d\phi\,\vec{a}_x - \int_0^{\pi/2} b\sin^2\phi\cos\phi\,d\phi\,\vec{a}_y + \int_0^{\pi/2} z\sin\phi\cos\phi\,d\phi\,\vec{a}_z\right]\frac{1}{(b^2+z^2)^{3/2}}$$



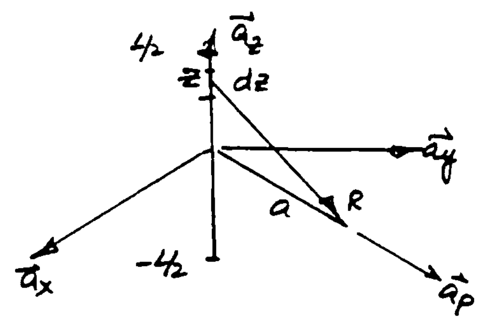
$$= \frac{\rho_{\ell 0} b}{4\pi\epsilon_0 (b^2+z^2)^{3/2}}\left[\frac{b}{3}\cos^3\phi\,\vec{a}_x - \frac{b}{3}\sin^3\phi\,\vec{a}_y + \frac{z}{2}\sin^2\phi\,\vec{a}_z\right]_0^{\pi/2}$$

$$= \frac{\rho_{\ell 0} b}{4\pi\epsilon_0 (b^2+z^2)^{3/2}}\left[-\frac{b}{3}\vec{a}_x - \frac{b}{3}\vec{a}_y + \frac{z}{2}\vec{a}_z\right].$$ Now substitute the values

at $z=1$ $\quad \vec{E} = -135.77\,\vec{a}_x - 135.77\,\vec{a}_y + 1018.3\,\vec{a}_z$ V/m

at $z=0$ $\quad \vec{E} = -18000\,(\vec{a}_x + \vec{a}_y)$ V/m

Problem 3.16 $\vec{R} = a\,\vec{a}_\rho - z\,\vec{a}_z \quad R = \sqrt{a^2+z^2}$



$$\vec{E} = \frac{\rho_{l0}}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{z\,dz}{(a^2+z^2)^{3/2}}\left[a\vec{a}_\rho - z\,\vec{a}_z\right]$$

$$= \vec{a}_\rho\,\frac{\rho_{l0}\,a}{4\pi\epsilon_0}\cancel{\left[-\frac{1}{\sqrt{z^2+a^2}}\right]_{-\frac{L}{2}}^{\frac{L}{2}}} \xrightarrow{} 0$$

$$-\,\vec{a}_z\,\frac{\rho_{l0}}{4\pi\epsilon_0}\left[-\frac{z}{\sqrt{a^2+z^2}} + \ln\left[z+\sqrt{a^2+z^2}\right]\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$\frac{L}{2} = 10\text{m}$

$a = 2\text{m}$

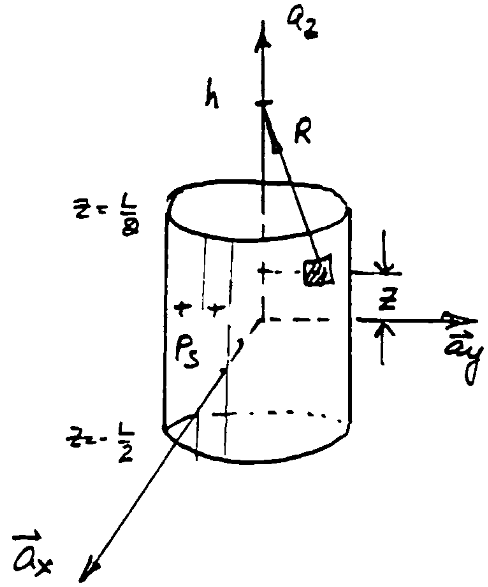
$\rho_{l0} = 100\ \text{nc/m}$

$$E_z = -\frac{\rho_{l0}}{4\pi\epsilon_0}\left[-\frac{L}{\sqrt{a^2+L^2/4}} + \ln\left[\frac{\frac{L}{2}+\sqrt{a^2+\frac{L^2}{4}}}{-\frac{L}{2}+\sqrt{a^2+\frac{L^2}{4}}}\right]\right]$$

$= -2397.51$ V/m when values are substituted.

Problem 3.17 $\vec{R} = -b\vec{a}_\rho + (h-z)\,\vec{a}_z$

$R = \sqrt{b^2+(h-z)^2}$



$$\vec{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_{-\frac{L}{2}}^{L/2}\int_0^{2\pi} \frac{b\,d\phi\,dz\left[-b\vec{a}_\rho - (z-h)\vec{a}_z\right]}{\left[b^2+(z-h)^2\right]^{3/2}}$$

$\vec{a}_\rho = \vec{a}_x\cos\phi + \vec{a}_y\sin\phi$ and

$$\int_0^{2\pi}\cos\phi\,d\phi = 0\,, \quad \int_0^{2\pi}\sin\phi\,d\phi = 0$$

Thus, $E_x = 0$, $E_y = 0$

$$E_z = -\frac{2\pi b\rho_s}{4\pi\epsilon_0}\int_{-\frac{L}{2}}^{L/2}\frac{(z-h)\,dz}{\left[b^2+(z-h)^2\right]^{3/2}}$$

$$= \frac{2\pi b\rho_s}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{b^2+(z-h)^2}}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{b\rho_s}{2\epsilon_0}\left[\frac{1}{\sqrt{b^2+(\frac{L}{2}-h)^2}} - \frac{1}{\sqrt{b^2+(\frac{L}{2}+h)^2}}\right]$$

when $h=0$, $E_z = 0$

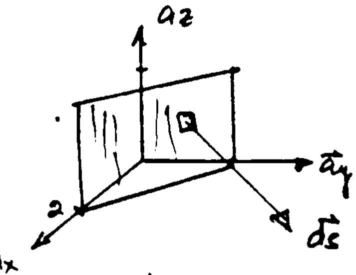
when $h = L/2$,

$$E_z = \frac{b\rho_s}{2\epsilon_0}\left[\frac{1}{b} - \frac{1}{\sqrt{b^2+L^2}}\right]$$

when $h = -L/2$,

$$E_z = \frac{b\rho_s}{2\epsilon_0}\left[\frac{1}{\sqrt{b^2+L^2}} - \frac{1}{b}\right]$$

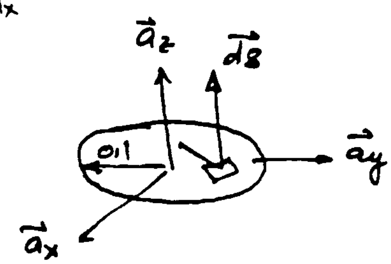
Problem 3.18 $\vec{D} = 6y\,\vec{a}_x + 2x\,\vec{a}_y + 14xy\,\vec{a}_z \;\; mC/m^2$



a) $\vec{ds} = dy\,dz\,\vec{a}_x + dx\,dz\,\vec{a}_y$

$$\int_s \vec{D}\cdot\vec{ds} = \int_0^2 6y\,dy\int_0^2 dz + \int_0^2 2x\,dx\int_0^2 dz$$

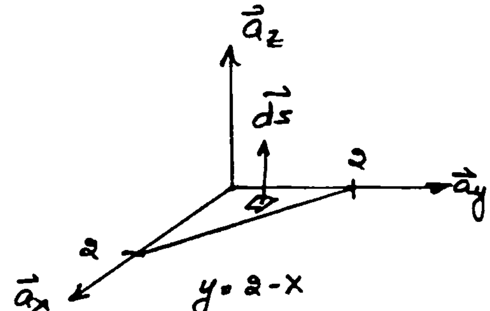
$$= 32\,mC$$



b) $\vec{ds} = \rho\,d\rho\,d\phi\,\vec{a}_z$

$$\int_s \vec{D}\cdot\vec{ds} \quad \int_0^{0.1}\int_0^{2\pi} 14xy\,\rho\,d\rho\,d\phi \qquad x = \rho\cos\phi \quad y = \rho\sin\phi$$

$$= \int_0^{0.1}\rho^3 d\rho\int_0^{2\pi}\sin\phi\cos\phi\,d\phi = 0$$



c) $\vec{ds} = dx\,dy\,\vec{a}_z$

$$\int_s \vec{D}\cdot\vec{ds} = \int_0^2 14x\,dx\int_2^{2-x} y\,dy$$

$y = 2-x$

$dy = -dx$

$$= 7\int_0^2 x\left[(2-x)^2 - 4\right]dx \Rightarrow$$

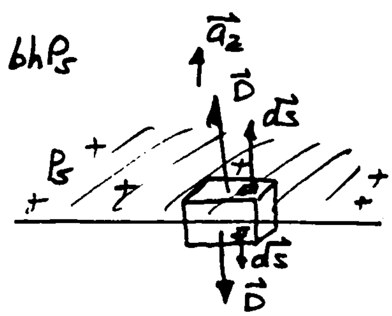
$$= \left[\frac{7x^4}{4} - \frac{28x^3}{3}\right]_0^2 = -46.67\,mC$$

Problem 3.19 $\oint_s \vec{D}\cdot\vec{ds} = Q_{enc} \Rightarrow D_\rho = \dfrac{a\rho_s}{\rho}$ $\qquad a = 0.2\,m$ $\quad \rho_s = 10\,mC/m^2$

$$\psi = \int_s \vec{D}\cdot\vec{ds}\Big|_{at\ \rho=2} = \int_{\pi/4}^{3\pi/4}\frac{a\rho_s}{\rho}\,\rho\,d\phi\int_2^4 dz = a\pi\rho_s = 6.28\,mC.$$

Problem 3.20 $\rho < b:\ \vec{E} = 0 \qquad \rho > b:\ \vec{E} = \dfrac{b\rho_s}{\epsilon_0\rho}\,\vec{a}_\rho \qquad D_\rho = \dfrac{b\rho_s}{\rho}$

$$\psi = \int_s \vec{D}\cdot\vec{ds} = b\rho_s\int_0^{\pi/2}\frac{1}{\rho}\,\rho\,d\phi\int_0^h dz = \frac{\pi}{2}bh\rho_s$$



Problem 3.21

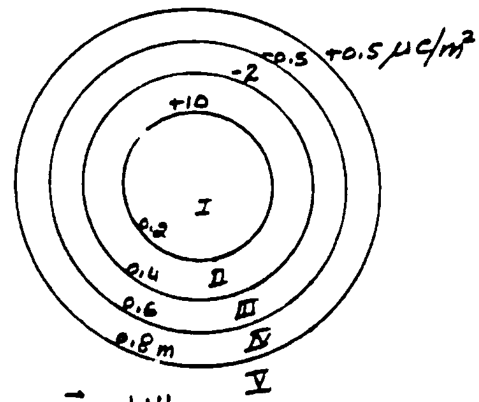
$$\oint_s \vec{D}\cdot\vec{ds} = Q_{enc} \Rightarrow 2D_zA = \rho_s A$$

$$D_z = \frac{\rho_s}{2}$$

$$E_z = \frac{\rho_s}{2\epsilon_0}$$

A= Area of Top and bottom surfaces.

Problem 3.22 $\vec{E} = \dfrac{Q_{enc}}{4\pi\epsilon_0 r^2}\vec{a}_r$



$\vec{E}\big|_{r=0.1} = 0 \qquad \because Q_{enc} = 0$

$\vec{E}\big|_{r=0.3} = \dfrac{4\pi\,(0.2)^2 \cdot 10\times 10^{-6}}{(0.3)^2}\, 9\times 10^9\, \vec{a}_r$

$= 502.65\, \vec{a}_r \quad kV/m$

$\vec{E}\big|_{r=0.5} = \dfrac{4\pi\,[0.2^2\times 10 - 0.4^2\times 2]10^{-6}}{(0.5)^2}\, 9\times 10^9\, \vec{a}_r = 36.19\,\vec{a}_r \text{ kV/m}$

$\vec{E}\big|_{r=0.7} = \dfrac{4\pi\,[0.2^2\times 10 - 0.4^2\times 2 - 0.6^2\times 0.5]10^{-6}}{(0.7)^2}\, 9\times 10^9\, \vec{a}_r = -23.08\,\vec{a}_r \text{ kV/m}$

$\vec{E}\big|_{r=1} = 4\pi\,[0.2^2\times 10 - 0.4^2\times 2 - 0.6^2\times 0.5 + 0.8^2\times 0.5]10^{-6}\times 9\times 10^9\, \vec{a}_r = 24.88\,\vec{a}_r \text{ kV/m}$

Problem 3.23 $\vec{D} = \dfrac{Q}{4\pi r^2}\vec{a}_r \Rightarrow \Psi = \int_S \vec{D}\cdot d\vec{s} = \dfrac{Q}{4\pi}\int_0^{\theta_0} \dfrac{1}{r^2} r^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi$

$= \dfrac{Q}{2}(1 - \cos\theta_0)$

Problem 3.24 $\rho < a,\ \vec{D} = 0\ \ \vec{E} = 0$

$a \le \rho \le b$: $Q_{enc} = k(\rho - a)2\pi h \Rightarrow \quad 2\pi\rho D_\rho = k(\rho - a)2\pi h \Rightarrow D_\rho = \dfrac{k(\rho - a)}{\rho}, \quad E_\rho = \dfrac{k}{\epsilon_0}\left(1 - \dfrac{a}{\rho}\right)$

$b \le \rho \le c \quad \vec{D} = 0 \quad \vec{E} = 0$

$\rho \ge c$: $\quad D_\rho 2\pi\rho h = k(b-a)2\pi h \Rightarrow D_\rho = \dfrac{k(b-a)}{\rho}, \quad E_\rho = \dfrac{k(b-a)}{\epsilon_0 \rho}$

Problem 3.25 $r < b$: $\quad Q_{enc} = k\int_0^r dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi r k$

$\oint_S \vec{D}\cdot d\vec{s} = 4\pi r^2 D_r \Rightarrow \quad D_r = \dfrac{k}{r}$

$r > b$: $\quad Q_{enc} = 4\pi b k \Rightarrow \quad D_r = \dfrac{kb}{r^2}$

Problem 3.26 $\quad Q_{enc} = 2\pi a L \rho_s \qquad a = 0.05\text{m}$

$\oint_S \vec{D}\cdot d\vec{s} = 2\pi\rho L D_\rho \Rightarrow D_\rho = \dfrac{a\rho_s}{\rho}$

$E_\rho = \dfrac{a\rho_s}{\epsilon_0 \rho} \quad$ at $\rho = 1\, m$, $E_\rho = 100\times 10^3$ V/m $\Rightarrow$

$\rho_s = 100\times 10^3 \times \dfrac{10^{-9}}{36\pi}\cdot\dfrac{1}{0.05} = 17.68\ \mu C/m^2$

Problem 3.27 $\vec{R} = \rho\,\vec{a}_\rho - z\,\vec{a}_z \quad R = \sqrt{\rho^2 + z^2}$ where $\rho = a$

$$V = \int_0^L \frac{\rho_l\, dz}{4\pi\epsilon_0\sqrt{\rho^2+z^2}} = \frac{\rho_l}{4\pi\epsilon_0} \ln\left[z + \sqrt{z^2+\rho^2}\right]\Big|_0^L$$

$$= \frac{\rho_l}{4\pi\epsilon_0} \ln\left[\frac{L+\sqrt{L^2+\rho^2}}{\rho}\right] \qquad \text{at } \rho = a,\ V(a) = \frac{\rho_l}{4\pi\epsilon_0}\ln\left[\frac{L+\sqrt{L^2+a^2}}{a}\right]$$

$$\vec{E} = -\nabla V = \frac{\rho_l}{4\pi\epsilon_0}\left[\frac{L/\rho}{\sqrt{L^2+\rho^2}}\right]\vec{a}_\rho \qquad \text{at } \rho = a, \quad \vec{E}(a) = \frac{\rho_l}{4\pi\epsilon_0 a}\left[\frac{L}{\sqrt{L^2+\rho^2}}\right]$$

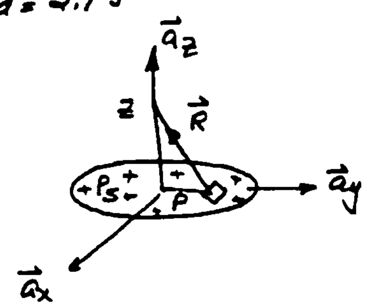
Problem 3.28

$$W = \frac{Q_1 Q_2}{4\pi\epsilon_0 R} = \frac{500\times10^{-9}\times(-600)\times10^{-9}}{1\times10^{-3}} \cdot 9\times10^{9} = -2.7\,J$$

The electric field due to 500 nC is attracting the charge of -600 nC towards it. Thus, energy released = 2.7 J

Problem 3.29

$$V = \frac{\rho_s}{4\pi\epsilon_0}\int_0^b\int_0^{2\pi}\frac{\rho\, d\phi\, d\rho}{\sqrt{\rho^2+z^2}}$$



or

$$V = \frac{\rho_s}{2\epsilon_0}\int_0^b \frac{\rho\, d\rho}{\sqrt{\rho^2+z^2}} = \frac{\rho_s}{2\epsilon_0}\left[\sqrt{\rho^2+z^2}\right]_0^b$$

$$= \frac{\rho_s}{2\epsilon_0}\left[\sqrt{b^2+z^2} - z\right]$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial z}\vec{a}_z = -\frac{\rho_s}{2\epsilon_0}\left[\frac{z}{\sqrt{b^2+z^2}} - 1\right]\vec{a}_z$$

Problem 3.30 From Problem 3.29, by changing the lower limit to a, we get

$$V = \frac{\rho_s}{2\epsilon_0}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right]$$

and

$$\vec{E} = \frac{\rho_s z}{2\epsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]\vec{a}_z$$

Problem 3.31 $V = \dfrac{Q}{4\pi\epsilon_0 R} \Rightarrow Q = 4\pi\epsilon_0 R V$

when $V = 9\,kV$ at $R = 0.2\,m$, $Q = \dfrac{0.2\times9\times10^{3}}{9\times10^{9}} = 200\,nC$

(a) when $V = 18\,kV$, $R = \dfrac{200\times10^{-9}\times9\times10^{9}}{18\times10^{3}} = 0.1\,m$ or $10\,cm$

(b) when $V = 3\,kV$, $R = 0.6\,m$ or $60\,cm$.

Problem 3.32 $\vec{E} = 10\,\vec{a}_x + 20\,\vec{a}_y + 20\,\vec{a}_z$ kV/m

From $(0,0,0)$ to $(3,0,0)$: $W_1 = -q\int_C \vec{E}\cdot d\vec{l} = -0.1\times10^{-9}\times10\times10^{3}\times3 = -3\,\mu J$

From $(3,0,0)$ to $(3,4,0)$: $W_2 = -0.1\times10^{-9}\times20\times10^{3}\times4 = -8\,\mu J$

Total work done: $W = -11\,\mu J$

Directly from $(0,0,0)$ to $(3,4,0)$: $dW = -q\vec{E}\cdot d\vec{l} = -qE_x dx - qE_y dy$

$$W = -0.1\times10^{-9}\int_0^3 10\times10^{3}\,dx \; -0.1\times10^{-9}\int_0^4 20\times10^{3}\,dy = -11\,\mu J$$

Problem 3.33 $\vec{E} = 10\times10^{3}\,\vec{a}_x$ V/m $\quad V=0$ at $x=0$, at any point $P(x,y,z)$

$$V_p = -\int_0^x \vec{E}\cdot d\vec{l} = -10\times10^{3}x \;.\quad \text{Thus,}\quad V_p(x,y,z) = -10x \text{ kV}$$

Problem 3.34 $\quad V = 10x^2 + 20y^2 + 5z$ V .

$$\vec{E} = -\nabla V = -[20x\,\vec{a}_x + 40y\,\vec{a}_y + 5] \text{ V/m} \qquad \nabla\times\vec{E} = 0$$

This potential function can exist because $\nabla\times\vec{E} = 0$

Problem 3.35 : Inside: $Q_{enc} = \int_0^\rho \rho_v\,\rho\,d\rho\int_0^{2\pi}d\phi\int_0^h dz = \pi\rho^2 h\,\rho_v$

$$\oint_S \vec{D}\cdot d\vec{s} = 2\pi\rho h D_\rho \Rightarrow D_\rho = \frac{\rho\,\rho_v}{2} \Rightarrow E_\rho = \frac{\rho\,\rho_v}{2\epsilon_0} \quad \rho\le a$$

outside: $\quad Q_{enc} = \pi a^2 h\,\rho_v$ and $E_\rho = \dfrac{a^2\rho_v}{2\rho\,\epsilon_0} \quad \rho\ge a$

Potential: outside: Let $V=0$ when $\rho = b$, then

$$V(\rho) = -\int_b^\rho \frac{a^2\rho_v}{2\epsilon_0}\frac{1}{\rho}\,d\rho = \frac{\rho_v a^2}{2\epsilon_0}\ln(b/\rho)$$

when $\quad \rho = a$: $\quad V(a) = \dfrac{\rho_v a^2}{2\epsilon_0}\ln(b/a)$

Potential distribution inside $\rho<a$:

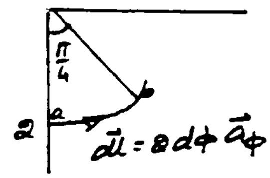
$$V(\rho) - V(a) = -\int_a^\rho \frac{\rho_v}{2\epsilon_0}\,\rho\,d\rho = +\frac{\rho_v}{4\epsilon_0}[a^2-\rho^2]$$

Thus, $\quad V(\rho) = \dfrac{\rho_v}{4\epsilon_0}[a^2-\rho^2] + \dfrac{\rho_v a^2}{2\epsilon_0}\ln(b/a)$

Problem 3.36 $\vec{E} = 10y\,\vec{a}_x + 10x\,\vec{a}_y + 2z\,\vec{a}_z$ kV/m

$\nabla\times\vec{E} = 0$

$\vec{E}$ can exist.



$$\vec{E}\cdot\vec{dl} = -20y\,d\phi\,\vec{a}_x\cdot\vec{a}_\phi + 20x\,d\phi\,\vec{a}_y\cdot\vec{a}_\phi \quad \text{kV}$$

$\vec{a}_x\cdot\vec{a}_\phi = -\sin\phi$, $\vec{a}_y\cdot\vec{a}_\phi = \cos\phi$, $x = 2\cos\phi$, $y = 2\sin\phi$

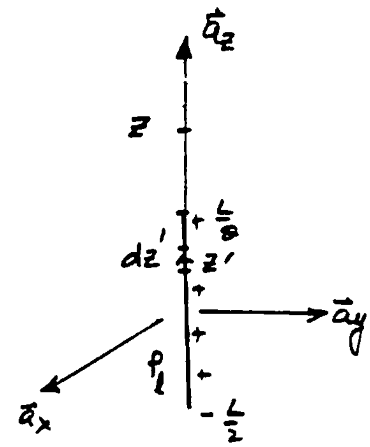
$$\vec{E}\cdot\vec{dl} = (-40\sin^2\phi + 40\cos^2\phi)\,d\phi \text{ kV} = 40\times10^{3}\cos\phi\,d\phi$$

Hence $$V_{ba} = -\int_a^b \vec{E}\cdot\vec{dl} = -40\times10^{3}\int_0^{\pi/4}\cos 2\phi\,d\phi = -40\text{kV}$$

$$W_{ba} = q V_{ba} = -0.5\times10^{-3}\times 40\times10^{3} = -20 \text{ J}$$

$\vec{E}$ field is doing the work.

Problem 3.37



$$V = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\rho_\ell}{4\pi\epsilon_0}\frac{dz'}{z-z'} = -\frac{\rho_\ell}{4\pi\epsilon_0}\Big[\ln(z-z')\Big]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= -\frac{\rho_\ell}{4\pi\epsilon_0}\left[\ln\left(\frac{z-L/2}{z+L/2}\right)\right]$$

$$\vec{E} = -\nabla V = \frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{1}{z-\frac{L}{2}} - \frac{1}{z+\frac{L}{2}}\right]\vec{a}_z$$

$$= \frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{L}{z^2 - L^2/4}\right]\vec{a}_z$$

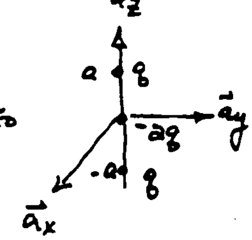
Problem 3.38 $d = 1\,\mu m$ $\quad \vec{P} = q\,d\,\vec{a}_z = 10\times10^{-9}\times1\times10^{-6}\vec{a}_z = 10^{-14}\vec{a}_z$ C m.

$$V_p = \frac{\vec{P}\cdot\vec{a}_r}{4\pi\epsilon_0 r^2} \quad r = 1\text{ m},\ \vec{a}_r = \vec{a}_z \Rightarrow V_p = \frac{10^{-14}\times9\times10^{9}}{1^2} = 90\,\mu V$$

$$\vec{E}_p = \frac{P}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\vec{a}_r + \cancel{\sin\theta\,\vec{a}_\theta}^{\,0}\right] \quad \because \theta = 0$$

$$= 10^{-14}\times 9\times10^{9}\times 2\,\vec{a}_z = 180\,\vec{a}_z \;\mu V/m$$

Problem 3.39



$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{z-a} - \frac{2}{z} + \frac{1}{z+a}\right] = \frac{2a^2 q}{z(z^2-a^2)4\pi\epsilon_0}$$

for $z >> a$ $$V \cong \frac{2a^2 q}{4\pi\epsilon_0 z^3}$$

$$\vec{E} = -\nabla V = \frac{6a^2 q}{4\pi\epsilon_0 z^4}\vec{a}_z$$

Problem 3.40 $V = \dfrac{\vec{P}\cdot\vec{a}_r}{4\pi\epsilon_0 r^2} + \dfrac{q}{4\pi\epsilon_0 r^2} = \dfrac{1}{4\pi\epsilon_0 r^2}\left[q + \vec{P}\cdot\vec{a}_r\right]$

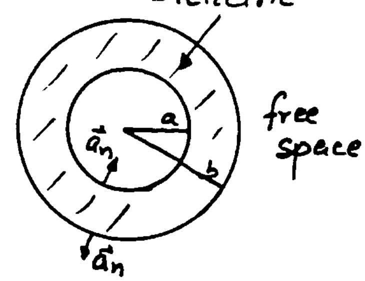
$$\vec{E} = \frac{qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right] + \frac{q\,\vec{a}_r}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0 r^3}\left[(2d\cos\theta + r)\vec{a}_r + d\sin\theta\,\vec{a}_\theta\right]$$

Problem 3.41 Inside the inner conductor: $\vec{E} = 0$ $\rho < a$

$a \le \rho \le b$ $E_\rho = \dfrac{a\rho_s}{\rho\epsilon_0}$ and $E_\rho = 0$ for $\rho > b$ when the outer conductor is grounded.

When the outer conductor is not grounded,

$E_\rho = 0$ $b \le \rho \le c$ and $E_\rho = \dfrac{a\rho_s}{\rho\epsilon_0}$ $\rho > c$



Problem 3.42 $\vec{E} = 0$ $\rho < a$.

$a \le \rho \le b$ $E_\rho = \dfrac{a\rho_s}{\rho\epsilon}$, $D_\rho = \dfrac{a\rho_s}{\rho}$, $w = \dfrac{1}{2}\dfrac{a^2\rho_s^2}{\epsilon\rho^2}$

$\vec{P} = \vec{D} - \epsilon_0\vec{E} \Rightarrow P_\rho = \dfrac{a\rho_s}{\rho}\left(\dfrac{\epsilon_r - 1}{\epsilon_r}\right)$

$\rho_{vb} = -\nabla\cdot\vec{P} = 0$ $\rho_{sb}\big|_{\rho=a} = \vec{P}\cdot\vec{a}_n = -\rho_s\left(\dfrac{\epsilon_r-1}{\epsilon_r}\right)$, $\rho_{sb}\big|_{\rho=b} = \dfrac{a}{b}\rho_s\left(\dfrac{\epsilon_r-1}{\epsilon_r}\right)$

$\rho > b$: $E_\rho = \dfrac{a\rho_s}{\epsilon_0\rho}$, $D_\rho = \dfrac{a\rho_s}{\rho}$, $w = \dfrac{1}{2}\dfrac{a^2\rho_s^2}{\epsilon_0\rho^2}$

Problem 3.43 ϵ is a function of location. i.e $\epsilon(x, y, z)$

$\nabla\cdot\vec{D} = 0 \Rightarrow \nabla\cdot\epsilon\vec{E} = 0 \Rightarrow \epsilon\nabla\cdot\vec{E} + \vec{E}\cdot\nabla\epsilon = 0$

Thus: $\nabla\cdot\vec{E} = -\dfrac{\vec{E}\cdot\nabla\epsilon}{\epsilon}$

Problem 3.44 $\epsilon = \alpha z^n$, Given $\vec{E} = E\,\vec{a}_z$

From Problem 3.43, $\nabla\cdot\vec{E} = -\dfrac{\vec{E}\cdot\nabla\epsilon}{\epsilon}$

However, $\nabla\epsilon = \dfrac{\partial}{\partial z}(\alpha z^n)\,\vec{a}_z = n\alpha z^{n-1}\,\vec{a}_z = \dfrac{\alpha n}{z} z^n a_z$
$= \dfrac{n}{z}\epsilon\,\vec{a}_z$

and $\nabla\cdot\vec{E} = -\dfrac{nE}{z}$

where E is the magnitude of the electric field intensity.

Problem 3.45 If ρ_s is the surface charge density, $Q = 4\pi a^2 \rho_s$.

From Gauss' Law: $D_r = \dfrac{a^2\rho_s}{r^2}$, $\epsilon = \epsilon_0\left[\dfrac{a+r}{r}\right] \Rightarrow E_r = \dfrac{a^2\rho_s}{\epsilon_0 r(a+r)}$

$$P_r = D_r - \epsilon_0 E_r = \frac{a^3\rho_s}{r^2(a+r)}, \quad \rho_{sb}\Big|_{r=a} = -\frac{\rho_s}{2}, \quad \rho_{sb}\Big|_{r=b} = \frac{a^3\rho_s}{b^2(a+b)}$$

$$\rho_{vb} = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) = \frac{a^3\rho_s}{r^2(a+r)^2}, \quad w = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\rho_s^2 a^4}{\epsilon_0 r^3(a+r)}$$

$$V_r = -\int_\infty^r \vec{E}\cdot d\vec{l} = -\frac{a^2\rho_s}{\epsilon_0}\int \frac{dr}{r(a+r)} = -\frac{Q}{4\pi\epsilon_0 a}\int_\infty^r \frac{1}{r}\,dr + \frac{Q}{4\pi\epsilon_0 a}\int_\infty^r \frac{dr}{r+a}$$

$$= \frac{Q}{4\pi\epsilon_0 a}\ln\left(\frac{a+r}{r}\right)$$

Problem 3.46 : $W = \dfrac{Q_1 Q_2}{4\pi\epsilon r} = \dfrac{10\times10^{-6}\times 10\times10^{-6}\times 9\times10^{9}}{5.5\times 10\times 10^{-3}} = 16.36\,J$

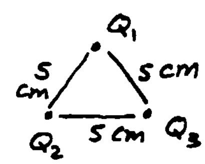
Problem 3.47 $E_z = -\dfrac{\rho_s}{\epsilon}$, $w = \dfrac{1}{2}\dfrac{\rho_s^2}{\epsilon}$ $\quad A = 400\times10^{-4}\,m^2 \quad d = 1\times10^{-3}\,m$

$$W = \frac{1}{2}\frac{\rho_s^2}{\epsilon}Ad = \frac{1}{2}\times(250\times10^{-9})^2\times\frac{36\pi\times10^9}{2}\times 400\times10^{-4}\times 1\times10^{-3} = 70.7\,nJ$$

Problem 3.48 $R = \sqrt{4^2+3^2} = 5$

$$W = \frac{Q_1 Q_2}{4\pi\epsilon_0 R} = \frac{100\times10^{-9}\times 300\times10^{-9}\times 9\times10^{9}}{5} = 54\,\mu J$$

Problem 3.49 $V_1 = \dfrac{Q_2+Q_3}{4\pi\epsilon_0 R}$, $V_2 = \dfrac{Q_1+Q_3}{4\pi\epsilon_0 R}$, $V_3 = \dfrac{Q_1+Q_2}{4\pi\epsilon_0 R}$



$R = 5$ cm

$$W = \frac{1}{2}\sum Q_i V_i = \frac{1}{4\pi\epsilon_0 R}\left[Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1\right]$$

$$= \frac{9\times10^9}{5}\left[100\times200 + 200\times300 + 300\times100\right]10^{-18} = 19.8\,mJ$$

Problem 3.50 $E_\rho = \dfrac{100}{\rho}$ $\quad D_\rho = \dfrac{100\epsilon}{\rho}$ $\quad w = \dfrac{1}{2}\vec{D}\cdot\vec{E} = \dfrac{1}{2}\epsilon\left(\dfrac{100}{\rho}\right)^2$ $\quad a = 0.2\,m$, $b = 0.5\,m$

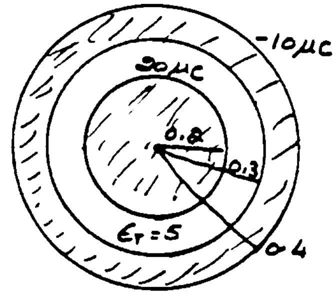
$$W = \frac{\epsilon}{2}(100)^2\int_{0.2}^{0.5}\frac{1}{\rho}\,d\rho\int_0^{2\pi}d\phi\int_0^1 dz = \frac{\epsilon}{2}(100)^2\ln\left(\frac{0.5}{0.2}\right)2\pi\,(1) = 1.4\,\mu J$$

where $\epsilon = 5.5\epsilon_0$.

Problem 3.51 When Q is the total charge on the sphere, the potential on the surface of the sphere: $V = \frac{Q}{4\pi\epsilon a}$

$$dW = V\,dQ \Rightarrow W = \int_0^Q \frac{Q}{4\pi\epsilon a}\,dQ = \frac{1}{2}\frac{Q^2}{4\pi\epsilon a} = \frac{1}{2}QV$$

Problem 3.52



$0.2 \le r \le 0.3$: $D_r = \frac{Q}{4\pi r^2}$, $E_r = \frac{Q}{4\pi\epsilon r^2}$ $\qquad \epsilon = 5\epsilon_0$

$w_1 = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{Q^2}{(4\pi)^2\epsilon r^4}$ $\qquad$ where $Q = 20\times10^{-6}$ C

$$W_1 = \frac{1}{2}\frac{Q^2}{(4\pi)^2\epsilon}\int_{0.2}^{0.3}\frac{1}{r^2}dr\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi = \frac{1}{2}\frac{Q^2}{16\pi^2}\cdot\frac{4\pi}{\epsilon}\left[\frac{1}{0.2}-\frac{1}{0.3}\right] = 0.6\,J$$

$r > 0.4$ m: $\quad Q_{enc} = 10\,\mu C \Rightarrow W_2 = \frac{1}{2}\frac{Q_{enc}^2}{16\pi^2}\cdot\frac{4\pi}{\epsilon_0}\left[\frac{1}{0.4}-\frac{1}{\infty}\right] = 1.125\,J$

$$W = W_1 + W_2 = 1.725\,J$$

Problem 3.53 $\vec{E}_1 = 12\vec{a}_x + 24\vec{a}_y - 36\vec{a}_z$ V/m, $\epsilon_1 = 4\epsilon_0$, Interface at $x = 5$

$E_{tan1} = E_{tan2} \Rightarrow E_{2y} = 24$ V/m $\quad E_{2z} = -36$ V/m

$D_{1n} = D_{2n} \Rightarrow \epsilon_1 E_{1x} = \epsilon_2 E_{2x} \Rightarrow E_{2x} = \frac{4}{16}\cdot 12 = 3$ V/m

$$\vec{E}_2 = 3\vec{a}_x + 24\vec{a}_y - 36\vec{a}_z \quad V/m$$

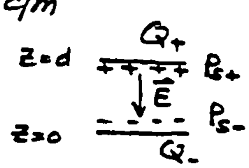
Problem 3.54 $E_r\big|_{r=20cm} = 10\times10^6$ V/m $\Rightarrow D_r = 10\times10^6\times\frac{10^{-9}}{36\pi} = 88.42\ \mu C/m^2$

Thus, $\rho_s\big|_{r=20cm} = 88.42\ \mu C/m^2$.

Problem 3.55 $E_z = -10\times10^3$ V/m $\quad A = 25\times10^{-4}\,m^2$, $d = 1\times10^{-3}$ m

$\epsilon_r = 3.6$

$D_z = \epsilon E_z = -\frac{36\times10^{-9}}{36\pi}\times10\times10^3 = -31.83\,\mu C/m^2$



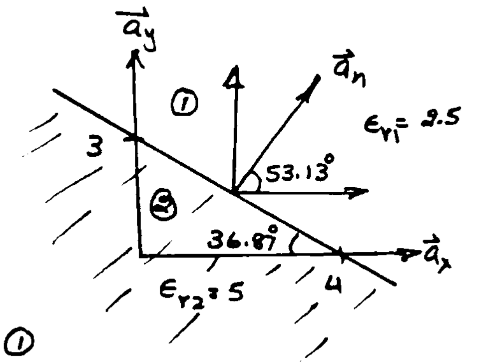
Thus, $\rho_{s+} = 318.3$ nC/m²

and $\rho_{s-} = -318.3$ nC/m²

$Q_+ = \rho_{s+}A = 795.7$ PC $\qquad Q_- = \rho_{s-}A = -795.7$ PC $\qquad$ PC $= 10^{-12}$ C

Problem 3.56 $\vec{a}_n = \cos(53.13°)\,\vec{a}_x + \sin(53.13°)\,\vec{a}_y$

$$= 0.6\,\vec{a}_x + 0.8\,\vec{a}_y$$



$$\vec{E}_1 = 25\,\vec{a}_x + 50\,\vec{a}_y + 25\,\vec{a}_z \quad V/m,$$

Let $\vec{E}_2 = E_{x2}\,\vec{a}_x + E_{y2}\,\vec{a}_y + E_{z2}\,\vec{a}_z$

$$\vec{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = 0 \Rightarrow 0.8\,E_{y2} + 0.6\,E_{x2} = 27.5 \quad ①$$

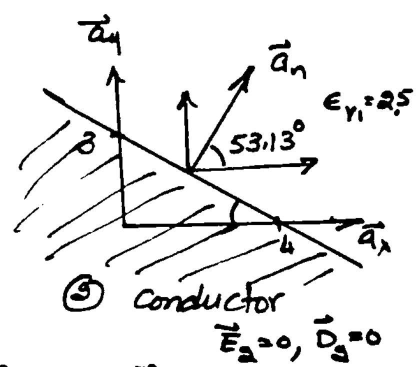
$$\vec{a}_n \times (\vec{E}_1 - \vec{E}_2) = \vec{a}_x\,[25 - E_{z2}]\,0.8 + \vec{a}_y\,[E_{z2} - 25]\,0.6 + \vec{a}_z\,(0.8\,E_{x2} - 0.6\,E_{y2} + 10) = 0$$

Thus, $E_{z2} = 25$ and $0.6\,E_{y2} - 0.8\,E_{x2} = 10$ ②

From ① and ②: $E_{x2} = 8.5$ and $E_{y2} = 28$.

Thus, $\vec{E}_2 = 8.5\,\vec{a}_x + 28\,\vec{a}_y + 25\,\vec{a}_z \quad V/m$

Problem 3.57 $\vec{a}_n = 0.6\,\vec{a}_x + 0.8\,\vec{a}_y$



Let $\vec{E}_1 = E_{x1}\,\vec{a}_x + E_{y1}\,\vec{a}_y + E_{z1}\,\vec{a}_z\,, \quad E_{y1} = 50\ V/m$

$$\vec{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0 \Rightarrow \vec{a}_n \times \vec{E}_1 = 0$$

or $0.6\,E_{y1} = 0.8\,E_{x1} \Rightarrow E_{x1} = \frac{0.6}{0.8} \times 50 = 37.5\ V/m.$

and $E_{z1} = 0$ $\quad \vec{a}_n \cdot \vec{D}_1 = \rho_s \Rightarrow \rho_s = (0.6\,\vec{a}_x + 0.8\,\vec{a}_y) \cdot (37.5\,\vec{a}_x + 50\,\vec{a}_y)\,2.5\,\epsilon_0$

$$= 1.38\ nC/m^2$$

$$\vec{E}_1 = 37.5\,\vec{a}_x + 50\,\vec{a}_y \quad V/m.$$

Problem 3.58 $C = C_1 + C_2 + C_3 = \frac{A}{d}\,\epsilon_0\,[\epsilon_{r1} + \epsilon_{r2} + \epsilon_{r3})$

$$= \frac{100 \times 10^{-4}}{10^{-3}} \cdot \frac{10^{-9}}{36\pi}\,[2 + 3.6 + 9] = 1.29\ nF$$

Problem 3.59 $C_1 = \frac{\epsilon_1 A_1}{d_1} = \frac{9 \times 10^{-9}}{36\pi} \cdot \frac{100 \times 10^{-4}}{0.5 \times 10^{-3}} = 1.592\ nF$

$$C_2 = \frac{\epsilon_2 A_2}{d_2} = \frac{3.6 \times 10^{-9}}{36\pi} \cdot \frac{100 \times 10^{-4}}{0.5 \times 10^{-3}} = 0.637\ nF$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = 0.455\ nF$$

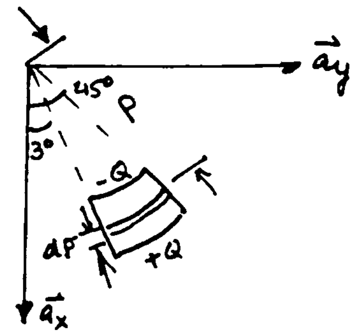
Problem 3.60 On a per-unit length basis:

$$C_1 = \frac{\epsilon_1 \pi}{\ln(b/a)} = \frac{\pi \times 6 \times 10^{-9}}{36\pi \ln(1.5)} = 0.343 \ nF/m$$

$$C_2 = \frac{\pi \epsilon_2}{\ln(b/a)} = \frac{\pi \times 25 \times 10^{-9}}{36\pi \ln(1.5)} = 0.171 \ nF/m, \qquad C = C_1 + C_2 = 0.514 \ nF/m$$

when $L = 10$, $C_T = 5.14 \ nF$

Problem 3.61 At a distance ρ, the area is



$\epsilon = 3.6 \epsilon_0$

$$A = \int_{\pi/6}^{\pi/4} \rho \, d\phi \int_0^{0.1} dz = 26.18 \times 10^{-3} \rho \ \ m^2$$

$$\frac{1}{C} = \int_c \frac{dl}{\epsilon A} = \int_{0.1}^{0.3} \frac{d\rho}{26.18 \times 10^{-3} \rho \epsilon}$$

$$= \frac{38.2}{\epsilon} \ln(3) = 1.318 \times 10^{12} \Rightarrow C = 0.76 \ PF$$

Problem 3.62

$$C_1 = \frac{4\pi\epsilon_1 ac}{c-a} = \frac{4\pi \times 5 \times 10^{-9} \times 10 \times 20 \times 10^{-4}}{36\pi \times 10 \times 10^{-2}} = 111.11 \ PF$$

$$C_2 = \frac{4\pi\epsilon_2 bc}{b-c} = \frac{4\pi \times 10 \times 10^{-9}}{36\pi} \cdot \frac{20 \times 30 \times 10^{-4}}{10 \times 10^{-2}} = 666.67 \ PF$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = 95.24 \ PF$$

Problem 3.63 Outside: $\nabla^2 V = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_o}{\partial r}\right) = 0 \Rightarrow V_o = -\frac{c_1}{r} + c_2$
$(r > b)$

Note: $\rho = \rho_v$

As $r \to \infty$, $V_o \to 0 \Rightarrow c_2 = 0$. Thus, $V_o(r) = -\frac{c_1}{r}$, $\vec{E}_o = -\nabla V_o = -\frac{c_1}{r^2}\vec{a}_r$

Inside, $r < b$: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_i}{\partial r}\right) = -\frac{\rho}{\epsilon_0} \Rightarrow \frac{d}{dr}\left(r^2 \frac{\partial V_i}{\partial r}\right) = -\frac{\rho}{\epsilon_0} r^2$

or $r^2 \frac{dV_i}{dr} = -\frac{\rho}{3\epsilon_0} r^3 + c_3 \Rightarrow \frac{dV_i}{dr} = -\frac{\rho}{3\epsilon_0} r + \frac{c_3}{r^2}$

As $r \to 0$, $\vec{E}_i$ can't be ∞.

Since $\vec{E}_i = -\nabla V_i = -\vec{a}_r\left(\frac{\partial V_i}{\partial r}\right) = -\left(\frac{\rho}{3\epsilon_0} r + \frac{c_3}{r^2}\right)\vec{a}_r$. Thus, $c_3 = 0$

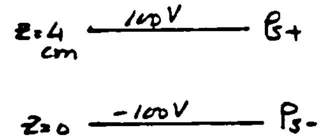
Thus, $\vec{E}_i = \frac{\rho r}{3\epsilon_0}\vec{a}_r$. At $r = b$, $D_{n1} = D_{n2} \Rightarrow \epsilon_0 \frac{\rho b}{3\epsilon_0} = -\epsilon_0 \frac{c_1}{b^2} \Rightarrow c_1 = -\frac{\rho b^3}{3\epsilon_0}$

Hence: $\frac{dV_i}{dr} = -\frac{\rho r}{3\epsilon_0} \Rightarrow V_i = -\frac{\rho r^2}{6\epsilon_0} + c_4$. At $r = b$, $V_i = V_o \Rightarrow -\frac{\rho b^2}{6\epsilon_0} + c_4 = \frac{\rho b^2}{3\epsilon_0}$

Hence $c_4 = \frac{\rho b^2}{2\epsilon_0}$. Thus, $V_i = -\frac{\rho r^2}{6\epsilon_0} + \frac{\rho b^2}{2\epsilon_0} = \frac{\rho}{2\epsilon_0}\left[b^2 - \frac{r^2}{3}\right]$, $\vec{E}_i = \frac{\rho r}{3\epsilon_0}\vec{a}_r$

and $V_o = \frac{\rho b^3}{3\epsilon_0 r}$ and $\vec{E}_o = \frac{\rho b^3}{3\epsilon_0} \cdot \frac{1}{r^2}\vec{a}_r$

Problem 3.64 $\nabla^2 V = 0 \Rightarrow \dfrac{\partial^2 V}{\partial z^2} = 0$ or $V = a_1 z + a_2$



At $z=0$ $V(0) = -100 \Rightarrow a_2 = -100$

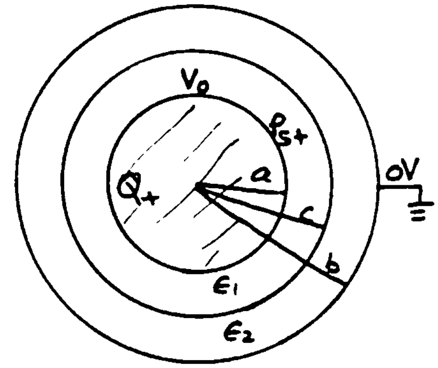
At $z = 0.04$ m, $V(0.04) = 100 \Rightarrow a_1 = 5000$

Hence $V(z) = 5000z - 100$

$\vec{E} = -\nabla V = -5000\,\vec{a}_z$ V/m $\quad \vec{D} = \epsilon_0 \vec{E} = -44.21\,\vec{a}_z$ nC/m^2

$\rho_{s+} = 44.21$ nC/m^2 and $\rho_{s-} = -44.21$ nC/m^2

Problem 3.65 $\nabla^2 V = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial V}{\partial \rho}\right) = 0 \Rightarrow$



Region-1: $V_1 = A \ln \rho + B \qquad a \le \rho \le c$

Region-2: $V_2 = G \ln \rho + H \qquad c \le \rho \le b$

$V_1 = V_0$ at $\rho = a$ and $V_2 = 0$ at $\rho = b \Rightarrow$

$V_1 = A \ln(\rho/a) + V_0$ and $V_2 = G \ln(\rho/b)$ $\qquad V_1 = V_2$ at $\rho = c$ and $D_{\rho 1} = D_{\rho 2}$ at $\rho = c$

$E_{\rho 1} = -\dfrac{A}{\rho}, \quad E_{\rho 2} = -\dfrac{G}{\rho}$

$D_{\rho 1} = -\dfrac{\epsilon_1 A}{\rho}, \quad D_{\rho 2} = -\dfrac{\epsilon_2 G}{\rho}, \quad D_{\rho 1} = D_{\rho 2}\Big|_{\rho = c} \Rightarrow A = \dfrac{\epsilon_2}{\epsilon_1} G$ ①

$V_1 = V_2$ at $\rho = c \Rightarrow \quad V_0 + A \ln(c/a) = G \ln(c/b)$ ②

From ① and ②: $\quad G = \dfrac{\epsilon_1 V_0}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}, \quad A = \dfrac{\epsilon_2 V_0}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}$

Hence, $V_1 = V_0 \left[\dfrac{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(\rho/c)}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}\right]$ and $V_2 = \dfrac{V_0 \epsilon_1 \ln(\rho/b)}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}$

$$\vec{E}_1 = \frac{-\epsilon_2 V_0 \vec{a}_\rho}{\rho\left[\epsilon_1 \ln(c/b) + \epsilon_2 \ln\left(\frac{a}{c}\right)\right]} = \frac{\epsilon_2 V_0 \vec{a}_\rho}{\rho\left[\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)\right]}$$

$$\vec{E}_2 = \frac{\epsilon_1 V_0 \vec{a}_\rho}{\rho\left[\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)\right]}, \qquad \rho_{s+} = \vec{D}_1 \cdot \vec{a}_\rho \Big|_{\rho = a} = \frac{\epsilon_1 \epsilon_2 V_0}{a\left[\epsilon_1 \ln\left(\frac{b}{c}\right) + \epsilon_2 \ln\left(\frac{c}{a}\right)\right]}$$

$Q_+ = 2\pi a \rho_{s+} \Rightarrow \quad C = \dfrac{Q_+}{V_0} = \dfrac{2\pi \epsilon_1 \epsilon_2}{\epsilon_1 \ln\left(\frac{b}{c}\right) + \epsilon_2 \ln\left(\frac{c}{a}\right)}$ F/m.
(per unit length)

Problem 3.66 Substitute $a = 0.1$ m, $b = 0.2$m, $c = 0.15$m, $\epsilon_{r1} = 3$, and $\epsilon_{r2} = 9$ in the solution of Problem 3.65 and obtain

$V_0 = 100$V

$V_1 = -199.458 \ln(\rho/a) + 100$, $V_2 = -66.486 \ln(\rho/b)$

$\vec{E}_1 = \frac{199.458}{\rho} \vec{a}_\rho$, $\vec{D}_1 = \frac{5.291}{\rho} \vec{a}_\rho$ nC/m^2 $\rho_{s+}|_{\rho=a} = 52.91$ nC/m^2

$\vec{E}_2 = \frac{66.486}{\rho} \vec{a}_\rho$ $\vec{D}_2 = \frac{5.291}{\rho} \vec{a}_\rho$ nC/m^2 $\rho_{s-}|_{\rho=b} = 26.46$ nC/m^2

$Q_+ = 2\pi a \rho_{s+}(1) = 33.244$ nC/m.

$$C = \frac{Q_+}{100} = 332.44 \text{ PF/m}$$

Problem 3.67 $\nabla^2 V = 0$ $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \Rightarrow$ $V = -\frac{c_1}{r} + c_2$

at $r = b$, $V = 0 \Rightarrow c_2 = \frac{c_1}{b}$ Thus, $V = c_1\left[\frac{1}{b} - \frac{1}{r}\right]$

at $r = a$, $V = V_0 \Rightarrow c_1 = \frac{V_0}{\frac{1}{b} - \frac{1}{a}}$ $\Rightarrow$ $V = \frac{-V_0}{\frac{1}{b} - \frac{1}{a}}\left(\frac{1}{b} - \frac{1}{r}\right) = \frac{V_0 ab}{(b-a)r} - \frac{V_0 a}{b-a}$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\vec{a}_r = \frac{V_0}{\frac{1}{b}-\frac{1}{a}} \cdot \frac{1}{r^2}\vec{a}_r = \frac{V_0 ab}{(b-a)r^2}\vec{a}_r, \quad \vec{D} = \frac{V_0 \epsilon ab}{(b-a)r^2}\vec{a}_r$$

$$\rho_{s+}|_{r=a} = \frac{V_0 \epsilon b}{a(b-a)}, \quad Q_+ = 4\pi a^2 \rho_{s+} = \frac{4\pi ab\epsilon V_0}{b-a}$$

$$C = \frac{Q_+}{V_0} = \frac{4\pi\epsilon ab}{b-a} = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$$

Problem 3.68 Substitute $a = 0.05$ m, $b = 0.1$m, $V_0 = 500$V and $\epsilon = 9\epsilon_0$ in Problem 3.67 and obtain

$V(r) = \frac{50}{r} - 500$, $\vec{E} = \frac{50}{r^2}\vec{a}_r$, $\vec{D} = \frac{3.98}{r^2}\vec{a}_r$ nC/m^2

$\rho_{s+} = \frac{3.98 \times 10^{-9}}{(0.05)^2} = 1.59\ \mu$C/m^2, $Q_+ = 1.59 \times 10^{-6} \times 4\pi (0.05)^2$
$= 50$ nC

$$C = \frac{Q_+}{V_0} = 100 \text{PF}$$

Problem 3.69 $\nabla^2 V = -\frac{A}{\epsilon\rho} \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = -\frac{A}{\epsilon\rho}$

Integrating $\frac{\partial V}{\partial\rho} = -\frac{A}{\epsilon} + \frac{c_1}{\rho}$ and $V = -\frac{A}{\epsilon}\rho + c_1 \ln\rho + c_2$ ①

At $\rho = b$, $V = 0 \Rightarrow c_2 = \frac{Ab}{\epsilon} - c_1 \ln b$ or $V = -\frac{A}{\epsilon}\rho + c_1 \ln(\rho/b) + \frac{Ab}{\epsilon}$

At $\rho = a$, $V = V_0 \Rightarrow c_1 = \frac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}$ and $c_2 = \frac{Ab}{\epsilon} - \left[\frac{A}{\epsilon}(b-a) - V_0\right]\frac{\ln b}{\ln(b/a)}$

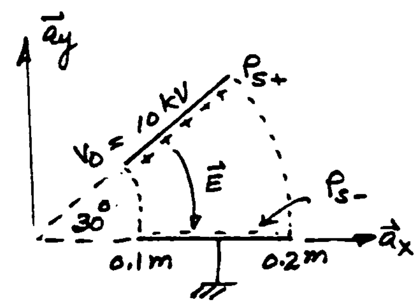
Thus, $V = -\frac{A\rho}{\epsilon} + \frac{A}{\epsilon}b + \left[\frac{A}{\epsilon}(b-a) - V_0\right]\frac{\ln(\rho/b)}{\ln(b/a)}$

$\vec{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\vec{a}_\rho = \left[\frac{A}{\epsilon} - \frac{\frac{A}{\epsilon}(b-a) - V_0}{\rho\ln(b/a)}\right]\vec{a}_\rho$, $\vec{D} = \epsilon\vec{E} = \left[A - \frac{A(b-a) - \epsilon V_0}{\rho\ln(b/a)}\right]\vec{a}_\rho$

$\rho_{s+}\Big|_{\rho=a} = A - \frac{A(b-a) - \epsilon V_0}{a\ln(b/a)}$, $Q_+ = 2\pi a\rho_{s+} = 2\pi a\left[A - \frac{A(b-a) - \epsilon V_0}{a\ln(b/a)}\right]$

Thus, $C = \frac{Q_+}{V_0} = \frac{2\pi a}{V_0}\left[A - \frac{A(b-a) - V_0\epsilon}{a\ln(b/a)}\right]$, when $A \to 0$ $C = \frac{2\pi\epsilon}{\ln(b/a)}$

Problem 3.70



$\nabla^2 V = 0 \Rightarrow \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0 \Rightarrow$

$V = c_1\phi + c_2$ at $\phi = 0$ $V = 0 \Rightarrow c_2 = 0$

At $\phi = \frac{\pi}{6}$, $V = 10{,}000 \Rightarrow c_1 = 19098.59$

Hence, $V = 19098.59\,\phi$

$\vec{E} = -\nabla V = -\frac{1}{\rho}\frac{\partial V}{\partial\phi} = -\frac{19098.59}{\rho}\vec{a}_\phi$, $\vec{D} = \epsilon_0\vec{E} = -\frac{168.87}{\rho}\vec{a}_\phi$ nC/m^2

$\rho_{s+} = \frac{168.87}{\rho}$ nC/m^2. The charge on the top-plate per-unit length in the z-direction is

$$Q_+\Big|_{\text{per-unit length}} = \int_{0.1}^{0.2}\frac{168.87\times10^{-9}}{\rho}d\rho\,(1) = 117.05\ \text{nC/m}$$

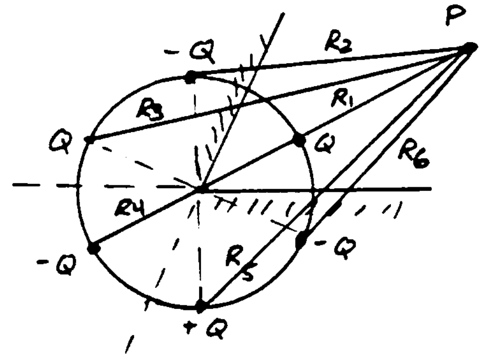
Thus, $C = \frac{Q_+|_{\text{per-unit length}}}{V_0} = 11.71$ PF/m

$w = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1.613}{\rho^2}$ mJ/m^3 or $W = \frac{1}{2}CV_0^2$

$= \frac{1}{2}\times 11.71\times10^{-12}\times 10{,}000^2$

$W = \int_{0.1}^{0.2}\frac{1.613}{\rho^2}\rho\,d\rho\int_0^{\pi/6}d\phi\,(1) = 585.26\ \mu$J/m $= 585.26\ \mu$J/m

Problem 3.71 $P(7, 30°, 10)$, Q is at $(5, 30°, 10)$



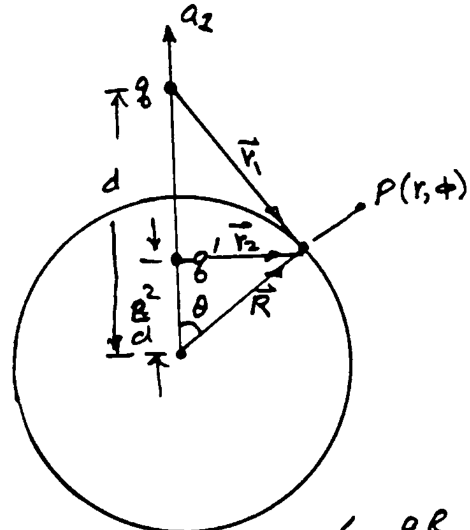
$R_1 = 2\,m$, $R_2 = 6.24\,m$, $R_3 = 10.44\,m$

$R_4 = 12\,m$, $R_5 = 10.44\,m$ $R_6 = 6.24\,m$

$$V_P = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4} + \frac{1}{R_5} - \frac{1}{R_6}\right]$$

when $Q = 500\,nC$, $V_P \cong 1295\,V$

Problem 3.72



$\vec{r}_1 = R\vec{a}_r - d\vec{a}_z$

$r_1^2 = \vec{r}_1 \cdot \vec{r}_1 = R^2 + d^2 - 2dR\,\vec{a}_r \cdot \vec{a}_z$

$= R^2 + d^2 - 2dR\cos\theta$

$r_1 = \sqrt{R^2 + d^2 - 2dR\cos\theta}$

$r_2 = \sqrt{R^2 + \frac{R^4}{d^2} - \frac{2R^3}{d}\cos\theta}$

Potential at any point $P(r,\phi)$ is

$q' = q\frac{R}{d}$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r^2 + d^2 - 2rd\cos\theta)^{1/2}} - \frac{\frac{R}{d}}{\left(r^2 + \frac{R^4}{d^2} - \frac{2rR^2}{d}\cos\theta\right)^{1/2}}\right]$$

$$\vec{E}\Big|_{r=R} = -\nabla V\Big|_{r=R} = \frac{q}{4\pi\epsilon_0 R}\left[\frac{R^2 - d^2}{r_1^3}\right]\vec{a}_r$$ [we need to compute $\vec{a}_r$ only]

$$\rho_s\Big|_{r=R} = \epsilon_0 \vec{E}\Big|_{r=R} \cdot \vec{a}_r = \frac{q}{4\pi R} \cdot \frac{R^2 - d^2}{r_1^3}$$

$$dQ = \rho_s\,ds = \frac{q}{4\pi R}\,\frac{R^2 - d^2}{r_1^3}\,R^2 \sin\theta\,d\theta\,d\phi$$

$$Q = \frac{q(R^2 - d^2)R^2}{4\pi R}\int_0^{\pi}\frac{1}{r_1^3}\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= \frac{q}{2d}\left[R^2 - d^2\right]\int_{d-R}^{d+R}\frac{dr_1}{r_1^2} = -q\frac{R}{d}$$

$r_1^2 = R^2 + d^2 - 2Rd\cos\theta$

$2r_1\,dr_1 = +2Rd\sin\theta\,d\theta$

$dr_1 = \frac{Rd}{r_1}\sin\theta\,d\theta$

or $\sin\theta\,d\theta = \frac{r_1\,dr_1}{Rd}$

when $\theta = 0$ $r_1 = d - R$

when $\theta = \pi$ $r_1 = d + R$