Basic Transforms

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Homogeneous Coordinate

- Homogeneous coordinates are coordinate systems representing n-dimensional projective spaces as n+1 coordinates. Homogeneous coordinates simply represent (x, y) as (x, y, 1).

- Ex)
  - The homogeneous coordinate representation of (1, 2, 3) is (1h, 2h, 3h, h). (h is a real number)
  - (1, 2, 3, 1) = (2, 4, 6, 2) are all equivalent to the three-dimensional (1, 2, 3) coordinates.

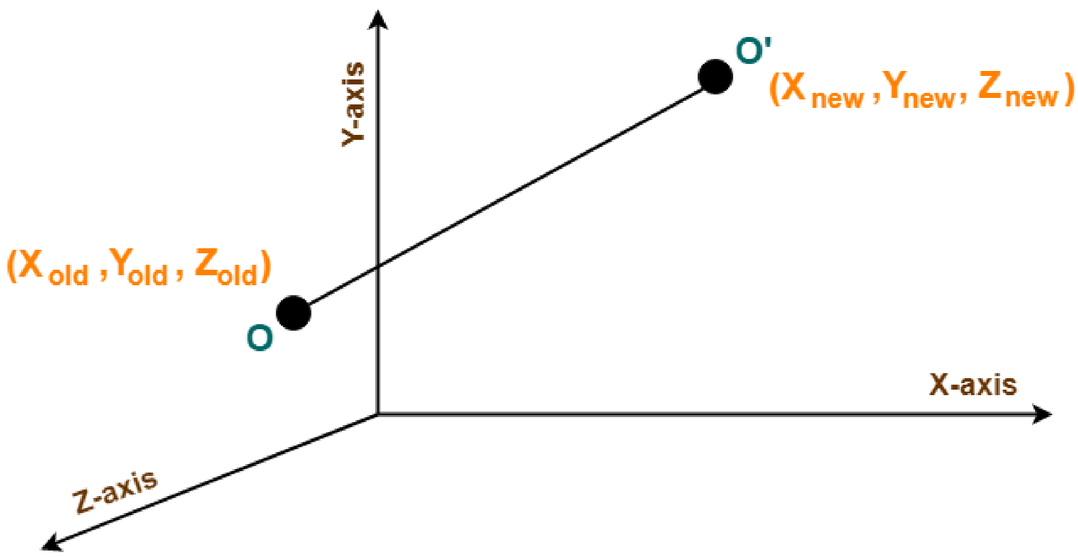
Homogeneous Coordinate

- Why use homogeneous coordinates?
  - It is used to express the Shearing, Rotation, Reflection, and Scaling Matrices, which will be described in the future, as a single matrix with the Translation Matrix.

  - Suppose you want to move a point at (x,y) to (x+dx, x+dy) and rotate along z-axis by θ
    - If you use inhomogeneous coordinates, you can not represent this with matrix operation
    - However, with homogeneous coordinates
      - $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} cos\theta & -sin\theta & dx \\ sin\theta & cos\theta & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Translation Matrix

- It is used when point object O needs to move from one position to another in the 3D plane.



Translation Matrix

- This translation is achieved by adding the translation coordinates to the old coordinates of the object as
  - $X_{new} = X_{old} + d_x$ (This denotes translation towards X axis)
  - $Y_{new} = Y_{old} + d_y$ (This denotes translation towards Y axis)
  - $Z_{new} = Z_{old} + d_z$ (This denotes translation towards Z axis)

Translation Matrix

- In Matrix form, the above translation equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

Rotation Matrix

- It is used when point object O has to be rotated from one angle to another in a 3D plane.

- Similar to the Shearing Matrix, there are three versions of Rotation because it is a three-dimensional plane.
  - X-axis Rotation
  - Y-axis Rotation
  - Z-axis Rotation

Rotation Matrix

- X-axis Rotation
- This rotation is achieved by using the following rotation equations
  - $X_{new} = X_{old}$
  - $Y_{new} = Y_{old} \times \cos\theta - Z_{old} \times \sin\theta$
  - $Z_{new} = Y_{old} \times \sin\theta + Z_{old} \times \cos\theta$

Rotation Matrix

- X-axis Rotation
- In Matrix form, the above rotation equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Rotation Matrix (For X-Axis Rotation)

Rotation Matrix

- Y-axis Rotation
- In Matrix form, the above rotation equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Rotation Matrix (For Y-Axis Rotation)

Rotation Matrix

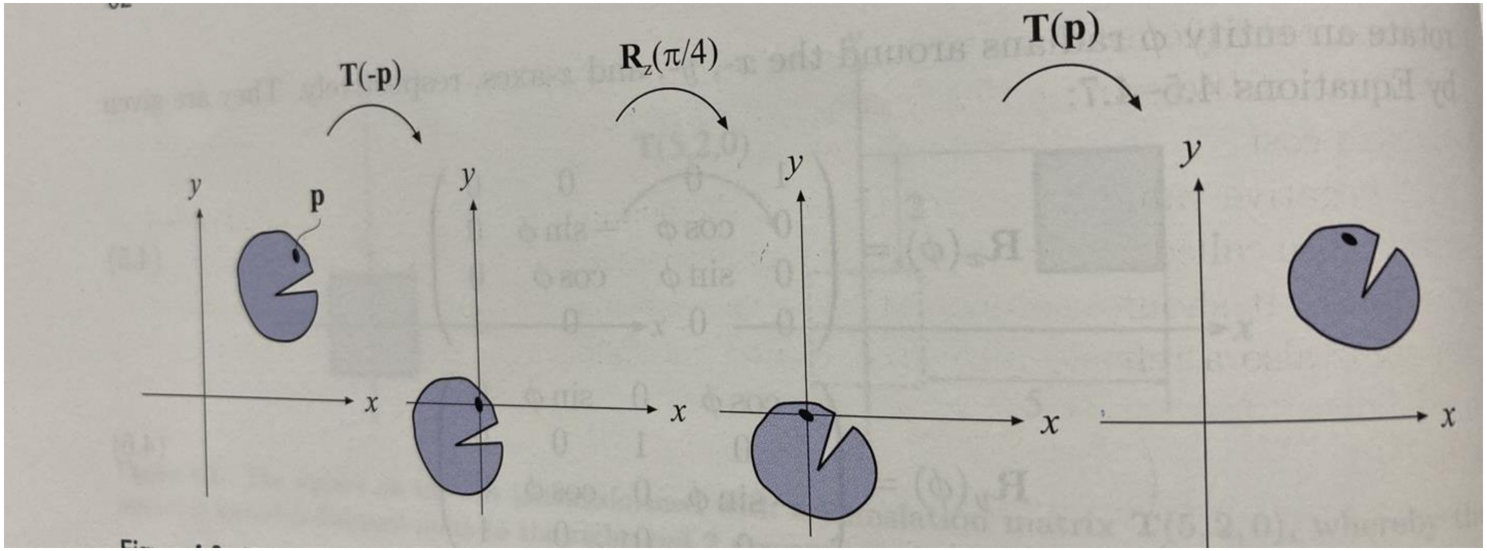
- Z-axis Rotation
- In Matrix form, the above rotation equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Rotation Matrix (For Z-Axis Rotation)

Rotation Matrix

- Example: Rotation around a point
  - The transform starts by translating the object so that the point coincides with the origin
  - Thereafter the actual rotation follows
  - The object has to be translated back to its original position



Scaling Matrix

- Scaling may be used to increase or reduce the size of object.
  - If scaling factor > 1, then the object size is increased.
  - If scaling factor < 1, then the object size is reduced.
- In Matrix form, the above scaling equations may be represented as

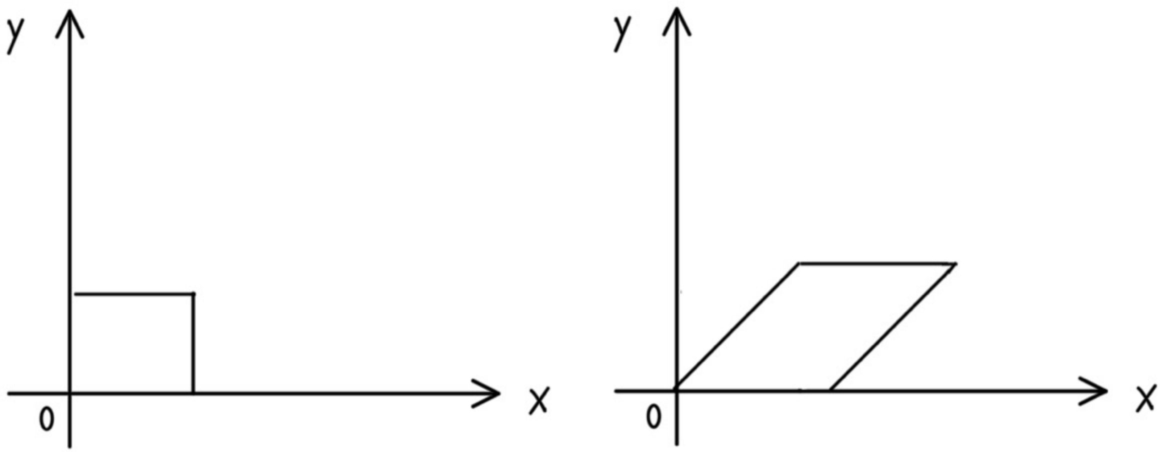
$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Scaling Matrix

Shearing Matrix

- In a three-dimensional plane, the object size can be changed along X direction, Y direction as well as Z direction. So, there are three versions of shearing

  - Shearing in X-axis
  - Shearing in Y-axis
  - Shearing in Z-axis



(a) Original object (b) Object after x shear

Shearing Matrix

- Shearing in X-axis
- Shearing in X-axis is achieved by using the following shearing equations
  - $X_{new} = X_{old}$
  - $Y_{new} = Y_{old} + Sh_y \times X_{old}$
  - $Z_{new} = Z_{old} + Sh_z \times X_{old}$

Shearing Matrix

- Shearing in X-axis

- In Matrix form, the above shearing equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ sh_y & 1 & 0 & 0 \\ sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix (In X axis)

Shearing Matrix

- Shearing in Y-axis

- In Matrix form, the above shearing equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix (In Y axis)

Shearing Matrix

- Shearing in Z-axis

- In Matrix form, the above shearing equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix (In Z axis)

Reflection Matrix

- Reflection is a kind of rotation where the angle of rotation is 180 degree. And the size of reflected object is same as the size of original object.
- In 3 dimensions, there are 3 possible types of reflection
  - Reflection relative to XY plane
  - Reflection relative to YZ plane
  - Reflection relative to XZ plane

Reflection Matrix

- Reflection Relative to XY Plane
- This reflection is achieved by using the following reflection equations
  - $X_{new} = X_{old}$
  - $Y_{new} = Y_{old}$
  - $Z_{new} = -Z_{old}$
- In Matrix form, the above reflection equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Reflection Matrix (Reflection Relative to XY Plane)

Reflection Matrix

- Reflection Relative to YZ Plane
- In Matrix form, the above reflection equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Reflection Matrix (Reflection Relative to YZ Plane)

Reflection Matrix

- Reflection Relative to ZX Plane
- In Matrix form, the above reflection equations may be represented as

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Reflection Matrix (Reflection Relative to XZ Plane)

Concatenation of Transforms

- Due to the noncommutativity of the multiplication operation on matrices, the order in which the matrices occur matters
- The order commonly used is
  - $C = TRS, T: Translation, R: Rotation, S: Scaling$