第4章　插值方法(3)

Hermite插值多项式

- 构造 $H_{2n+1}(\mathrm{x})$

已知： $x_i,\ y_i = f(x_i),\ y_i' = f'(x_i),\quad i = 0,1,\cdots n$

要求：$H_{2n+1}(x_i) = y_i,\qquad H_{2n+1}'(x_i) = y_i' \qquad (i = 0,1,2,\ldots n)$

假设：

$$H_{2n+1}(x) = \sum_{j=0}^{n} \alpha_j(x) y_j + \sum_{j=0}^{n} \beta_j(x) y_j'$$

Charles Hermite

Charles Hermite (December 24, 1822 – January 14, 1901) was a French mathematician who did research on number theory, quadratic forms, invariant theory, orthogonal polynomials, elliptic functions, and algebra.

Hermite polynomials, Hermite interpolation, Hermite normal form, Hermitian operators, and cubic Hermite splines are named in his honor. One of his students was Henri Poincaré.

He was the first to prove that *e*, the base of natural logarithms, is a transcendental number (超越数). His methods were later used by Ferdinand von Lindemann to prove that π is transcendental.

In a letter to Thomas Stieltjes in 1893, Hermite famously remarked: "I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives."

插值基函数 $\alpha_0(x)$

x	$x_0,$	$x_1,$	$\cdots,$	x_n
y	$1,$	$0,$	$\cdots,$	0
y'	$0,$	$0,$	$\cdots,$	0

$$\alpha_0(x)=(ax+b)l_0^2(x)=(ax+b)\frac{(x-x_1)^2(x-x_2)^2\ldots(x-x_n)^2}{(x_0-x_1)^2(x_0-x_2)^2\ldots(x_0-x_n)^2}$$

插值基函数　$\beta_0(x)$

x	$x_0,$	$x_1,$	$\cdots,$	x_n
y	0,	0,	$\cdots,$	0
y'	1,	0,	$\cdots,$	0

$$\beta_0(x) = (Ax+B)l_0^2(x) = (Ax+B)\frac{(x-x_1)^2(x-x_2)^2\ldots(x-x_n)^2}{(x_0-x_1)^2(x_0-x_2)^2\ldots(x_0-x_n)^2}$$

$$\alpha_0(x)=(ax+b)l_0^2(x)$$

$$\alpha_0(x_0)=1 \qquad \alpha_0(x_0)=(ax_0+b)l_0^2(x_0)=ax_0+b=1$$

$$\alpha_0'(x_0)=0 \qquad \alpha_0'(x_0)=al_0^2(x_0)+2l_0(x_0)l_0'(x_0)(ax_0+b)=a+2l_0'(x_0)(ax_0+b)=0$$

$$l_0(x)=\frac{\omega_{n+1}(x)}{(x-x_0)\omega_{n+1}'(x_0)}$$

$$\omega_{n+1}(x)=(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$$

$$\omega_{n+1}'(x)=(x-x_1)\cdots(x-x_n)+(x-x_0)*\gamma(x)$$

$$\omega_{n+1}'(x_0)=(x_0-x_1)\cdots(x_0-x_n)$$

$$l_0'(x_0)=\frac{1}{\omega_{n+1}'(x_0)}\left[\frac{\omega_{n+1}'(x)}{(x-x_0)}-\frac{\omega_{n+1}(x)}{(x-x_0)^2}\right]_{x=x_0}$$

$$=\frac{1}{\omega_{n+1}'(x_0)}\left[\frac{\omega_{n+1}'(x)}{(x-x_0)}-\frac{(x-x_1)(x-x_2)...(x-x_n)}{(x-x_0)}\right]_{x=x_0} \qquad =\frac{1}{\omega_{n+1}'(x_0)}\left[\frac{\omega_{n+1}'(x)-(x-x_1)(x-x_2)...(x-x_n)}{(x-x_0)}\right]_{x=x_0}$$

$$=\frac{\gamma(x_0)}{\omega_{n+1}'(x_0)}$$

$$ax_0 + b = 1$$

$$a + 2l_0'(x_0)(ax_0 + b) = 0$$

$$a = -2l_0'(x_0)$$

$$b = 1 + 2x_0 l_0'(x_0)$$

$$\begin{aligned}\alpha_0(x) &= \left[-2l_0'(x_0)x + 1 + 2x_0 l_0'(x_0)\right] l_0^2(x) \\ &= \left[1 + 2(x_0 - x)l_0'(x_0)\right] l_0^2(x)\end{aligned}$$

插值基函数 $\beta_0(x)$

x	$x_0,$	$x_1,$	$\cdots,$	x_n
y	$0,$	$0,$	$\cdots,$	0
y'	$1,$	$0,$	$\cdots,$	0

$$\beta_0(x)=(Ax+B)l_0^2(x)=(Ax+B)\frac{(x-x_1)^2(x-x_2)^2\dots(x-x_n)^2}{(x_0-x_1)^2(x_0-x_2)^2\dots(x_0-x_n)^2}$$

$$\beta_0(x) = (Ax + B)l_0^2(x)$$

$$\beta_0(x_0) = 0$$
$$\beta_0'(x_0) = 1$$

$\longrightarrow$

$$\beta_0(x_0) = (Ax_0 + B)l_0^2(x_0) = Ax_0 + B = 0$$
$$\beta_0'(x_0) = Al_0^2(x_0) + 2l_0(x_0)l_0'(x_0)(Ax_0 + B) = A + 2l_0'(x_0)(Ax_0 + B) = 1$$

$\longrightarrow$ $A = 1, \ B = -x_0$

$\longrightarrow$ $\beta_0(x) = (x - x_0)l_0^2(x)$

x	$x_0,\ \cdots,\ x_j,\ \cdots,\ x_n$
y	$0,\ \cdots,\ 1,\ \cdots,\ 0$
y'	$0,\ \cdots,\ 0,\ \cdots,\ 0$

x	$x_0,\ \cdots,\ x_j,\ \cdots,\ x_n$
y	$0,\ \cdots,\ 0,\ \cdots,\ 0$
y'	$0,\ \cdots,\ 1,\ \cdots,\ 0$

$$\alpha_j(x) = \left[1 + 2(x_j - x)l_j'(x_j)\right]l_j^2(x)$$

$$\beta_j(x) = (x - x_j)l_j^2(x)$$

$$H_{2n+1}(x) = \sum_{j=0}^{n} \alpha_j(x) y_j + \sum_{j=0}^{n} \beta_j(x) y_j'$$

Hermite插值余项

$$R(x) = f(x) - H_{2n+1}(x) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi)\omega_{n+1}^2(x)$$

其中，$\xi \in (a,b)$，$\omega_{n+1}^2 = (x-x_0)(x-x_1)\cdots(x-x_n)$

误差估计：

$$R(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2\cdots(x-x_n)^2$$

证明：设 $R(x)=K(x)(x-x_0)^2\cdots(x-x_n)^2$，则

$$g(t)=f(t)-K(x)(t-x_0)^2\cdots(t-x_n)^2$$

有2n+3个零点。根据中值定理，存在

$$g^{(2n+2)}(\xi)=0,\ \ a\le\xi\le b$$

于是 $K(x)=\dfrac{f^{(2n+2)}(\xi)}{(2n+2)!}$。

特例 (n=1)

$$H_3(x_0) = y_0 \qquad H_3(x_1) = y_1$$
$$H_3'(x_0) = y_0' \qquad H_3'(x_1) = y_1'$$

x	x_0	x_1
f	y_0	y_1
f'	y_0'	y_1'

$$\alpha_0(x) = \left(1 + 2\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_1}{x_0 - x_1}\right)^2 \qquad \beta_0(x) = (x - x_0)\left(\frac{x - x_1}{x_0 - x_1}\right)^2$$
$$\alpha_1(x) = \left(1 + 2\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_0}{x_1 - x_0}\right)^2 \qquad \beta_1(x) = (x - x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)^2$$

$$R_3(x) = f(x) - H_3(x) = \frac{1}{4!} f^{(4)}(\xi)(x - x_0)^2 (x - x_1)^2$$

例题

设 $f(x)=\sin x$ ，试用 $f(0)=0$,

$f(\pi/6)=\dfrac{1}{2}$, $f^{'}(0)=1$, $f^{'}(\pi/6)=\dfrac{\sqrt{3}}{2}$ 确定二点三

次 Hermite 插值多项式 $H_3(x)$ 并计算 $H_3(\dfrac{\pi}{12})$ 的值。

$$H_3(x)=\left[1+2\frac{x-0}{\pi/6}\times 0+(x-0)\times 1\right]\left[\frac{x-\pi/6}{\pi/6}\right]^2$$

$$+\left[\left(1-2\frac{x-\pi/6}{\pi/6}\right)\times\frac{1}{2}+(x-\pi/6)\times\frac{\sqrt{3}}{2}\right]\left[\frac{x-0}{\pi/6}\right]^2$$

$$=x\left(\frac{6x}{\pi}-1\right)^2+\left[\left(\frac{3}{2}-\frac{6x}{\pi}\right)+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)\right]\frac{36}{\pi^2}x^2$$

$$H_3\left(\frac{\pi}{12}\right)=\frac{\pi}{48}+\frac{1}{4}-\frac{\sqrt{3}}{96}\pi=0.258768616$$

$$\sin\frac{\pi}{12}=0.258819045$$

分段低次插值法

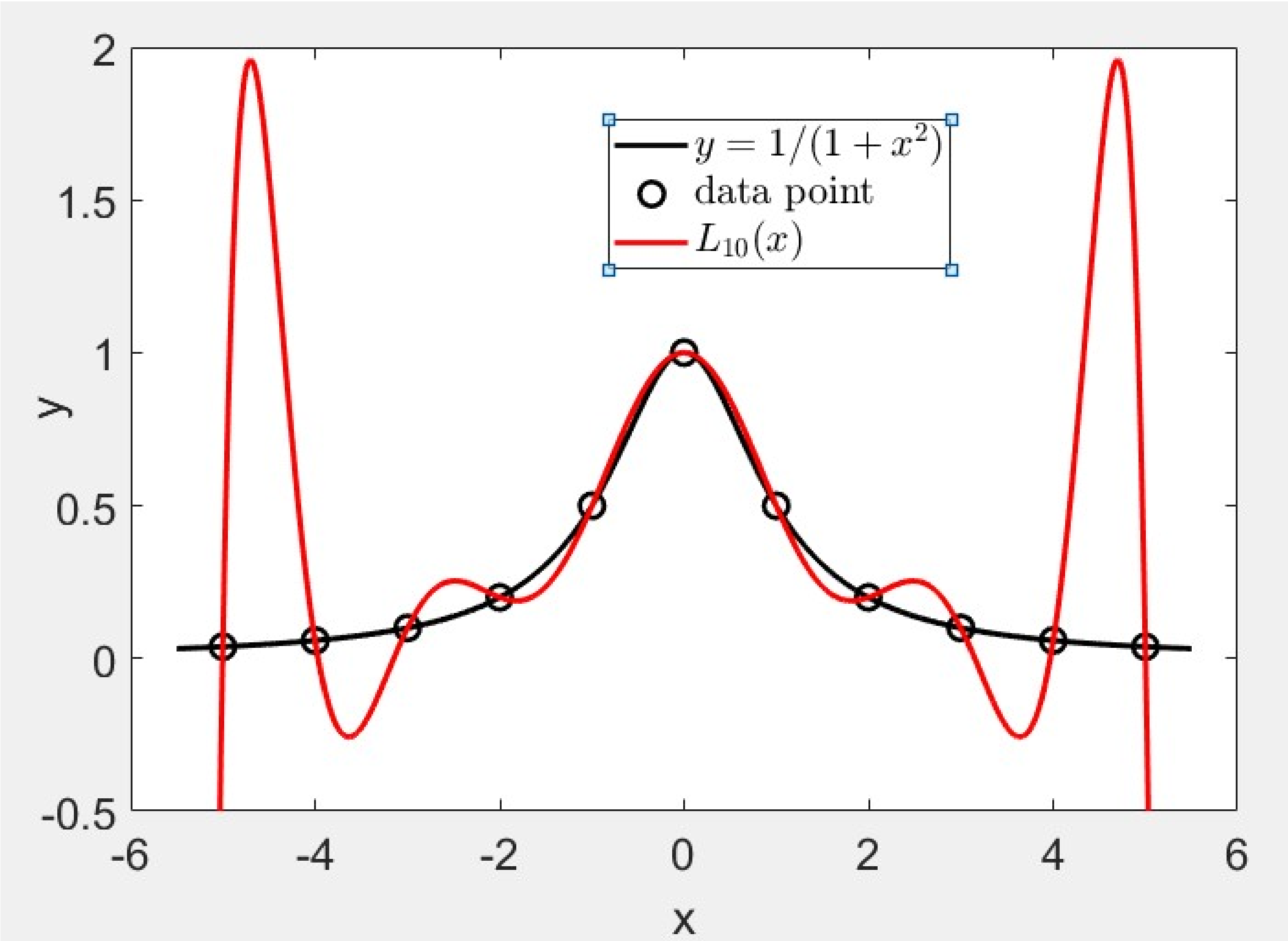
- 高次插值中的问题

一般地说，适当提高插值多项式的次数，有可能提高计算结果的准确程度，但决不可由此得出结论，认为插值多项式的次数越高越好。例如，对于函数 $f(x)=1/(1+x^2)$

Lagrange插值多项式

$$L_n(x)=\sum_{k=0}^{n} f(x_k)\left(\prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x-x_i}{x_k = x_i}\right)$$

Runge现象：并非插值点取得越多越好。



解决办法：分段插值

Carl Runge

Carl David Tolmé Runge (German: 1856–1927) was a German mathematician, physicist, and spectroscopist.
He was co-developer and co-eponym of the Runge–Kutta method in the field of what is today known as numerical analysis

In 1880, he received his Ph.D. in mathematics at Berlin, where he studied under Karl Weierstrass. In 1886, he became a professor in Hannover, Germany.

His interests included mathematics, spectroscopy, geodesy, and astrophysics. In addition to pure mathematics, he did a great deal of experimental work studying spectral lines of various elements (together with Heinrich Kayser), and was very interested in the application of this work to astronomical spectroscopy.

In 1904, on the initiative of Felix Klein he received a call to the Georg-August University of Göttingen, which he accepted. There he remained until his retirement in 1925.

The crater Runge (Runge火山) on the Moon is named after him.

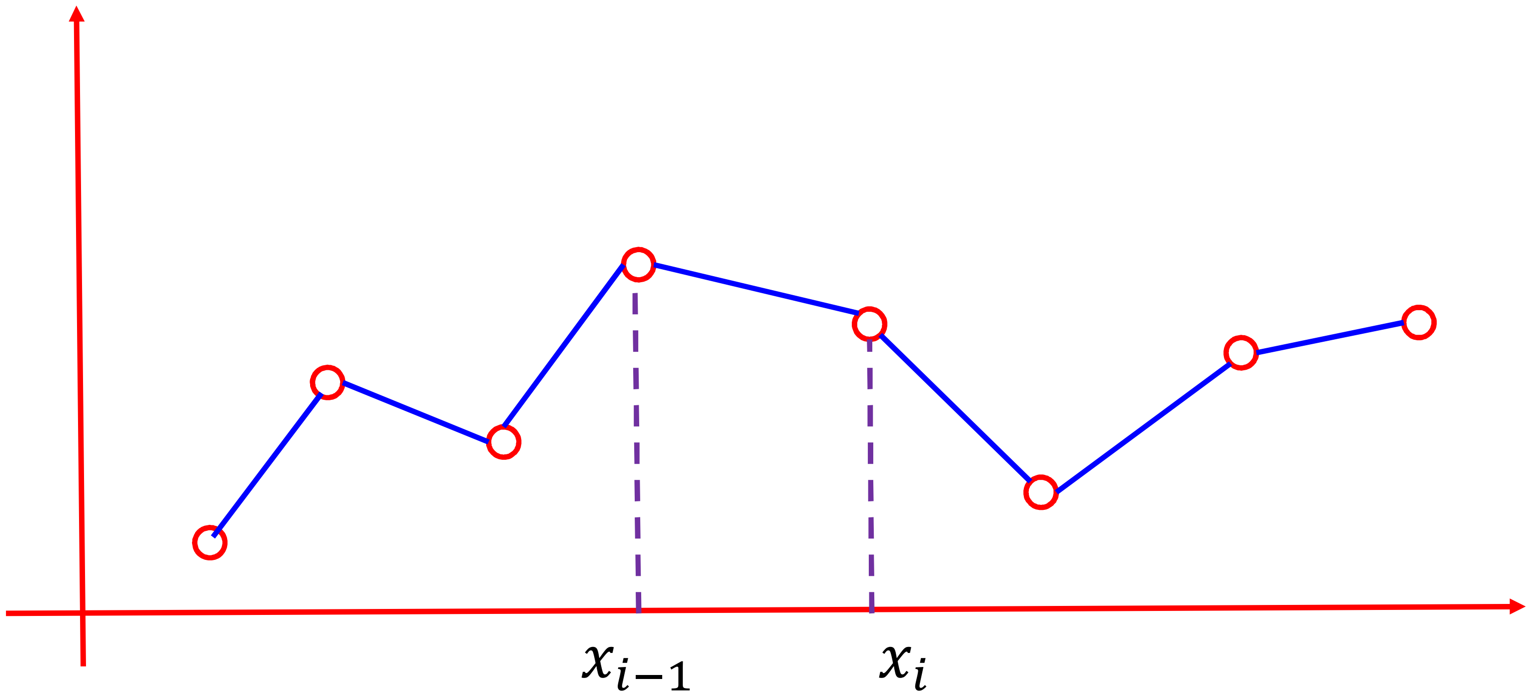
当 $n=10$ 时，$f(x)$ 与 $L_{10}(x)$ 偏差很大。出现这种现象的原因:

(1) 据Lagrange插值余项估计式(4.11)，当插值节点加密, n 增大时，有时 $M_{n+1}=\max\limits_{a\le x\le b}\left|f^{(n+1)}(x)\right|$ 可能非常大；特别当插值节点比较分散、插值区间较大时, $\left|\omega_{n+1}(x)\right|$也较大。

(2) 当 n 增大时, Lagrange插值多项式次数增大，计算量的增幅也是巨大的，这就加大了计算过程中的舍入误差。

分段线性插值

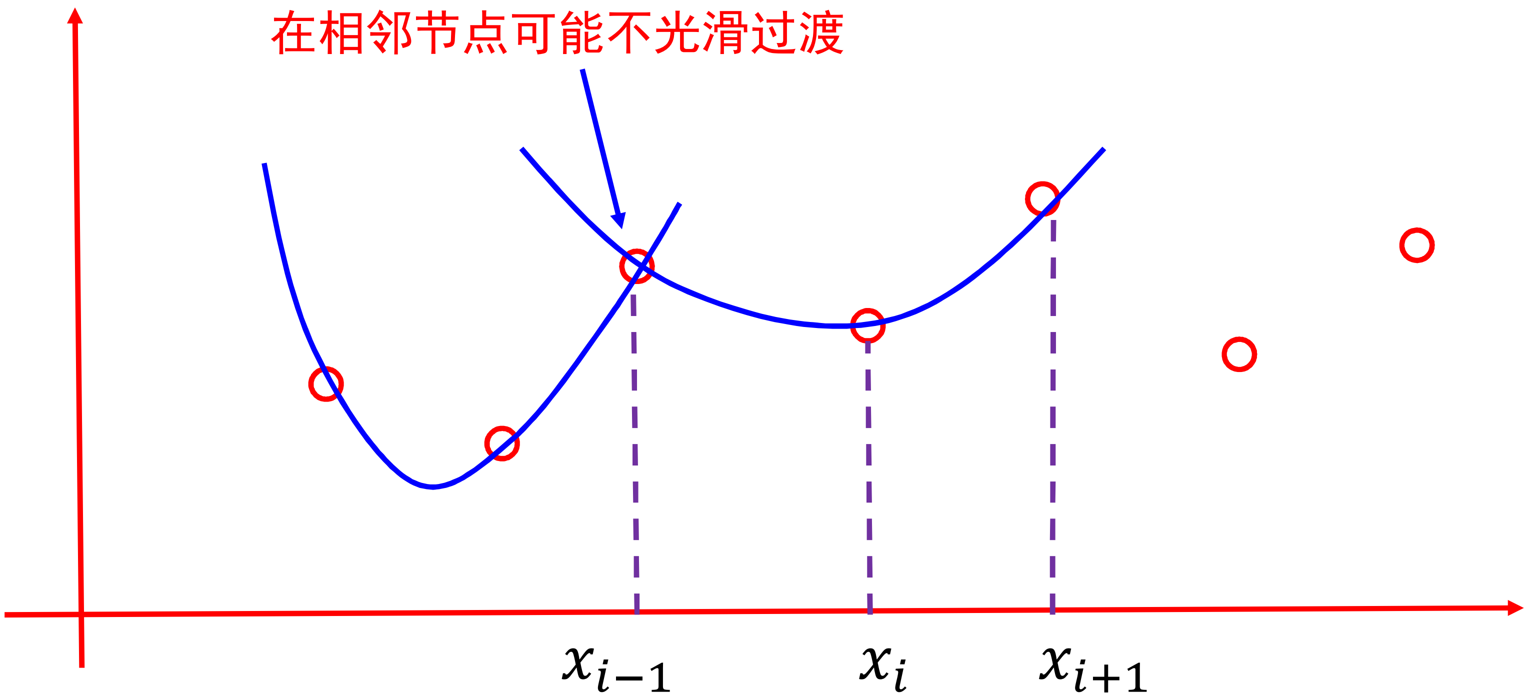
$$a = x_0 < x_1 < \cdots < x_n = b \qquad [x_{i-1}, x_i]\ (i = 1,2,\cdots,n)$$



$$L_{1i}(x) = \frac{x - x_i}{x_{i-1} - x_i} y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i \quad (x_{i-1} \le x \le x_i; i = 1,2,\cdots,n)$$

分段二次插值

- 对于插值节点x_{i-1}, x_i，x_{i+1}，在小区间 $[x_{i-1}, x_{i+1}]$ 内作二次插值

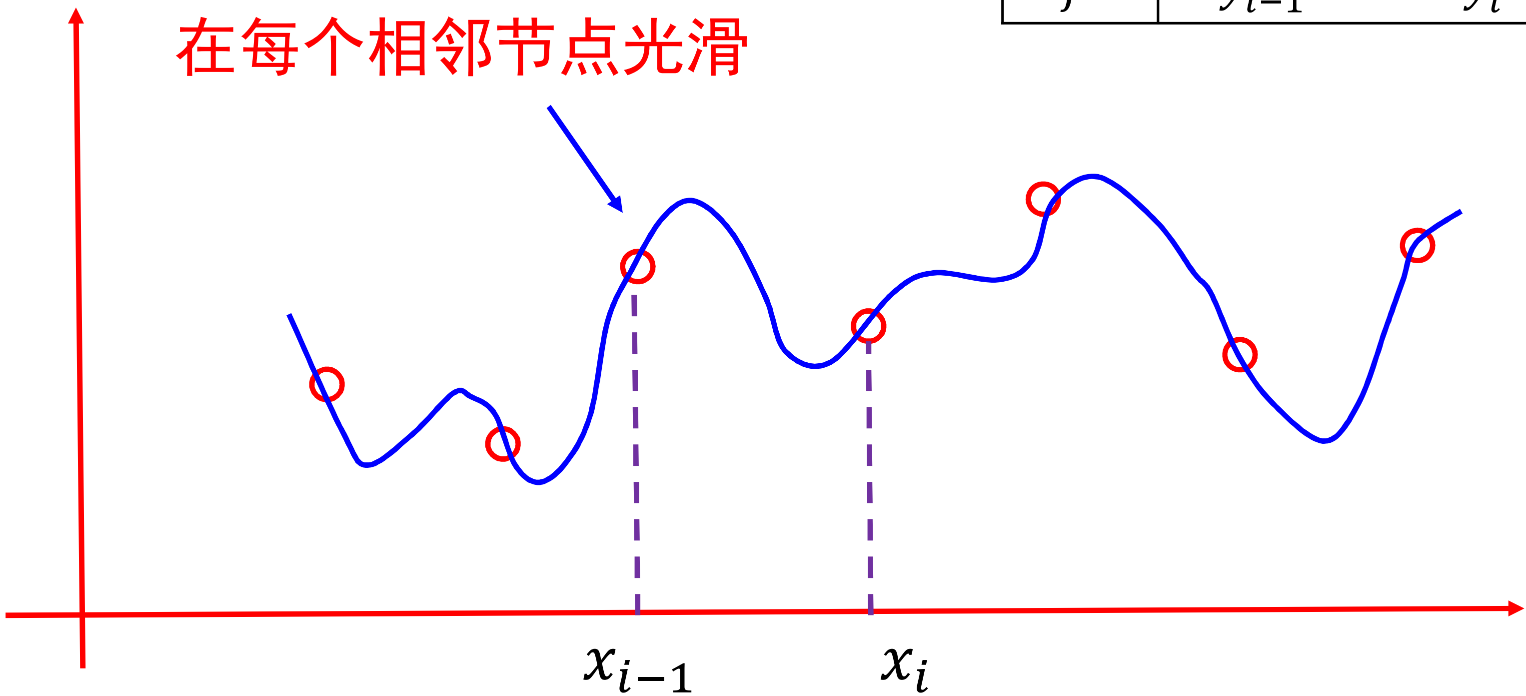


$$L_{2i}(x) = \frac{(x - x_i)(x - x_{i+1})}{(x_{i-1} - x_i)(x_{i-1} - x_{i+1})} y_{i-1} + \frac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})} y_i + \frac{(x - x_i)(x - x_{i-1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})} y_{i+1}$$

$$(x_{i-1} \le x \le x_{i+1})$$

分段 Hermite 插值

x	x_{i-1}	x_i
f	y_{i-1}	y_i
f'	y'_{i-1}	y'_i



$$H_3^i(x) = y_{i-1}\alpha_0^i(x) + y_i\alpha_1^i(x) + y'_{i-1}\beta_0^i(x) + y'_i\beta_1^i(x)$$

$$\alpha_0^i(x) = \left(1 + 2\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_i}{x_{i-1} - x_i}\right)^2 \qquad \alpha_1^i(x) = \left(1 + 2\frac{x - x_i}{x_{i-1} - x_i}\right)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 \qquad \beta_0^i(x) = (x - x_{i-1})\left(\frac{x - x_i}{x_{i-1} - x_i}\right)^2 \qquad \beta_1^i(x) = (x - x_i)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2$$

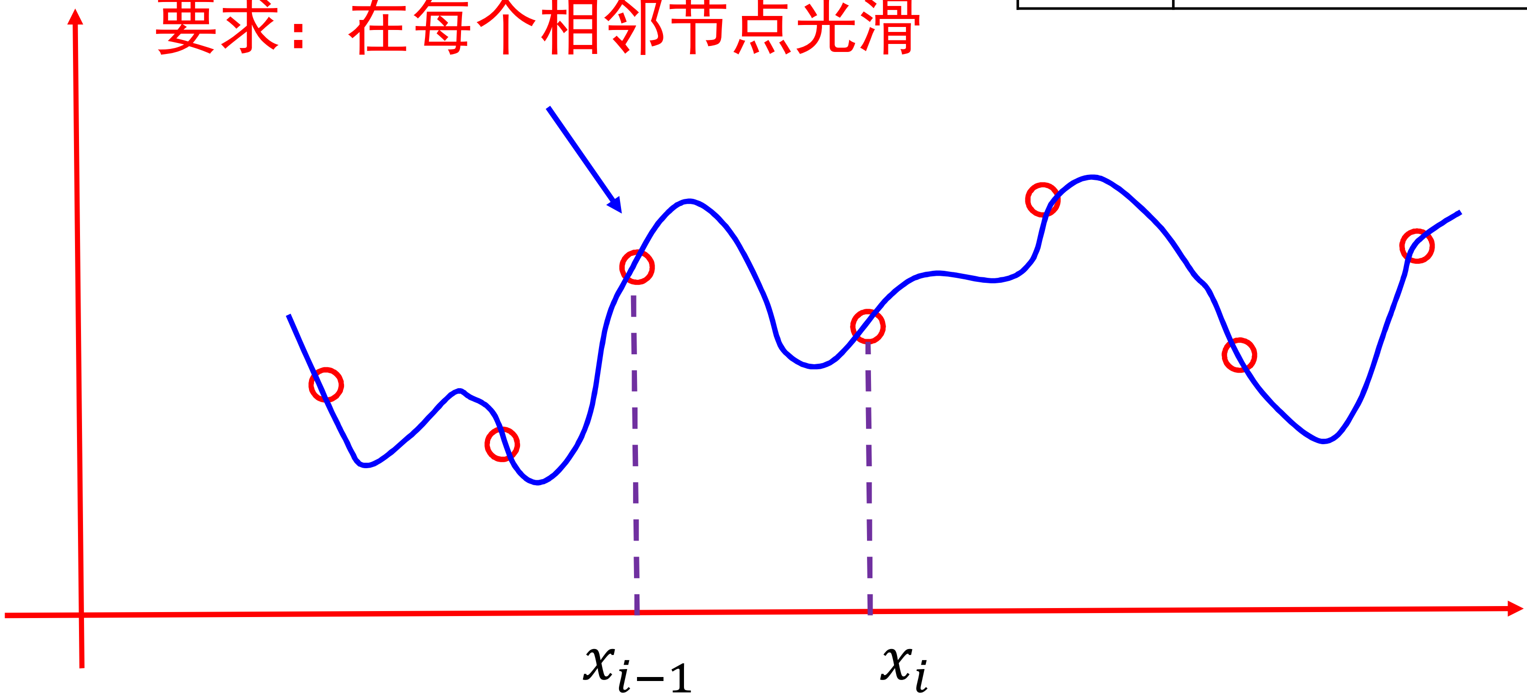
分段3次样条插值

Hermite 需要知道导数值，不知道时候怎么办？

方法：利用光滑性求出来

x	x_{i-1}	x_i
f	y_{i-1}	y_i
f'	?	?

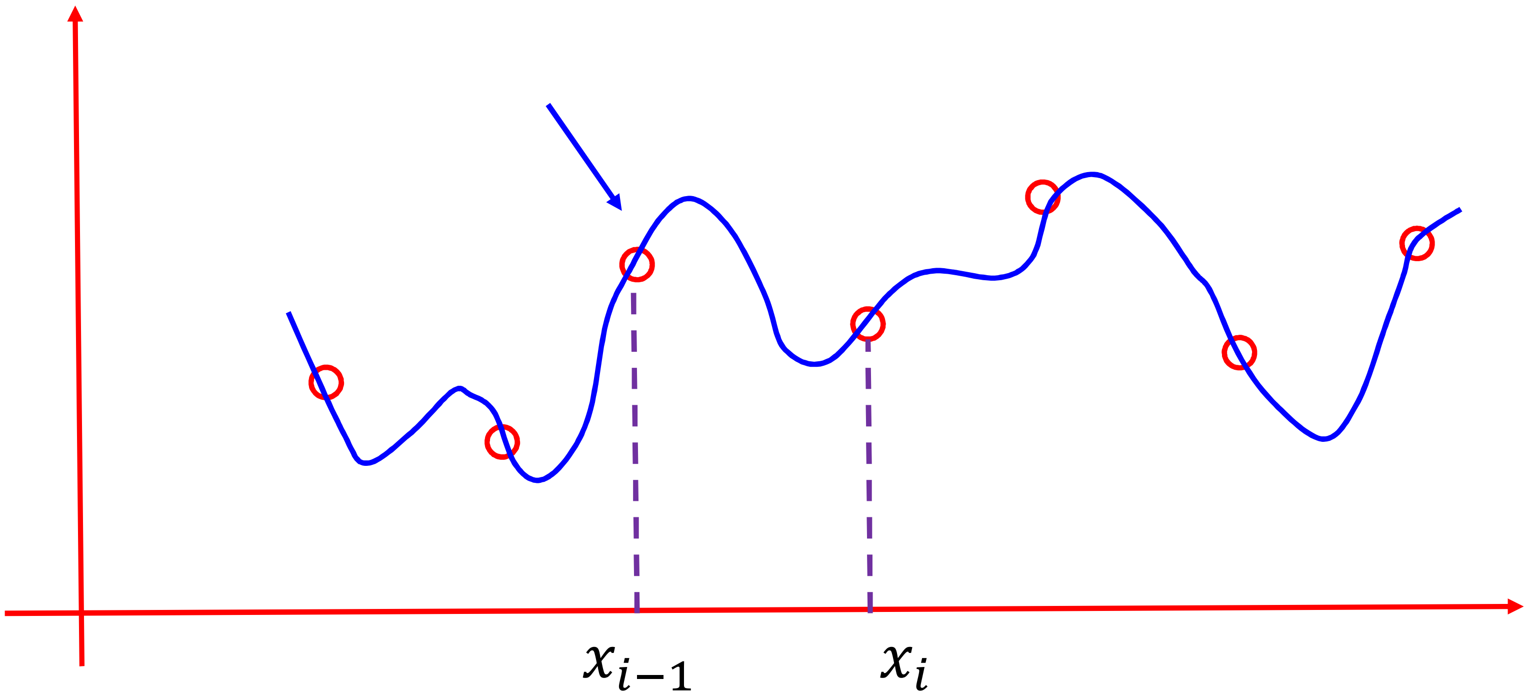
要求：在每个相邻节点光滑



分段3次样条插值

$$S_3^i(x) = y_{i-1}\alpha_0^i(x) + y_i\alpha_1^i(x) + m_{i-1}\beta_0^i(x) + m_i\beta_1^i(x)$$

要求：
$$\begin{cases} [S_3^{i-1}(x_{i-1})] = [S_3^i(x_{i-1})] = y_{i-1} \\ [S_3^{i-1}(x_{i-1})]' = [S_3^i(x_{i-1})]' = m_{i-1} \\ [S_3^{i-1}(x_{i-1})]'' = [S_3^i(x_{i-1})]'' \end{cases}$$



分段3次样条插值

$$(1-a_i)m_{i-1} + 2m_i + a_i m_{i+1} = b_i$$

$$a_i = \frac{h_{i-1}}{h_{i-1} + h_i}, \quad b_i = 3\left[(1-a_i)\frac{y_i - y_{i-1}}{h_{i-1}} + a_i \frac{y_{i+1} - y_i}{h_i}\right]$$

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Discrete unified gas kinetic scheme for all Knudsen number flows: Low-speed isothermal case

Zhaoli Guo,[1,*] Kun Xu,[2,†] and Ruijie Wang[3,‡]

[1] *State Key Laboratory of Coal Combustion, Huazhong University of Science and Technology, Wuhan 430074, China*

[2] *Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China*

[3] *Nano Science and Technology Program, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China*

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Based on the Boltzmann-BGK (Bhatnagar-Gross-Krook) equation, in this paper a discrete unified gas kinetic scheme (DUGKS) is developed for low-speed isothermal flows. The DUGKS is a finite-volume scheme with the discretization of particle velocity space. After the introduction of two auxiliary distribution functions with the inclusion of collision effect, the DUGKS becomes a fully explicit scheme for the update of distribution function. Furthermore, the scheme is an asymptotic preserving method, where the time step is only determined by the Courant-Friedricks-Lewy condition in the continuum limit. Numerical results demonstrate that accurate solutions in both continuum and rarefied flow regimes can be obtained from the current DUGKS. The comparison between the DUGKS and the well-defined lattice Boltzmann equation method (D2Q9) is presented as well.

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flows, numer

where $\bar{f}^+(\boldsymbol{x}_b,\boldsymbol{\xi},t_n)$ and the gradient $\boldsymbol{\sigma}_b = \nabla \bar{f}^+(\boldsymbol{x}_b,\boldsymbol{\xi},t_n)$ at the cell interface can be approximated by linear interpolations. For example, in the one-dimensional case shown in Fig. 1, the reconstructions become

$$\sigma_{j+1/2} = \frac{\bar{f}^+(x_{j+1},\xi,t_n) - \bar{f}^+(x_j,\xi,t_n)}{x_{j+1} - x_j}, \tag{17a}$$

$$\bar{f}^+(x_{j+1/2},\xi,t_n) = \bar{f}^+(x_j,\xi,t_n) + \sigma_{j+1/2}(x_{j+1/2} - x_j). \tag{17b}$$

Based on Eqs. (14) and (16), we can get

$$\bar{f}(\boldsymbol{x}_b,\boldsymbol{\xi},t_n+h) = \bar{f}^+(\boldsymbol{x}_b,\boldsymbol{\xi},t_n) - \boldsymbol{\xi} h \cdot \boldsymbol{\sigma}_b, \tag{18}$$

from which the conserved variables at the cell interface can be obtained,

$$\boldsymbol{W}(\boldsymbol{x}_b,t_n+h) = \int \boldsymbol{\psi} \bar{f}(\boldsymbol{x}_b,\boldsymbol{\xi},t_n+h)\, d\boldsymbol{\xi}. \tag{19}$$

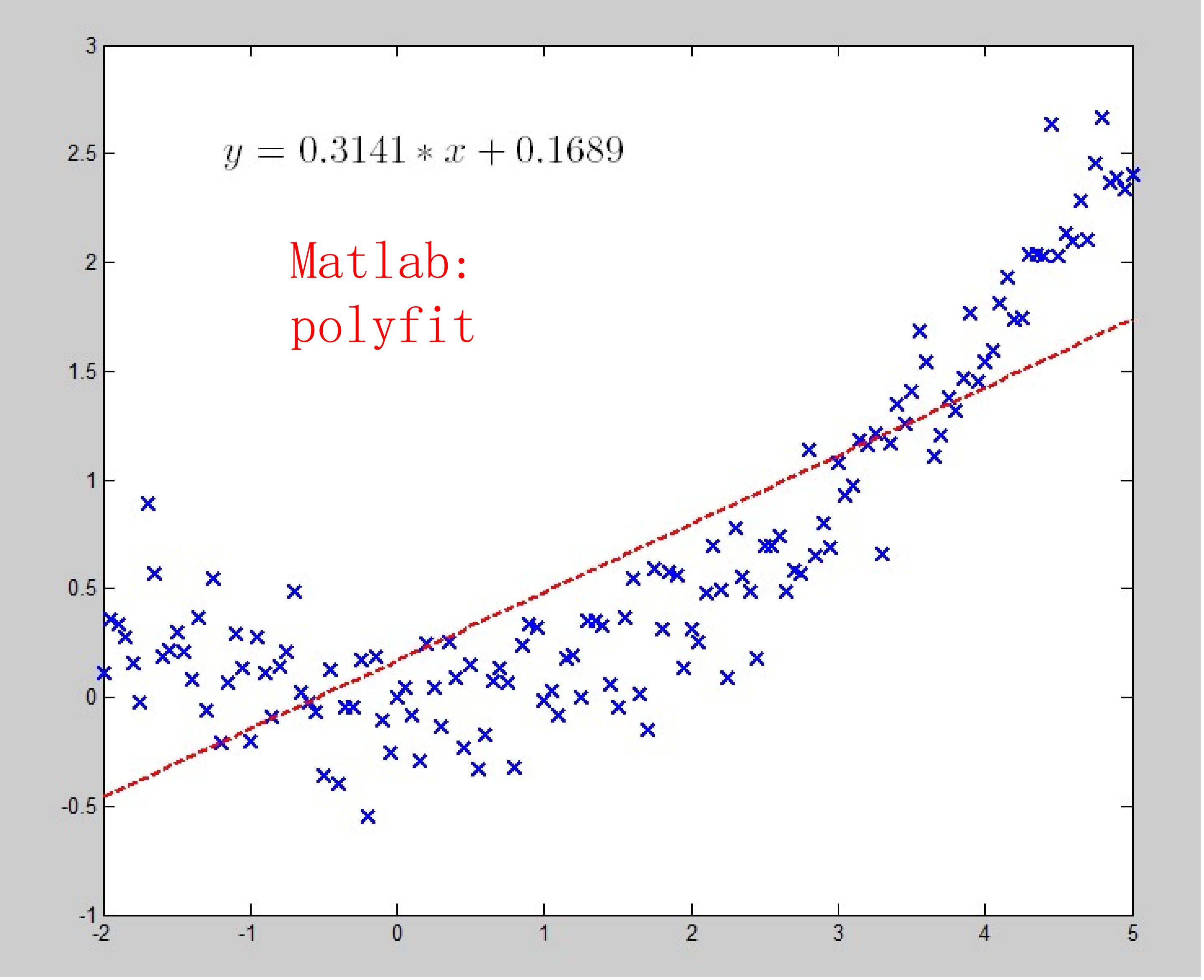
where $\boldsymbol{i} = (i_1,i_2,\ldots,i_D)$, and $\boldsymbol{N} = (N_1,N_2,\ldots,N_D)$, with N_i being positive integers.

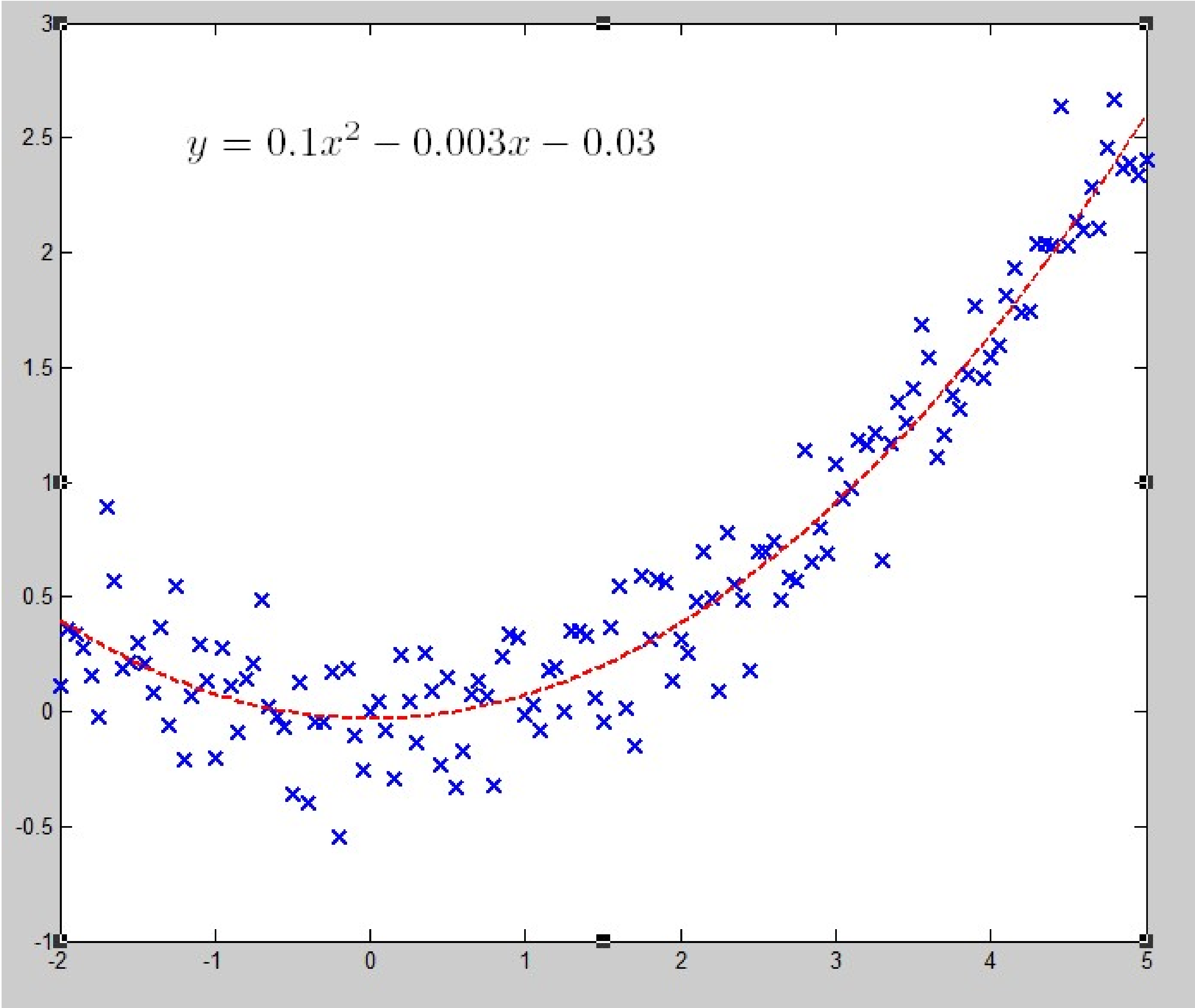
With the discrete velocity space, the moments of a continuous distribution function have to be expressed as discrete moments,

$$\boldsymbol{m} = \int \phi(\boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \sum_{\boldsymbol{i}=-\boldsymbol{N}}^{\boldsymbol{N}} w_{\boldsymbol{i}} \phi(\boldsymbol{\xi}_{\boldsymbol{i}}) f(\boldsymbol{\xi}_{\boldsymbol{i}}), \tag{23}$$

where ϕ is a polynomial of the particle velocity $\boldsymbol{\xi}$ and w_i is the weight of the numerical quadrature at the discrete velocity $\boldsymbol{\xi}_i$. In general, the quadrature cannot present the same results as the exact moments of a continuous distribution function due to numerical errors, and the difference can be reduced by increasing the number of discrete points. Particularly with discrete velocities, the BGK collision operator may not be fully conservative [23–25], i.e.,

最小二乘法





$y = 0.1x^2 - 0.003x - 0.03$
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-2
-1
0
1
2
3
4
5

原理：已知 $\boldsymbol{n+1}$ 组数据 $(x_0, y_0), (x_1, y_1), \ldots\ (x_n, y_n)$

用 $\boldsymbol{m+1}$ 个已知函数$\{\boldsymbol{\phi_0}, \boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_m}\}$的组合逼近，使得误差最小

组合函数：

$$P(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_m\phi_m(x) = \sum_{i=0}^{m} a_i\phi_i(x)$$

$\boldsymbol{P(x)}$在数据点 $\boldsymbol{x_j}$ 上的近似值

$$P(x_j) = a_0\phi_0(x_j) + a_1\phi_1(x_j) + \cdots + a_m\phi_m(x_j) = \sum_{i=0}^{m} a_i\phi_i(x_j)$$

与给定数据 $\boldsymbol{y_j}$的误差平方：

$$\left|P(x_j) - y_i\right|^2 = \left|\sum_{i=0}^{m} a_i\phi_i(x_j) - y_j\right|^2$$

所有给定数据的误差平方：

$$\sum_{j=0}^{n}\left|P(x_j)-y_i\right|^2=\sum_{j=0}^{n}\left|\sum_{i=0}^{m}a_i\phi_i(x_j)-y_j\right|^2$$

总体误差与组合系数有关，所以

$$F(a_0,a_1,\cdots,a_m)=\sum_{j=0}^{n}\left|P(x_j)-y_i\right|^2=\sum_{j=0}^{n}\left|\sum_{i=0}^{m}a_i\phi_i(x_j)-y_j\right|^2$$

要求总体误差最小

$$\frac{\partial}{\partial a_k}F(a_0,a_1,\cdots,a_m)=\frac{\partial}{\partial a_k}\left[\sum_{j=0}^{n}\left|\sum_{i=0}^{m}a_i\phi_i(x_j)-y_j\right|^2\right]=0$$

$$k=0,1,\ldots,m$$

$$\frac{\partial}{\partial a_k} F(a_0, a_1, \cdots, a_m) = \frac{\partial}{\partial a_k}\left[\sum_{j=0}^{n}\left|\sum_{i=0}^{m} a_i \phi_i(x_j) - y_j\right|^2\right] = 0$$

$$\frac{\partial}{\partial a_k}\left[\sum_{j=0}^{n}\left|\sum_{i=0}^{m} a_i \phi_i(x_j) - y_j\right|^2\right] = \sum_{j=0}^{n}\left[\frac{\partial}{\partial a_k}\left|\sum_{i=0}^{m} a_i \phi_i(x_j) - y_j\right|^2\right]$$

$$= \sum_{j=0}^{n}\left[2\left(\sum_{i=0}^{m} a_i \phi_i(x_j) - y_j\right)\frac{\partial}{\partial a_k}\left(\sum_{i=0}^{m} a_i \phi_i(x_j) - y_j\right)\right]$$

$$= \sum_{j=0}^{n}\left[2\left(\sum_{i=0}^{m} a_i \phi_i(x_j) - y_j\right)\phi_k(x_j)\right]$$

$$= \sum_{j=0}^{n}\left[2\left(\sum_{i=0}^{m} a_i \phi_i(x_j)\phi_k(x_j) - y_j \phi_k(x_j)\right)\right]$$

$$= 2\sum_{j=0}^{n}\sum_{i=0}^{m} a_i \phi_i(x_j)\phi_k(x_j) - 2\sum_{j=0}^{n} y_j \phi_k(x_j) = 0$$

$$\sum_{j=0}^{n}\sum_{i=0}^{m} a_i \phi_i(x_j)\phi_k(x_j) = \sum_{j=0}^{n} y_j \phi_k(x_j)$$

$$\sum_{i=0}^{m} a_i \left(\sum_{j=0}^{n} \phi_i(x_j)\phi_k(x_j) \right) = \sum_{j=0}^{n} y_j \phi_k(x_j)$$

$$\left(\phi_i, \phi_k\right) = \sum_{j=0}^{n} \phi_i(x_j)\phi_k(x_j), \;\; b_k = \sum_{j=0}^{n} y_j \phi_k(x_j)$$

$$\sum_{i=0}^{m} \left(\phi_i, \phi_k\right) a_i = b_k$$

$$\left(\phi_0, \phi_k\right) a_0 + \left(\phi_1, \phi_k\right) a_1 + \cdots + \left(\phi_m, \phi_k\right) a_m = b_k$$

$$\left(\phi_0,\phi_k\right)a_0+\left(\phi_1,\phi_k\right)a_1+\cdots+\left(\phi_m,\phi_k\right)a_m=b_k$$

$$\begin{bmatrix} \left(\phi_0,\phi_0\right) & \left(\phi_1,\phi_0\right) & \cdots & \left(\phi_m,\phi_0\right) \\ \left(\phi_0,\phi_1\right) & \left(\phi_1,\phi_1\right) & \cdots & \left(\phi_m,\phi_1\right) \\ \cdots & & & \\ \left(\phi_0,\phi_m\right) & \left(\phi_1,\phi_m\right) & \cdots & \left(\phi_m,\phi_m\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$P(x)=a_0\phi_0(x)+a_1\phi_1(x)+\cdots+a_m\phi_m(x)=\sum_{i=0}^{m}a_i\phi_i(x)$$