Lecture 17: Numerical Integration Contd..

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Error $= \int_a^b f(x)dx - \int_a^b p_k(x)dx$

We write

$$f(x) = p_k(x) + \underbrace{f[x_0, \ldots, x_k, x]\psi_k(x)}_{\text{Error}},$$

$E(f) = I(f) - I(p_k)$

where $\psi_k(x) = \prod_{j=0}^{k}(x - x_j)$. Then the error $E(f) = I(f) - I(p_k)$ is given by

$$E(f) = \int_a^b \overbrace{f[x_0, \ldots, x_k, x]}^{f}\overbrace{\psi_k(x)}^{g.}dx. \quad (1)$$

If $\psi_k(x)$ is of one sign in $[a, b]$ then by MVT

$$\int_a^b f[x_0, \ldots, x_k, x]\psi_k(x)dx = f[x_0, \ldots, x_k, \xi]\int_a^b \psi_k(x)dx \quad (2)$$

$f[x_0, .., x_k] = \frac{f^{(k)}(\xi)}{k!}$

for some $\xi \in (a, b)$. If $f(x) \in C^{k+1}(a, b)$, it then follows from (1) and (2) that

$$E(f) = \frac{1}{(k+1)!}f^{(k+1)}(\eta)\int_a^b \psi_k(x)dx \text{ for some } \eta \in (a, b). \quad (3)$$

Mean value Theorem for Integral:

Let $g(x)$ be a non-negative or non-positive integrable function on $[a, b]$ (i.e $g(x)$ is of one sign). If $f \in C([a, b])$, then

$$\int_a^b f(x)\,g(x)\,dx = f(\xi)\int_a^b g(x)\,dx. \quad \text{for some } \xi \in [a, b]$$

NOTE: The requirement of $g(x)$ is of one sign is essential.

Eg: $f(x) = g(x) = x$, $[a, b] = [-1, 1]$

$$\int_a^b f(x)\,g(x)\,dx = \int_{-1}^{1} x^2\,dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

$$f(\xi)\int_a^b g(x) = 0$$

Even if $\psi_k(x)$ is not of one sign, certain simplifications in the error $E(f)$ are possible. A particularly desirable instance of this kind occurs when

$$\int_a^b \psi_k(x)dx = 0. \tag{4}$$

In this case, we use the identity

$$f[x_0, \ldots, x_k, x] = f[x_0, \ldots, x_k, x_{k+1}] + f[x_0, \ldots, x_{k+1}, x](x - x_{k+1})$$

for arbitrary x_{k+1}. Then

$f[x_0, -, x_k] \int_a^b \psi_k(x) dx$

$= 0$

$$\begin{aligned} E(f) &= \int_a^b f[x_0, \ldots, x_{k+1}]\psi_k(x)dx \\ &\quad + \int_a^b f[x_0, \ldots, x_{k+1}, x](x - x_{k+1})\psi_k(x)dx \\ &= \int_a^b f[x_0, \ldots, x_{k+1}, x]\psi_{k+1}(x)dx, \end{aligned}$$

where $\psi_{k+1}(x) = (x - x_{k+1})\psi_k(x)$.

Since

$$\int_a^b f[x_0, \ldots, x_{k+1}]\psi_k(x)dx = f[x_0, \ldots, x_{k+1}] \int_a^b \psi_k(x)dx = 0$$

Choose x_{k+1} in such a way that $\psi_{k+1}(x) = (x - x_{k+1})\psi_k(x)$ is of one sign on (a, b). If $f(x) \in C^{(k+2)}(a, b)$ then

$$E(f) = \frac{1}{(k+2)!} f^{(k+2)}(\eta) \int_a^b \psi_{k+1}(x)dx \text{ for some } \eta \in (a, b). \tag{5}$$

Error in the rectangle rule: In this case, we have $x_0 = a$ and $\psi_0(x) = x - a$.

$I(f) \simeq I(P_0)$

$P_0(x) = f(x_0)$

$E(f) = \int_a^b f[x_0, x] \psi_0(x)\, dx$

$$E_R = E(f) = \int_a^b f[a, x](x - a)\, dx.$$

$\psi_0(x) = (x - x_0)$

Since $\psi_0(x) = x - a$ is of one sign on (a, b), using the formula (3), we obtain

$$E_R = f'(\eta) \int_a^b (x - a)dx = \frac{f'(\eta)(b - a)^2}{2}.$$

If $b - a = h$,

$E_R = \frac{h^2}{2} f'(\eta)$

$\rightarrow 0$ as $h \rightarrow 0$

Error in the mid-point Rule: In this case, $x_0 = (a+b)/2$, and $\psi_0(x) = (x - \frac{a+b}{2})$ fails to be one sign and

$$\int_a^b (x - \frac{a+b}{2})dx = 0.$$

But, with $x_1 = (a+b)/2$, $\psi_1(x) = (x - x_1)\psi_0(x) = (x - \frac{a+b}{2})^2$ is of one sign. Thus, using the formula (5), the error in the approximation is

$$E_M = \frac{f''(\eta)(b-a)^3}{24}, \quad \text{for some } \eta \in (a,b).$$

Error in the Trapezoidal rule: Note that, in this case, $I(f) \approx I(p_1)$. Thus, here $k = 1$, $x_0 = a$, $x_1 = b$, and

$E(f) = \int_a^b f[x_0, x_1, x]\, \psi_1(x)$

$= \underbrace{f[x_0, x_1, \xi]}\int_a^b \psi_1(x)dx.$

$p_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$

$$\psi_1(x) = (x - x_0)(x - x_1) = (x-a)(x-b).$$

$E(f) = \int_a^b f[x_0, x_1, x]\,(x-x_0)(x-x_1)\,dx$

As $\psi_1(x)$ is of one sign, use error formula (3) to have

$$E_T = \frac{f''(\eta)}{2}\int_a^b (x-a)(x-b)\,dx = \frac{f''(\eta)(b-a)^3}{12}, \quad \text{for some } \eta \in (a,b).$$

$k=2$

$I(f) \approx I(P_2)$.

$E(f) = \int_a^b f[x_0,x_1,x_2,x]\,\psi_2(x)\,dx$.

Error in the Simpson's rule:

$$E^S = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{iv}(\eta),\ \eta \in (a,b). \quad (\textit{Exercise})$$

Error in the Corrected Trapezoidal rule: Here, $k=3$ and $x_0 = x_1 = a$, $x_2 = x_3 = b$. Note that $\psi_3(x) = (x-a)^2(x-b)^2$ is of one sign on (a,b) and hence, the error can be expressed as

$I(f) \approx I(P_3)$

$I(P_3) = \int_a^b P_3(x)\,dx$.

$= \frac{b-a}{2}\left[f(a)+f(b)\right]$

$+ \frac{(b-a)^2}{12}\left[f'(a)-f'(b)\right]$

$$\begin{aligned} E^{CT} &= \frac{1}{4!} f^{iv}(\eta)\int_a^b (x-a)^2(x-b)^2\,dx \\ &= \frac{f^{iv}(\eta)(b-a)^5}{720}, \quad \eta \in (a,b). \end{aligned}$$

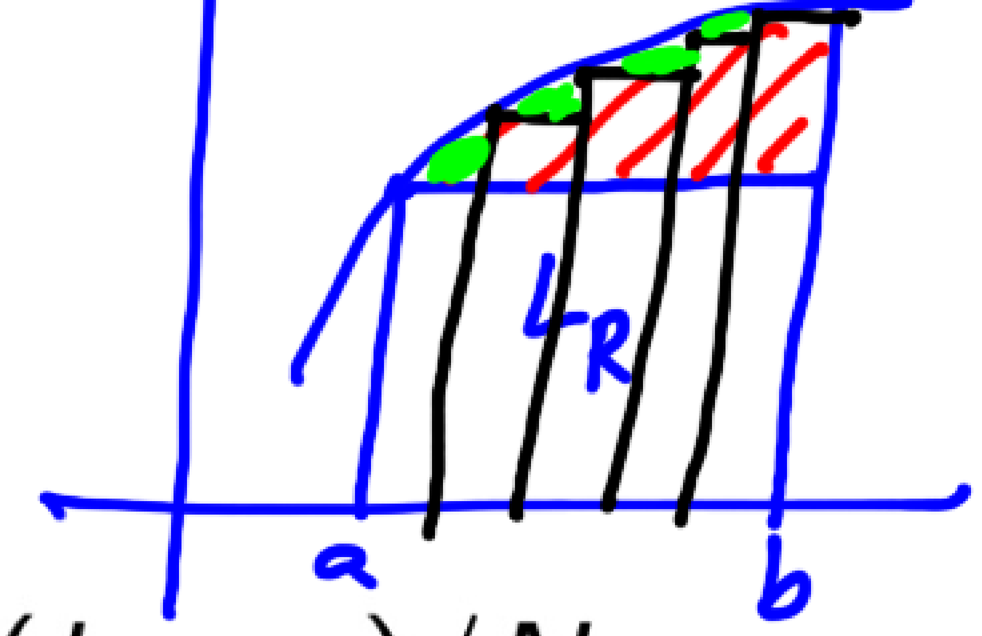
$E(f) = \int_a^b f[x_0,x_1,x_2,x_3,x]\,\psi_3(x)\,dx$

$x_0 = x_1 = a,\quad x_2 = x_3 = b,$

$\psi_3(x) = (x-a)^2(x-b)^2$

Composite rules

$L_R = (\ldots$



Divide $[a, b]$ into N subintervals

$x_0 = a$

$x_N = a + Nh = b.$

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b,$$

where $x_i = a + ih$, $i = 0, \ldots, N$ with uniform step-size $h = (b - a)/N$.
Set $f_i = f(x_i) = f(a + ih)$, $i = 0, \ldots, N$ and $f_{i-1/2} = f(x_i - h/2)$.
We apply the rectangular rule on each subinterval to have

$x_{i-1/2} = \frac{x_{i-1} + x_i}{2}$

$$\begin{aligned} \int_{x_{i-1}}^{x_i} f(x)dx &= (x_i - x_{i-1})f(x_{i-1}) + \frac{f'(\eta_i)(x_i - x_{i-1})^2}{2} \\ &= hf(x_{i-1}) + \frac{f'(\eta_i)h^2}{2}. \end{aligned}$$

$h = x_i - x_{i-1}$

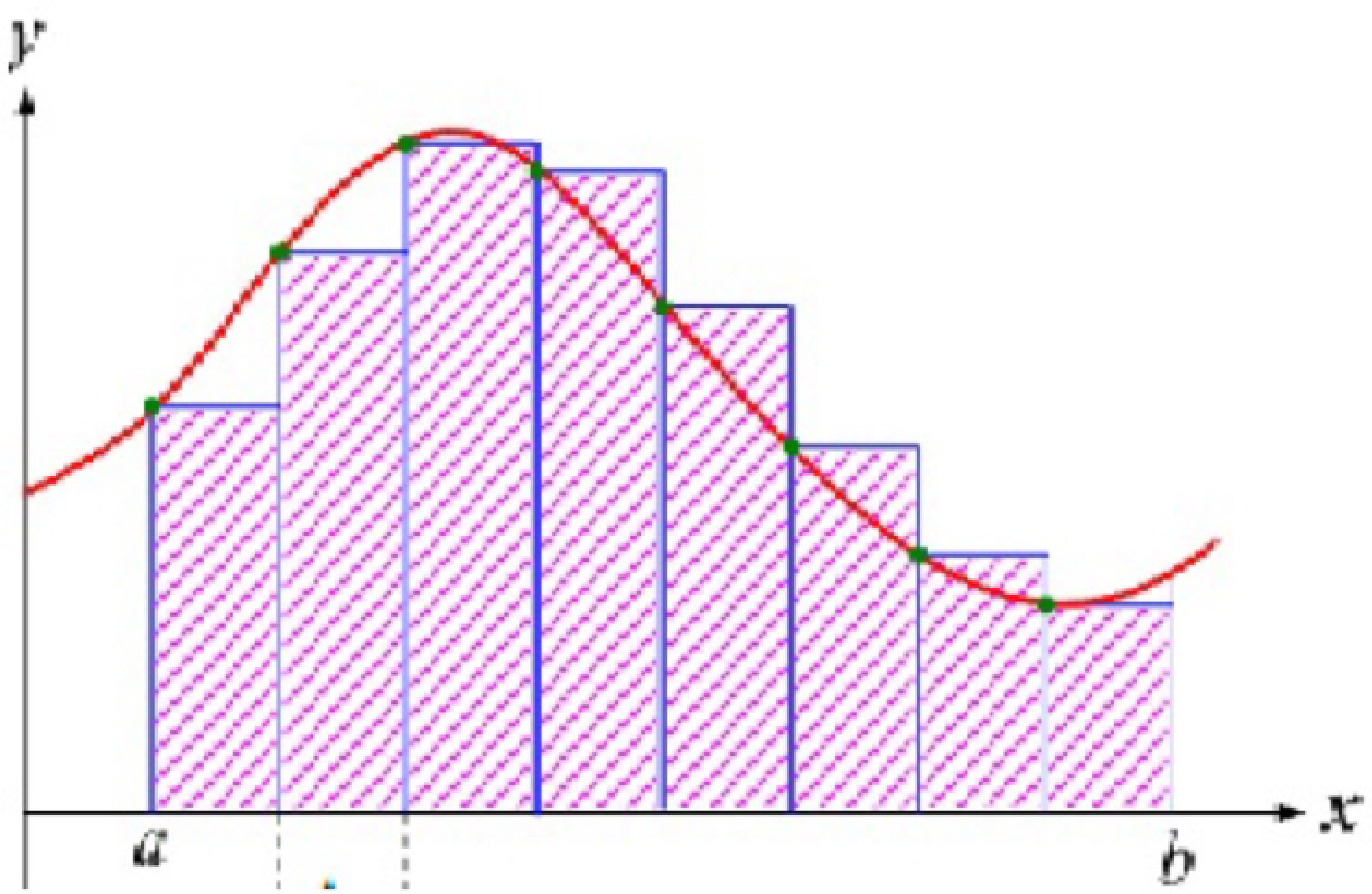
Summing over $i = 1$ to $i = N$ we obtain

$$I(f) = \int_a^b f(x)\,dx = h\sum_{i=1}^{N} f_{i-1} + \sum_{i=1}^{N} \frac{f'(\eta_i)h^2}{2} = R_C + E_C^R,$$

where

$$R_C = h\sum_{i=1}^{N} f_{i-1}$$

is the composite rectangle rule and the error



(Composite rectangle rule)

$$E_C^R = \sum_{i=1}^{N} \frac{f'(\eta_i)h^2}{2}, \quad \eta_i \in (x_{i-1}, x_i).$$

If $f'(x)$ is continuous, then we write

$$\sum_{i=1}^{N} \frac{f'(\eta_i)h^2}{2} = f'(\eta) \sum_{i=1}^{N} \frac{h^2}{2} = \frac{f'(\eta)Nh^2}{2}$$

With $Nh = b - a$,

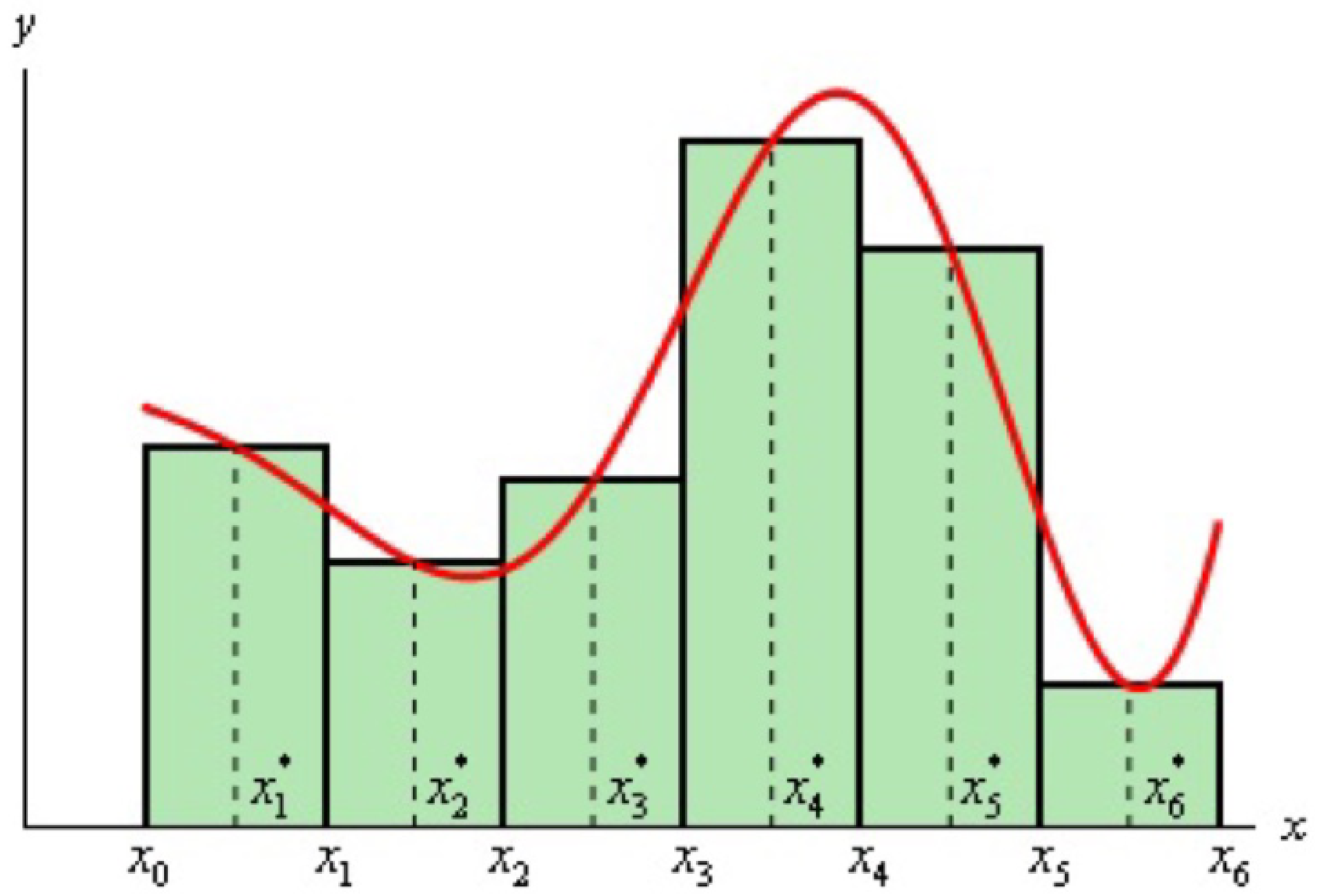
$E_C^R \to 0$ as $h \to 0$

$$E_C^R = \frac{f'(\eta)(b-a)h}{2}, \quad \text{for some } \eta \in (a,b).$$

Composite mid-point rule:

$$\int_a^b f(x)dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx h \sum_{i=1}^{N} f(x_{i-\frac{1}{2}}) = h \sum_{i=1}^{N} f_{i-1/2}.$$

$$I(f) \approx M_C = h \sum_{i=1}^{N} f_{i-1/2}.$$



(Composite mid-point rule)

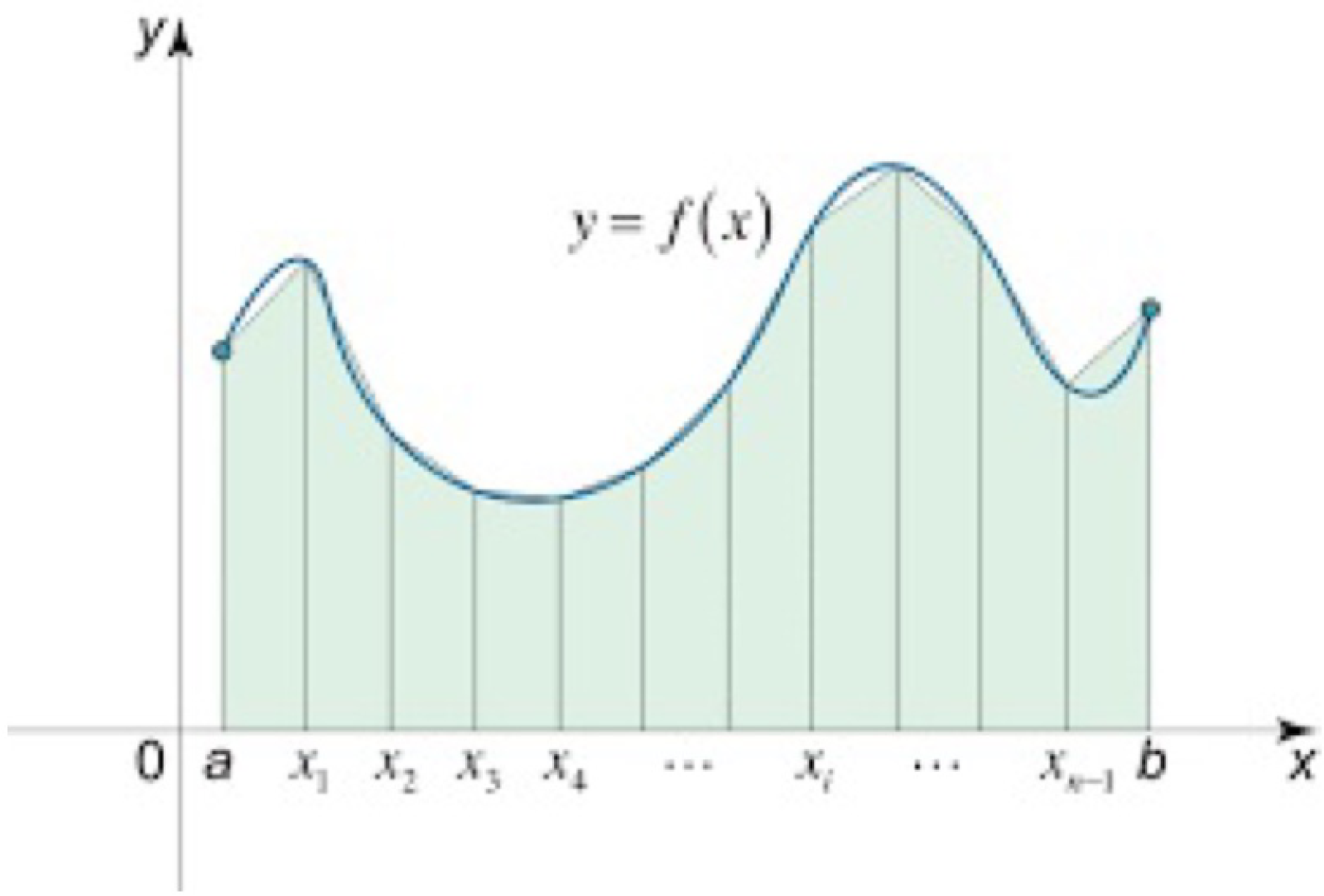
The error is given by

$$E_C^M = \frac{f''(\xi)(h)^2(b-a)}{24}$$

Composite trapezoid rule:

$$I(f) = \int_a^b f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \frac{h}{2}\sum_{i=1}^{N}[f(x_{i-1}) + f(x_i)].$$

$$I(f) \approx T_C = \frac{h}{2}\left[f_0 + 2\sum_{i=1}^{N-1} f_i + f_N\right].$$



(Composite trapezoid rule)

$$E_C^T = -\frac{f''(\eta)(h)^2(b-a)}{12}$$

Composite Simpson's rule:

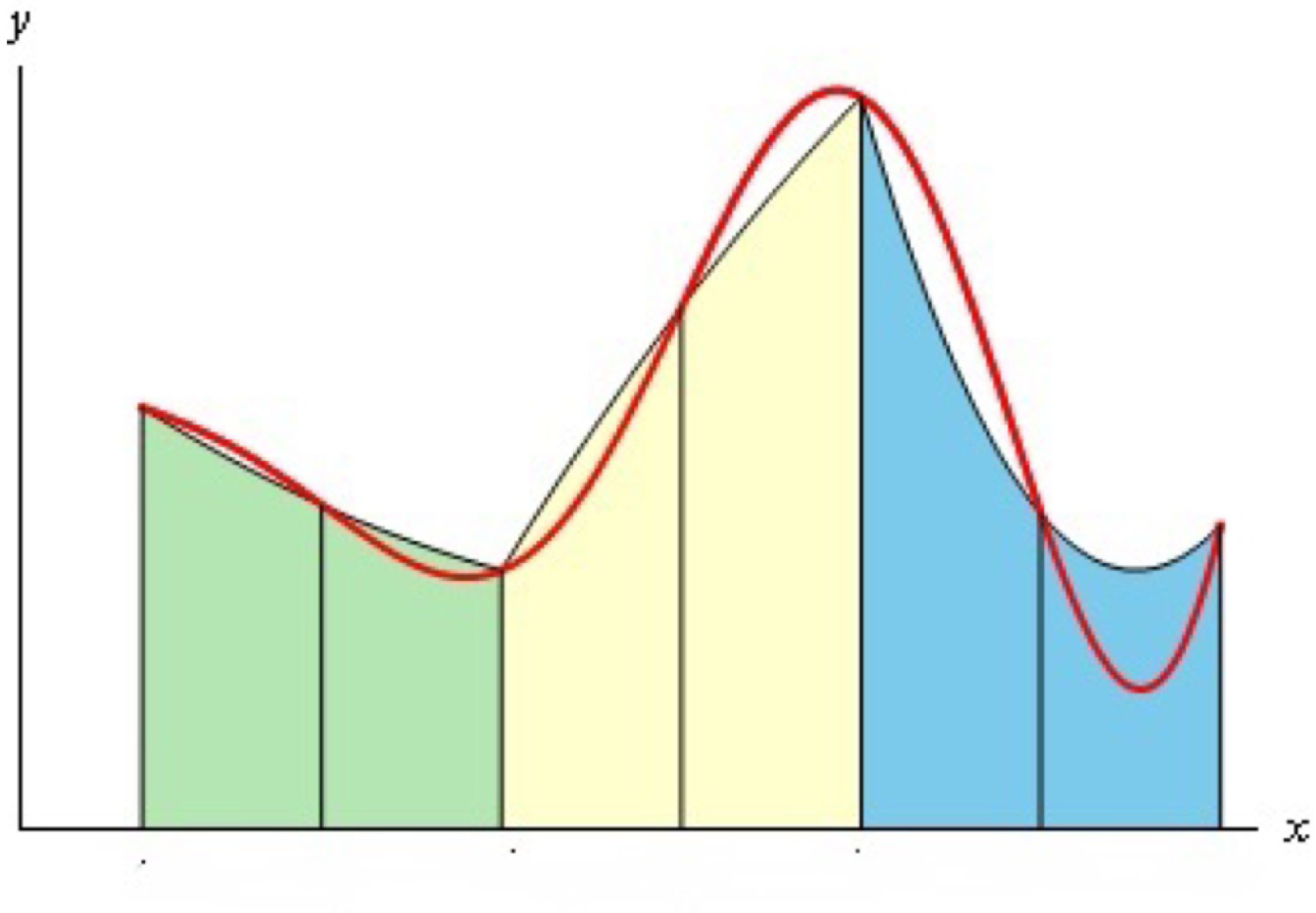
Letting $a = x_{i-1}$, $b = x_i$ and $x_i - x_{i-1} = h$, apply Simpson's rule over a single subinterval (x_{i-1}, x_i) to obtain

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} f(x)dx &= \overbrace{\frac{h}{6}[f(x_{i-1}) + 4f(x_i - h/2) + f(x_i)]}^{\text{Simpson's formula.}} - \overbrace{\frac{f^{iv}(\eta_i)(h/2)^5}{90}}^{\text{Error in } (x_{i-1}, x_i)} \\
&= \frac{h}{6}[f_{i-1} + 4f_{i-1/2} + f_i] - \frac{f^{iv}(\eta_i)(h/2)^5}{90}, \quad x_{i-1} < \eta_i < x_i.
\end{aligned}$$

Summing for $i = 1, \ldots, N$, we obtain

$$\begin{aligned}
I(f) &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)dx = \frac{h}{6}\sum_{i=1}^{N}[f_{i-1} + 4f_{i-1/2} + f_i] - \sum_{i=1}^{N} \frac{f^{iv}(\eta_i)(h/2)^5}{90} \\
&= S_C + E_C^S.
\end{aligned}$$

$$S_C = \frac{h}{6}\left[f_0 + f_N + 2\sum_{i=1}^{N-1} f_i + 4\sum_{i=1}^{N} f_{i-1/2}\right]$$



(Composite Simpson rule)

The error term is

$$E_C^S = \frac{f^{iv}(\xi)(h/2)^4(b-a)}{180}, \quad a < \xi < b$$

The corrected trapezoidal rule:

$$
\begin{aligned}
I(f) &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx \\
&\approx \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h^{2}}{12}[f'(x_0) - \cancel{f'(x_1)}] \\
&\quad + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h^{2}}{12}[\cancel{f'(x_1)} - f'(x_2)] \\
&\quad + \cdots + \frac{h}{2}[f(x_{N-1}) + f(x_N)] + \frac{h^{2}}{12}[\cancel{f'(x_{N-1})} - f'(x_N)]
\end{aligned}
$$

$x_0 = a$

$x_N = b$

Note that all the interior derivatives $f'(x_i), i = 1, \ldots, N-1$, cancel each other when the results of applying the corrected trapezoid rule on each subinterval are summed. The composite corrected trapezoidal rule is thus given by

$$
I(f) \approx C_C^T = \frac{h}{2}\left(f_0 + f_N + 2\sum_{i=1}^{N-1} f_i\right) + \frac{h^2}{12}[f'(a) - f'(b)].
$$

$$
E_C^{C^T} = \frac{f^{iv}(\eta)(h)^4(b-a)}{720}, \quad a < \eta < b.
$$