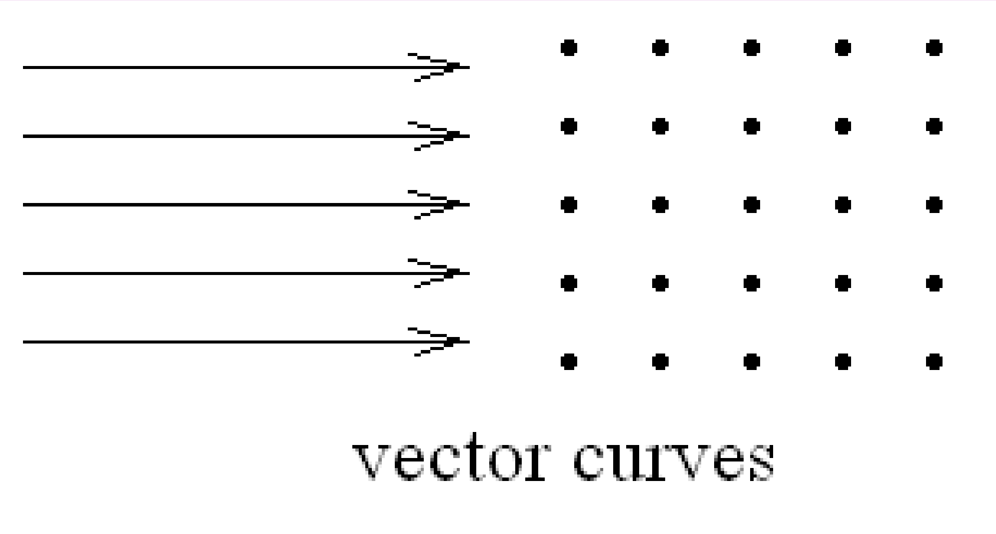
§ 2.9 The flux and divergence of a Vector field

➢1.the flux of a vector field



- （1）**differential surface element**

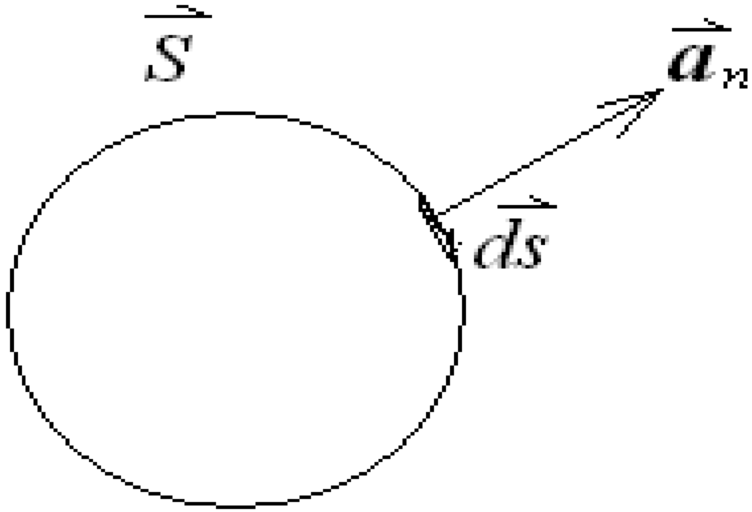
the differential length element

$$d\vec{l} = dx\vec{\mathbf{a}}_x + dy\vec{\mathbf{a}}_y + dz\vec{\mathbf{a}}_z$$

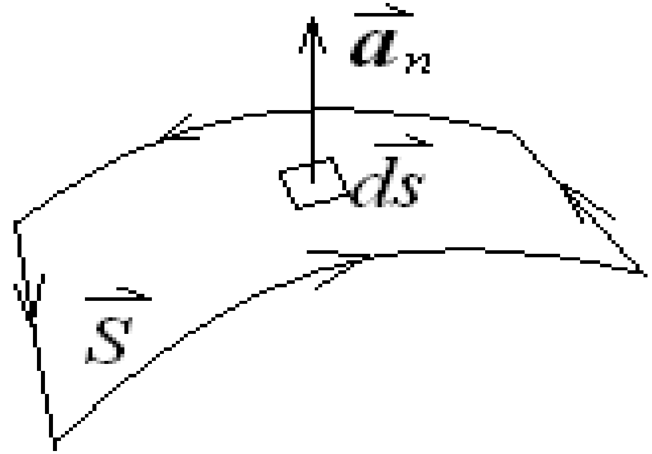
while a differential surface element can be written as

$$d\vec{s} = \vec{a}_n ds$$

$d\vec{s} = \vec{a}_n ds$ **is a vector, its magnitude is** ***ds*****; its direction is** $\vec{a}_n$**, which is normal to the surface** ***ds*****. Generally, for an enclosed surface, the outward direction** $\vec{a}_n$ **is positive; for an opened surface, right-hand rule Will define the direction of** $\vec{a}_n$



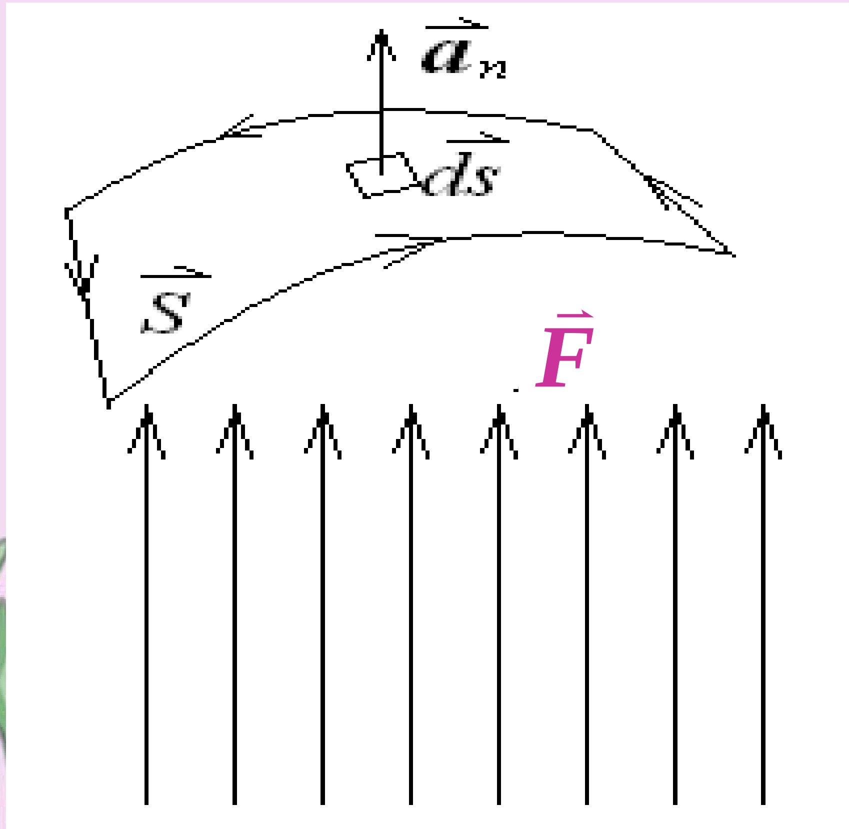
an enclosed surface



an opened surface

(2) Concept

The total of vector curves that are perpendicular to the surface $\vec{S}$ is called as the flux of a vector field $\vec{F}$



The flux can be expressed by $\int_s \vec{F} \bullet d\vec{s}$

where $\vec{F} \bullet d\vec{s}$ **defines the flow of the vector field** $\vec{F}$ **through the surface** $d\vec{s}$ **. thus, the integral of** $\int_s \vec{F} \bullet d\vec{s}$ **defines the flow of the vector field** $\vec{F}$ **through the whole surface** $\vec{S}$

- **(3) calculus and discussion**

$\vec{F} \bullet d\vec{s} = F\vec{a}_F \bullet \vec{a}_n ds = F ds \cos\theta$ **(dot product of vector fields)**

θ **is angle between the direction of** $\vec{F}$ **and the direction of the differential surface element** $d\vec{s}$ **,** F **and** ds **are their magnitude respectively.**

So $\int_s \vec{F} \bullet d\vec{s} = \int_s \vec{F} \bullet \vec{a}_n ds = \int_s F \cos\theta ds$

especially, for an enclosed surface, the flux is

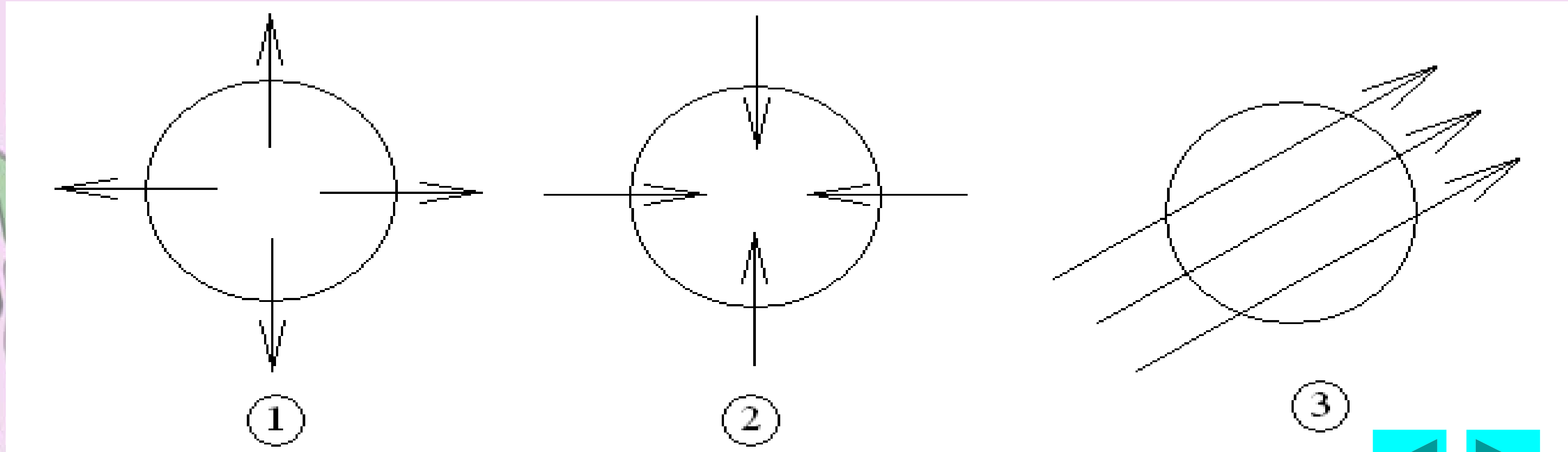
$$\oint_s \vec{F} \bullet d\vec{s} = \oint_s \vec{F} \bullet \vec{a}_n ds = \oint_s F \cos\theta ds$$

$$\oint_s \vec{F} \bullet d\vec{s} = \oint_s \vec{F} \bullet \vec{a}_n ds = \oint_s F \cos\theta ds$$

①>0 the net outward flow is positive, a source point as shown in the following figure.

②<0 the net outward flow is negative, a sink point as shown in the following figure.

③=0 no net outward flow, the vector is continuous. no sources or sinks.



$\oint_s \vec{F} \bullet d\vec{s}$ is a surface integral. It can illustrate the fact that a source exists or does not within an enclosed surface, but it can not illustrate the distribution of source points within an enclosed surface.

- 2. Divergence of a vector field

➢(1)concept

Let us consider a point *P* which is enclosed by volume ΔV bounded by an enclosed surface *s*. The flow of a vector field $\vec{F}$ through point *P* can be obtained by $\Delta V \to 0$, namely, we take the limit for $\oint_s \vec{F} \bullet d\vec{s}$

$$\lim_{\Delta v \to 0} \frac{\oint_s \vec{F} \bullet d\vec{s}}{\Delta v}$$

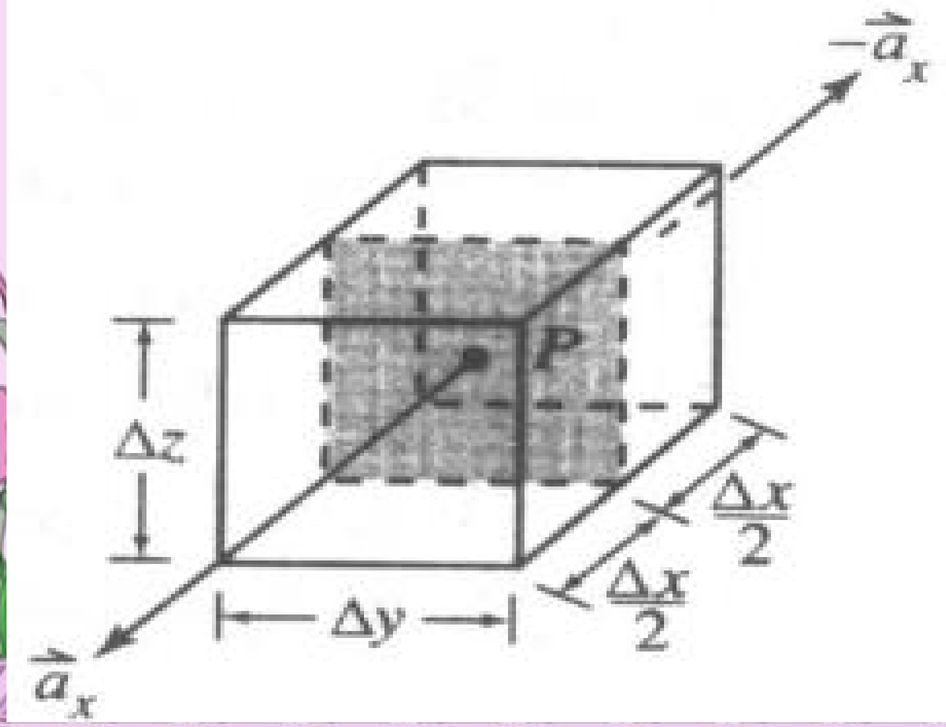
is called the divergence of the vector field $\vec{F}$

Namely, Div $\vec{F} = \lim_{\Delta V \to 0} \frac{\oint_s \vec{F} \bullet d\vec{s}}{\Delta V}$

(2) Some subjects about calculating

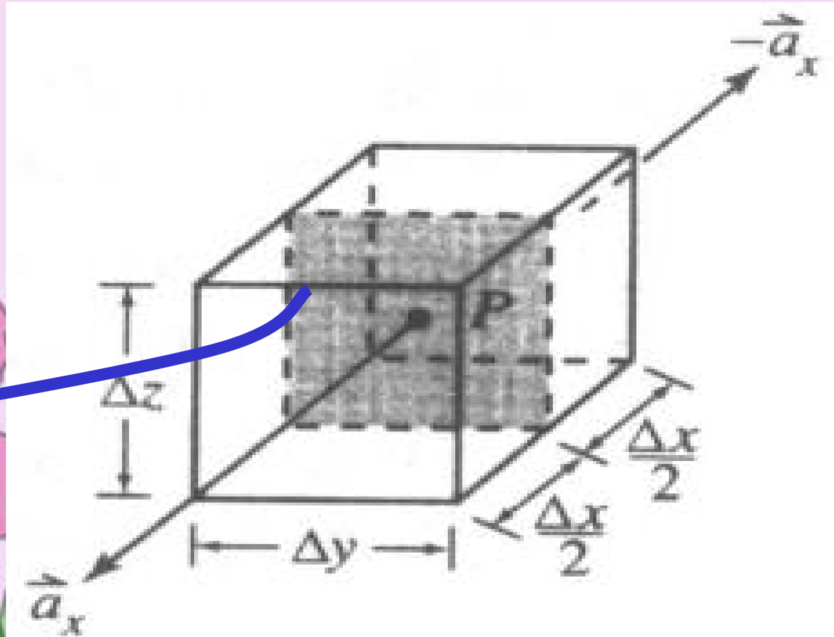
$$\oint_s \vec{F} \bullet d\vec{s}$$

- **Defines the outward flow of the vector field $\vec{F}$ through the surface $d\vec{s}$ as the unit normal to *ds* points away from the volume enclosed. Thus, it gives the net outward flow of flux of a vector field $\vec{F}$ from the volume Δv.**



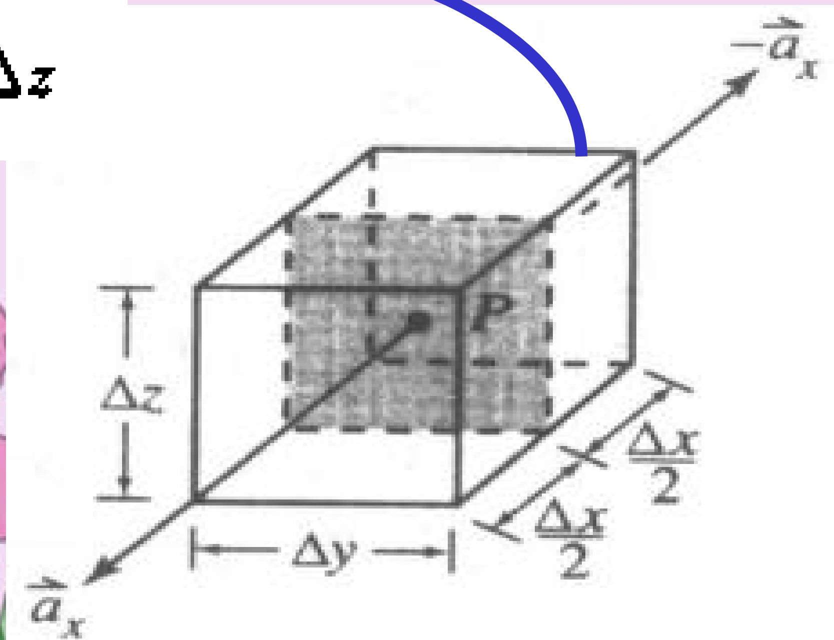
- **The outward flow of the field through the face in the positive *x* direction, using the Taylor series expansion and neglecting the higher-order terms, is**

$$\left[F_x + \frac{\partial F_x}{\partial x}\frac{\Delta x}{2}\right]\Delta y \Delta z$$



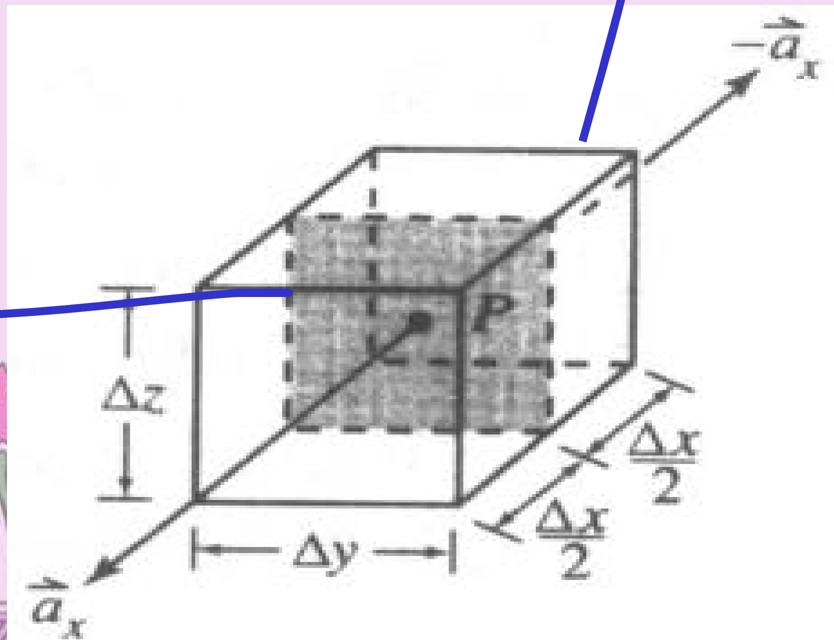
- **The outward flow of the field through the face in the negative *x* direction, using the Taylor series expansion and neglecting the higher-order terms, is**

$$-\left[F_x - \frac{\partial F_x}{\partial x}\frac{\Delta x}{2}\right]\Delta y \Delta z$$



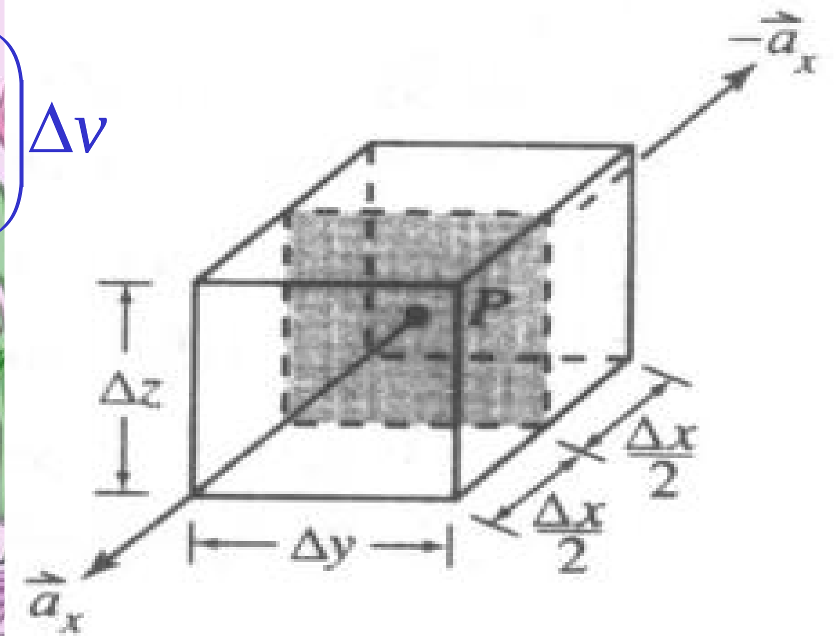
- **Therefore, the net outward flow of the vector field $\vec{F}$ through both the surfaces in the *x* direction is**

$$\frac{\partial F_x}{\partial x}\Delta x \Delta y \Delta z = \frac{\partial F_x}{\partial x}\Delta v$$



- **We can similarly obtain expressions for the net outward flow of the vector field $\vec{F}$ through the surfaces in the *y* and *z* directions.**
- **The net outward flow of the vector field $\vec{F}$ through all the surfaces enclosing the volume Δv then becomes**

$$\oint_s \vec{F} \bullet d\vec{s} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \Delta v$$



$$\textbf{Thus, div } \vec{F} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{F} \bullet d\vec{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \Delta v}{\Delta v}$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\mathrm{div}\,\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \right) \bullet \left(\vec{a}_x F_x + \vec{a}_y F_y + \vec{a}_z F_z \right)$$

$$= \nabla \bullet \vec{F}$$

We can write this divergence of $\vec{F}$ as $\nabla \bullet \vec{F}$

- Review:

for a vector field $\vec{F}$

$$\mathrm{div}\vec{F} = \nabla \bullet \vec{F}$$

in the rectangular coordinate system,

$$\nabla \bullet \vec{F} = \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \right) \bullet \vec{F}$$

$$= \vec{a}_x \bullet \frac{\partial}{\partial x} \vec{F} + \vec{a}_y \bullet \frac{\partial}{\partial y} \vec{F} + \vec{a}_z \bullet \frac{\partial}{\partial z} \vec{F}$$

$$
\begin{aligned}
&= \vec{\boldsymbol{a}}_x \bullet \frac{\partial}{\partial x}\left(\vec{\boldsymbol{a}}_x F_x + \vec{\boldsymbol{a}}_y F_y + \vec{\boldsymbol{a}}_z F_z\right) + \vec{\boldsymbol{a}}_y \bullet \frac{\partial}{\partial y}\left(\vec{\boldsymbol{a}}_x F_x + \vec{\boldsymbol{a}}_y F_y + \vec{\boldsymbol{a}}_z F_z\right) \\
&\qquad + \vec{\boldsymbol{a}}_z \bullet \frac{\partial}{\partial z}\left(\vec{\boldsymbol{a}}_x F_x + \vec{\boldsymbol{a}}_y F_y + \vec{\boldsymbol{a}}_z F_z\right)
\end{aligned}
$$

$$
= \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z
$$

$$
= \left(\vec{\boldsymbol{a}}_x \frac{\partial}{\partial x} + \vec{\boldsymbol{a}}_y \frac{\partial}{\partial y} + \vec{\boldsymbol{a}}_z \frac{\partial}{\partial z}\right) \bullet \left(\vec{\boldsymbol{a}}_x F_x + \vec{\boldsymbol{a}}_y F_y + \vec{\boldsymbol{a}}_z F_z\right)
$$

➢(3)the divergence theorem

It states that for a continuously differentiable vector field the net outward flux from a closed surface equals the integral of the divergence throughout the region bounded by that surface.

$$\oint_S \vec{F} \bullet d\vec{s} = \int_V \nabla \bullet \vec{F} dV$$

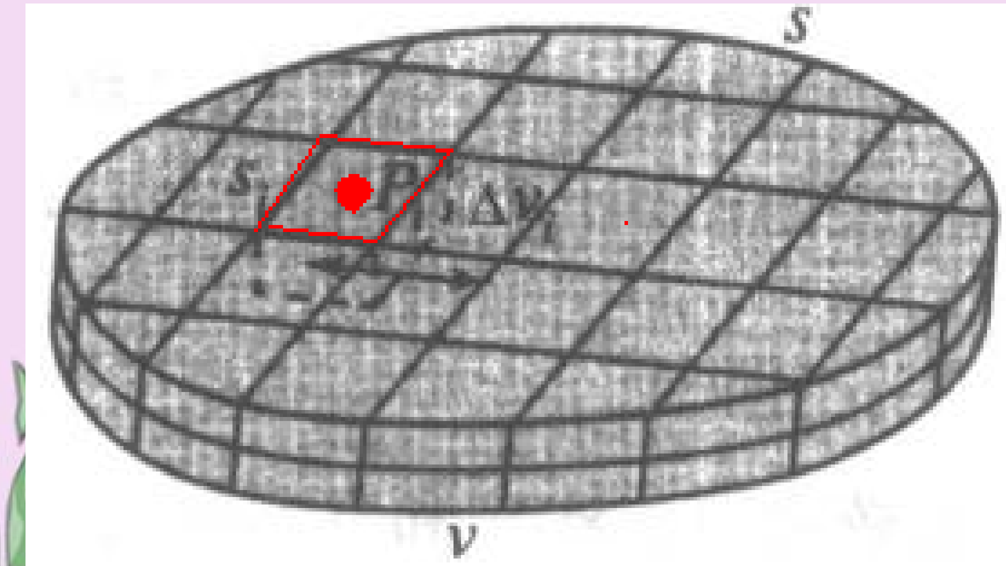
It is used to convert a closed surface integral into an equivalent volume integral and vice versa.

➢(3)the divergence theorem

if the vector field $\vec{F}$ is continuously differentiable in a region of volume V bounded by the surface s, the definition of divergence can be extended to cover the entire volume. This is done by

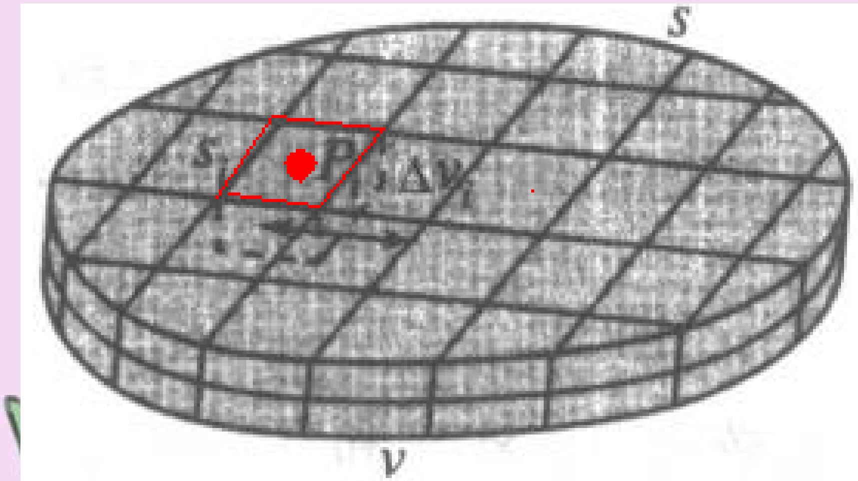
subdividing the volume *V* into *n* elementary volume (cells), all of which approach zero in the limit. That is, for an elementary volume Δv_i enclosing a point P_i and bounded by a surface S_i, the divergence of the vector field $\vec{F}$

at P_i is



$$\nabla\bullet\vec{F}_i = \lim_{\Delta v_i\to 0}\frac{\oint_{si}\vec{F}\bullet d\vec{s}}{\Delta v_i}$$

where $\vec{F}_i$ is the value of the vector field $\vec{F}$ at point P_i. From the knowledge of maths, We can rewrite the above equation as



$$\oint_{si}\vec{F}\bullet d\vec{s} = \nabla\bullet\vec{F}_i\Delta v_i + \varepsilon_i\Delta v_i$$

where $\varepsilon_i\to 0$ as $\Delta v_i\to 0$.

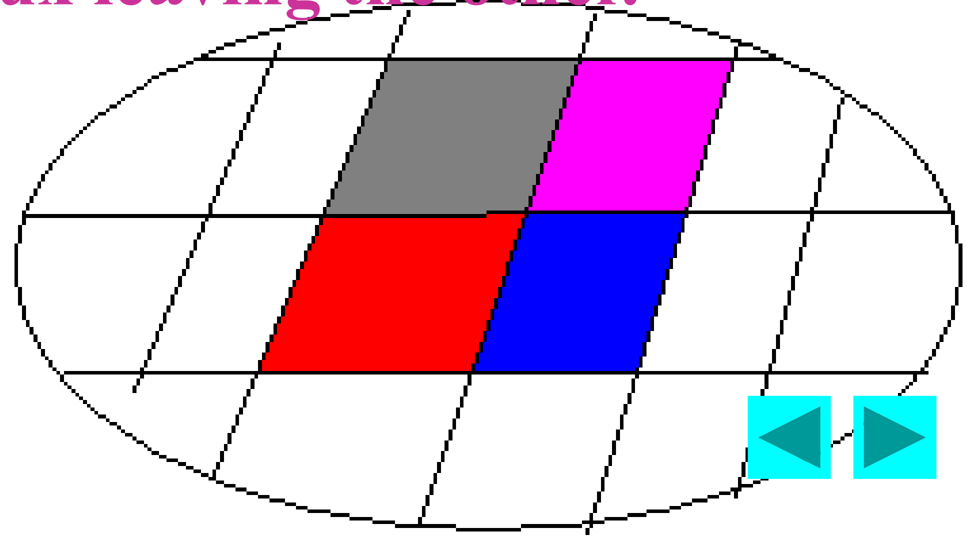
$$\oint F \bullet ds \quad \nabla \bullet F_i \Delta v_i + \varepsilon_i \Delta v_i$$

Summing for all cells, we obtain

$$\lim_{n\to\infty}\sum_{i}^{n}\oint_{si}\vec{F}\bullet d\vec{s} = \lim_{n\to\infty}\sum_{i}^{n}\nabla\bullet\vec{F}_i\Delta v_i + \lim_{n\to\infty}\sum_{i}^{n}\varepsilon_i\Delta v_i \quad \textbf{(2.9-1)}$$

•The left-hand side of the equation:

Observe that the surface integrals over the interfaces of the two cells within *v* vanish as the net flux leaving one cell cancels the net flux leaving the other.



Thus , the nonzero terms in the sum correspond to the outmost cells that belong to the surface *S*.

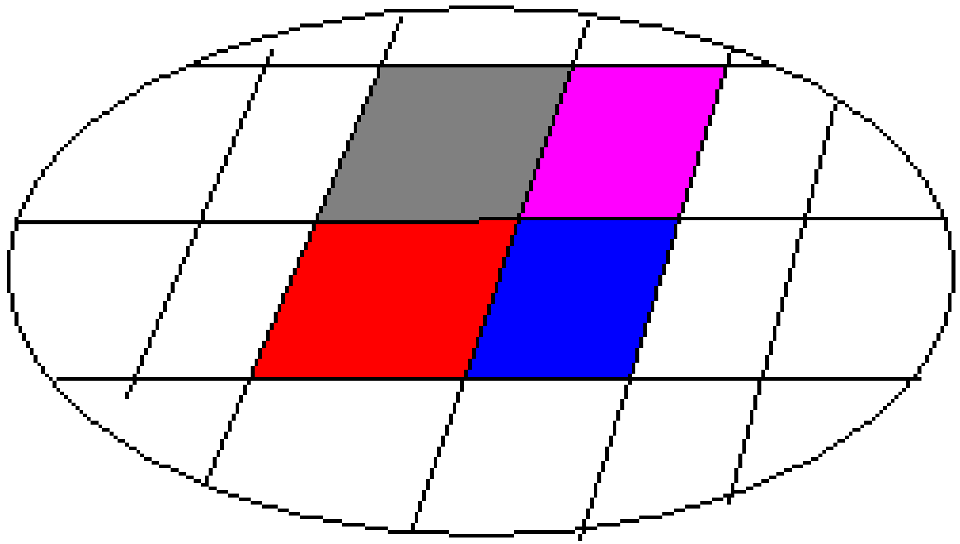
Hence, the left-hand side of the equation becomes

$$\lim_{n\to\infty}\sum_{i}^{n}\oint_{si}\vec{F}\bullet d\vec{s}=\oint_{s}\vec{F}\bullet d\vec{s}$$

•**the right-hand side of the equation:**

As the number of cells increases, the first term on the right-hand side of the equation, in the limit, becomes

$$\lim_{n\to\infty}\sum_{i}^{n}\nabla\bullet\vec{F}_i\Delta v_i=\int_V\nabla\bullet\vec{F}dV$$



the second term on the right-hand side of equation(2.9-1) involves the product of small quantities and vanishes as $n \to \infty$(infinity [in′finiti])

therefore, we can write (2.9-1) in the limit as

$$\oint_s \vec{F} \bullet d\vec{s} = \int_V \nabla \bullet \vec{F} dV \qquad (2.9\text{-}2)$$

Equation (2.9-2) is a mathematic definition of the divergence theorem.

It relates the volume integral of the divergence of a vector field to the surface integral of its normal component.

It states that for a continuously differentiable vector field the net outward flux from a closed surface equals the integral of the divergence throughout the region bounded by that surface.

$$\oint_S \vec{F} \bullet d\vec{s} = \int_V \nabla \bullet \vec{F} dV$$

It is used to convert a closed surface integral into an equivalent volume integral and vice versa.

Some examples:

example 1.

Verify the divergence theorem for a vector field $\vec{r} = \vec{a}_x x + \vec{a}_y y + \vec{a}_z z$ in the region bounded by the surface $x^2+y^2+z^2= r^2$

❖Solution

$$\nabla \bullet \vec{\mathbf{r}} = \left(\vec{\mathbf{a}}_x \frac{\partial}{\partial x} + \vec{\mathbf{a}}_y \frac{\partial}{\partial y} + \vec{\mathbf{a}}_z \frac{\partial}{\partial z} \right) \bullet \left(\vec{\mathbf{a}}_x x + \vec{\mathbf{a}}_y y + \vec{\mathbf{a}}_z z \right) = 3$$

$$\int_V \nabla \bullet \vec{\mathbf{r}} dV = \int_V 3dV = 3 \times \frac{4}{3} \pi r^3 = 4\pi r^3$$

$$\oint_s \vec{r} \bullet d\vec{s} = \oint_s r\vec{a}_r \bullet \vec{a}_n ds = \oint_s r\vec{a}_r \bullet \vec{a}_r ds$$

$$= r \times 4\pi r^2$$

$$= 4\pi r^3$$

Example 2: example 2.20

you read it by yourselves.

Exercises

page 66 : T2.35, T2.37;
Example 2.20