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Note

Disproofs of two conjectures on no hole anti-n-labeling of graphs ☆

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ABSTRACT

In this note, we disprove two conjectures on no hole anti-n-labeling of graphs proposed by Guan et al. (2019), and characterize a graph G with $\lambda_n^{nh}(G) = \alpha(G)$, where $\lambda_n^{nh}(G)$ and $\alpha(G)$ denote the no hole anti-n-labeling number and the independence number of G, respectively.

1. Introduction

All graphs considered in this paper are simple, undirected and finite. We refer to [2] for undefined terminology and notation. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set, respectively. We denote the number of vertices and edges of G by n and m, respectively. If the m is 0, then G is called an empty graph. For a vertex $v \in V(G)$, the *degree* of v, denoted by $d(v)$, is the number of edges incident with v in G. The maximum degree of G, denoted by $\Delta(G)$ (simply by Δ), is $\max\{d(v) : v \in V(G)\}$. The minimum degree of G, denoted by $\delta(G)$ (simply by δ), is $\min\{d(v) : v \in V(G)\}$. The *induced subgraph* of G induced by X, denoted by $G[X]$, is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. A *matching* in graph is a set of pairwise nonadjacent edges. A *perfect matching* is one which covers every vertex of the graph. A *clique* of a graph is a set of mutually adjacent vertices.

As widely known, the labeling problems of graphs arise in many networking and telecommunication problems [3,6,9,12–14]. Let $k \geq 1$ be an integer. An *anti-k-labeling* ψ of a graph is a mapping from $V(G)$ to $\{1, 2, \ldots, k\}$. An *anti-k-labeling* ψ of G is called a *no hole anti-k-labeling* if ψ is surjective. We define $\omega_\psi(e) = |\psi(u) - \psi(v)|$ for any edge $e = uv$, and $\omega_\psi(G) = \min\{\omega_\psi(e) : \ e \in E(G)\}$. The *no-hole anti-$k$-labeling number* of a graph G, denoted by $\lambda_k^{nh}(G)$, is $\max\{\omega_\psi(G) : \psi$ is no-hole anti-k-labeling $\}$. In this note, we focus on $\lambda_n^{nh}(G)$, the *no hole anti-n-labeling number* of a graph G. We start with two easy bounds for $\lambda_n^{nh}(G)$, due to Guan, Zhang, Li, Chen, Yang [8].

Lemma 1.1 ([8]). *For a connected graph G of order n,*

$$1 \leq \lambda_n^{nh}(G) \leq \min\{n - \triangle, \lfloor \frac{n-1}{\chi - 1} \rfloor, \lfloor \frac{n-\delta+1}{2} \rfloor\},$$

where χ denotes the chromatic number of G.

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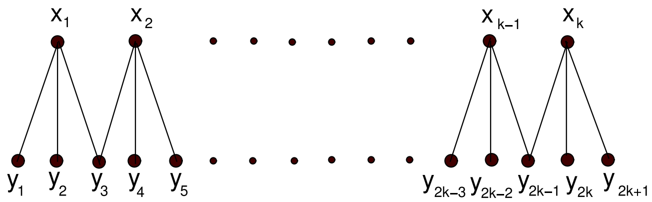

Fig. 1. T_k for an integer $k \geq 2$.

A set $S \subseteq V(G)$ is called an *independent set* of G if no two vertices of S are adjacent. An independent set in a graph is *maximum* if the graph contains no larger independent set. The *independence number*, denoted by $\alpha(G)$, is the cardinality of a maximum independent set in a graph G. On the other hand, a set $S \subseteq V(G)$ is called a *covering* of G if each edge of G has an end vertex in S. The *covering number* of G, denoted by $\beta(G)$, is the cardinality of a minimum covering of G.

Lemma 1.2 ([8]). *For a non-empty graph G, $\lambda_n^{nh}(G) \leq \alpha(G)$.*

Guan et al. [8] established the following lower bounds for $\lambda_n^{nh}(G)$ when G is a tree or 2-dimensional grids.

Theorem 1.3 ([8]). *For a tree T_n with bipartition (X_1, X_2), $|X_i| = q_i$, $i = 1, 2$, we have $\lambda_n^{nh}(T_n) \geq q = \min\{q_1, q_2\}$.*

Theorem 1.4 ([8]). *Let G be a 2-dimensional grid $P_m \times P_n (m \leq n)$. Then $\lambda_{mn}^{nh}(G) \geq \lfloor \frac{mn-m}{2} \rfloor$, where $P_m \times P_n$ denotes the cartesian product of P_m and P_n.*

Further, based on the above bounds, they proposed the following conjectures, which asserts that the bounds obtained above are exactly the no hole anti-n-labeling number of those graphs.

Conjecture 1.5 ([8]). *or a tree T_n with bipartition (X_1, X_2), $|X_i| = q_i$, $i = 1, 2$, we have $\lambda_n^{nh}(T_n) = q$, where $q = \min\{q_1, q_2\}$.*

Conjecture 1.6 ([8]). *Let G be a 2-Dimensional grid $P_m \times P_n (m \leq n)$. Then $\lambda_{mn}^{nh}(G) = \lfloor \frac{mn-m}{2} \rfloor$.*

The aim of the present note is to disprove the above two conjectures.

2. Counterexamples to Conjectures 1.5 and 1.6

A key tool we use here is due to Sakai and Wang [14]. Let G be a graph and k a positive integer. For an integer $k' > k$, a k-path with k' vertices is a sequence of k' distinct vertices $v_1, v_2, \ldots, v_{k'}$ of G such that $v_i, v_{i+1}, \ldots, v_{i+k}$ form a clique for each $i \in \{1, 2, \ldots, k' - k\}$. If $k' \leq k$, k-path with k' vertices is simply a clique of order k'. A Hamilton k-path of G is an k-path containing all vertices of G. Recall that a path or a cycle which contains every vertex of a graph is called a *Hamilton path or cycle* of the graph. By the above definition, a Hamilton 1-path is a Hamilton path. The *complement* $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$, in which two vertices u and v are adjacent if and only if $uv \notin E(G)$.

Theorem 2.1 (*Sakai and Wang [14]*). *For a graph G of order n, $\lambda_n^{nh}(G) \geq k + 1$ if and only if $\overline{G}$ has a Hamilton k-path.*

The *kth power* of a graph G, denoted by G^k, is the graph with $V(G^k) = V(G)$, in which two vertices being adjacent if and only if their distance in G is at most k. In particular, G^2 is known as the square of G. By the above definition, the theorem of Sakai and Wang can be reformulated equivalently as follows.

Theorem 2.2. *For a graph G of order n, $\lambda_n^{nh}(G) \geq k + 1$ if and only if $\overline{G}$ contains P_n^k as its spanning subgraph.*

Theorem 2.3. *For an integer $k \geq 2$, let T_k be the tree with vertex set $\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_{2k+1}\}$ and edge set $\{x_i y_{2i-1}, x_i y_{2i}, x_i y_{2i+1} : i \in \{1, \ldots, k\}\}$, as shown in Fig. 1. If k is a positive even integer, then $\lambda_n^{nh}(T_k) \geq \frac{3k}{2}$.*

Proof. Observe that the following sequence of the vertices,

$$x_{\frac{k}{2}+1} x_{\frac{k}{2}+2}, \ldots, x_k y_1 y_2, \ldots, y_k, y_{k+1}, \ldots, y_{2k+1} x_1 x_2, \ldots, x_{\frac{k}{2}},$$

is a Hamilton $(\frac{3k}{2} - 1)$-path of $\overline{T_k}$. By Theorem 2.1, we have

$$\lambda_n^{nh}(T_k) \geq \frac{3k}{2} > k = \min\{|X|, |Y|\}.$$

This disproves Conjecture 1.5. □

16	34	11	29	6	24	2	21
38	15	33	10	28	5	23	1
19	37	14	32	9	27	4	22
40	18	36	13	31	8	26	3
20	39	17	35	12	30	7	25

Fig. 2. Labels of $P_5 \times P_8$.

Theorem 2.4. *Let G be a 2-dimensional grid $P_m \times P_n (m \leq n)$. If m is odd and n is even, then $\lambda_{mn}^{nh}(G) \geq \lfloor \frac{mn-m}{2} \rfloor + 1$.*

Proof. Observe that $G = P_m \times P_n$ can be drawn on the plane as the chessboard with m rows and n columns. Since white and black squares alternate, as shown in Fig. 2, it is a bipartite graph. Since m is even, the number of vertices in G is even. By assigning the elements of $[1, \frac{mn}{2}]$ to "white" squares and the elements of $[\frac{mn}{2} + 1, mn]$ to "black" squares, (without loss of generality, the upper left corner is white) we obtain a no hole anti-n-labeling ψ in the following way:

(1) Put 1 in the last square of the second row;

(2) put $\frac{mn}{2} + 1$ in the far right square of the first row;

(3) put 2 to the second to last from the right of the first row;

(4) the label of each remaining vertex is assigned by the following rule:

(4.1) for a vertex v with the label i with $i \leq \frac{mn}{2}$, the vertices adjacent to v are labeled by $i + \lfloor \frac{mn-m}{2} \rfloor + 1, i + \lfloor \frac{mn-m}{2} \rfloor + 2, i + \lfloor \frac{mn-m}{2} \rfloor + 6, i + \lfloor \frac{mn-m}{2} \rfloor + 5$.

(4.2) for a vertex v with the label i with $i > \frac{mn}{2}$, the vertices adjacent to v are labeled by $i - \lfloor \frac{mn-m}{2} \rfloor - 1, i - \lfloor \frac{mn-m}{2} \rfloor - 2, i - \lfloor \frac{mn-m}{2} \rfloor - 6, i - \lfloor \frac{mn-m}{2} \rfloor - 5$.

Hence, $\lambda_{mn}^{nh}(G) \geq \lfloor \frac{mn-m}{2} \rfloor + 1$. □

3. The characterization

Corollary 3.1. *For a connected graph G of order n, $\lambda_n^{nh}(G) = 1$ if and only if $\overline{G}$ contains no a Hamilton path.*

Proof. It is an immediate consequence of Theorem 2.2 and Lemma 1.1. □

If G and H are vertex disjoint, then $G \vee H$ denotes the *join* of G and H, which is obtained from G and H by adding an edge between every vertex of G and every vertex of H.

Corollary 3.2 ([8]). *If $G = G_1 \vee G_2$, then $\lambda_n^{nh}(G) = 1$.*

Proof. It is clear that $\overline{G}$ is disconnected. Thus $\overline{G}$ has no a Hamilton path. By Corollary 3.1, we have $\lambda_n^{nh}(G) = 1$. □

Corollary 3.3 ([8]). *If $G = K_{n_1,\ldots,n_k}$ be a complete k-partite graph with $k \geq 2$, then $\lambda_n^{nh}(G) = 1$.*

Proof. It is a special case of Corollary 3.2. □

The following result can be found in page 139 in the book of Lovász and Plummer [11].

Lemma 3.4 (*Lovász and Plummer [11]*). *Let $H = (X, Y)$ be a bipartite graph, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. If H has a unique perfect matching $M = \{x_k y_k : k \in \{1, \ldots, n\}\}$, then the vertices of H can be labeled such that for every edge $x_i y_j \in E(H)$, $i \leq j$.*

Let us recall a classical theorem of Gallai.

Theorem 3.5 (*Gallai [7]*). *For any graph G of order n, $\alpha(G) + \beta(G) = n$.*

We also use a fundamental theorem, due independently to König and Egerváry.

Theorem 3.6 (*König [10] and Egerváry [5]*). *For any bipartite graph G, $\alpha'(G) = \beta(G)$.*

Corollary 3.7. *Let G be a bipartite graph of order n with bipartition (X, Y) and α an integer. Then*
(1) if $|X| = |Y| = \alpha$, $\alpha(G) = \alpha$ if and only if G has a perfect matching;
(2) if $\alpha = |X| > |Y|$, $\alpha(G) = \alpha$ if and only if G has a matching which covers Y.

Proof. By Theorem 3.6, Theorem 3.5,

$$\alpha(G) + \alpha'(G) = \alpha(G) + \beta(G) = n = |X| + |Y| = \alpha + |Y|,$$

implying that

$$\alpha(G) = \alpha \text{ if and only if } \alpha'(G) = |Y|. \quad \square$$

Theorem 3.8. *For a nonempty graph G of order n, $\lambda_n^{nh}(G) = \alpha(G)$ if and only if one of the following holds:*
(1) *If $\alpha(G) \geq \frac{n}{2}$, then G consists of a bipartite graph H with a unique perfect matching and with possibly some additional isolated vertices.*
(2) *If $\alpha(G) < \frac{n}{2}$, then $V(G)$ can be partitioned into $k+1$ independent sets $V_1, \ldots, V_k, V_{k+1}$ with $|V_i| = \alpha(G)$ for each $i \in \{1, \ldots, k\}$ and $|V_{k+1}| = l \leq \alpha(G)$, and with the additional properties that*
(i) $G[V_i \cup V_{i+1}]$ is a bipartite graph with a unique perfect matching M_i for each $i \leq k-1$, $G[V_k \cup V_{k+1}]$ has a unique matching M_k which covers V_{k+1};
(ii) $G[V_i \cup V_j]$ is a bipartite graph with a perfect matching for any pairs $i, j \in \{1, \ldots, k\}$ with $|i-j| \geq 2$, and $G[V_i \cup V_{k+1}]$ has a matching which covers V_{k+1} for each $i \leq k-1$;
(iii) the vertices of V_i are labeled as $v_1^i, v_2^i, \ldots, v_\alpha^i$ for each $i \in \{1, \ldots, k\}$ and the vertices of V_{k+1} are labeled as $v_1^{k+1}, v_2^{k+1}, \ldots, v_l^{k+1}$ such that
$s \leq t$ for each $v_s^j v_t^{j+1} \in E(G[V_j \cup V_{j+1}])$, where $j \leq k-1$ and $s, t \in \{1, \ldots, \alpha\}$, and
$p \leq q$ for each $v_p^k v_q^{k+1} \in E(G[V_k \cup V_{k+1}])$, where $p, q \in \{1, \ldots, l\}$.
Therefore, $M_i = \{v_1^i v_1^{i+1}, v_2^i v_2^{i+1}, \ldots, v_\alpha^i v_\alpha^{i+1}\}$ for each $i \leq k-1$, and $M_k = \{v_1^k v_1^{k+1}, v_2^k v_2^{k+1}, \ldots, v_l^k v_l^{k+1}\}$.

Proof. To prove the sufficiency, assume that G is a graph as given in the statement of the theorem. By Lemma 1.2, $\lambda_n^{nh}(G) \leq \alpha(G)$. It remains to prove that $\lambda_n^{nh}(G) \geq \alpha(G)$.

Let us first consider the case that $\alpha(G) \geq \frac{n}{2}$. Since G consists of a bipartite graph H with a unique perfect matching and with possibly some additional isolated vertices. By Lemma 3.4, the vertices of H are labeled as $x_1, \ldots, x_s$ and $y_1, \ldots, y_s$ such that $i \leq j$ for any edge $x_i y_j \in E(H)$. Moreover, the some additional isolated vertices are labeled as $z_1, \ldots, z_t$, where $t = \alpha(G) - s$. Note that $x_1, \ldots, x_s, z_1, \ldots, z_t, y_1, \ldots, y_s$ is a Hamilton $(\alpha(G)-1)$-path in $\overline{G}$. By Theorem 2.1, $\lambda_n^{nh}(G) \geq \alpha(G)$.

Now we consider the case that $\alpha(G) < \frac{n}{2}$. Let $v_1^i, v_2^i, \ldots, v_\alpha^i$ be the ordering of the vertices of V_i, as given in the assumption for each $i \in \{1, \ldots, k\}$ and $v_1^{k+1}, v_2^{k+1}, \ldots, v_l^{k+1}$ that of the vertices of V_{k+1} with the additional property as described in the statement in the theorem. Observe that

$$v_1^1, v_2^1, \ldots, v_\alpha^1, v_1^2, v_2^2, \ldots, v_\alpha^2, \ldots, v_1^k, v_2^k, \ldots, v_\alpha^k, v_1^{k+1}, v_2^{k+1}, \ldots, v_l^{k+1}$$

is a Hamilton $(\alpha(G)-1)$-path in $\overline{G}$. By Theorem 2.1, $\lambda_n^{nh}(G) \geq \alpha(G)$.

To prove the necessity, let G be a nonempty graph with $\lambda_n^{nh}(G) = \alpha(G)$. We consider two cases in terms of the value of $\alpha(G)$.

Case 1. $\alpha(G) \geq \frac{n}{2}$

Since $\lambda_n^{nh}(G) = \alpha(G)$, by Theorem 2.1, $\overline{G}$ has a Hamilton $(\alpha(G)-1)$-path:

$x_1, \ldots, x_s, z_1, \ldots, z_t, y_1, \ldots, y_s$ is a Hamilton $(\alpha(G)-1)$-path in $\overline{G}$, where $s + t = \alpha(G)$. By the definition of Hamilton $(\alpha(G)-1)$-path, both $x_1, \ldots, x_s, z_1, \ldots, z_t$ and $z_1, \ldots, z_t, y_1, \ldots, y_s$ are maximum independent sets of G, and $G[X \cup Y]$ is a bipartite graph with a perfect matching, where $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_s\}$. Moreover, for any edge $x_i y_j \in E(G)$, $i \leq j$. It follows that $H = G[X \cup Y]$ is a bipartite graph with a unique perfect matching and $\{z_1, \ldots, z_t\}$ is a set of isolated vertices of G.

Case 2. $\alpha(G) < \frac{n}{2}$

Let $v_1^1, v_2^1, \ldots, v_\alpha^1, v_1^2, v_2^2, \ldots, v_\alpha^2, \ldots, v_1^k, v_2^k, \ldots, v_\alpha^k, v_1^{k+1}, v_2^{k+1}, \ldots, v_l^{k+1}$ be a Hamilton $(\alpha(G)-1)$-path in $\overline{G}$, $1 \leq l \leq \alpha(G)$. Put $V_i = \{v_1^i, v_2^i, \ldots, v_\alpha^i\}$ for each $i \in \{1, \ldots, k\}$ and $V_{k+1} = \{v_1^{k+1}, v_2^{k+1}, \ldots, v_l^{k+1}\}$. By the definition of Hamilton $(\alpha(G)-1)$-path, we have V_i is independent set for each $i \in \{1, \ldots, k+1\}$. Furthermore,

(i) $G[V_i \cup V_{i+1}]$ is a bipartite graph with a unique perfect matching for each $i \leq k-1$, $G[V_k \cup V_{k+1}]$ has a unique matching which covers V_{k+1};

Since $\alpha(G) = \alpha$, both V_i and V_j are independent sets of cardinality α, by Corollary 3.7, it follows that

(ii) $G[V_i \cup V_j]$ is a bipartite graph with a perfect matching for any pairs $i, j \in \{1, \ldots, k\}$ with $|i-j| \geq 2$, and $G[V_i \cup V_{k+1}]$ has a matching which covers V_{k+1} for each $i \leq k$;

Moreover, (iii) follows by the definition of the Hamilton $(\alpha(G)-1)$-path. $\square$

4. Concluding remarks

A classical result of Dirac [4] asserts that G contains a Hamilton cycle if $\delta(G) \geq \frac{n}{2}$. As a natural generalization of Dirac's theorem, Seymour conjecture [15] that G contains the kth power of a Hamilton cycle if $\delta(G) \geq \frac{kn}{k+1}$. Aigner and Brandt [1] verified that G contains the square of a Hamilton path if $\delta(G) \geq \frac{2n-1}{3}$. In view of Theorem 2.2, one naturally poses the following problem.

Problem 1. Determine the minimum integer $f(n, k)$ such that every graph G of order n with $\delta(G) \geq f(n, k)$ contains a P_n^k.

Problem 2. Determine the minimum integer $g(n, k)$ such that every graph G of order n and size $m \geq g(n, k)$ contains a P_n^k.

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