# Lecture 22: Solutions of Linear Difference Equations

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# Linear Difference Equations

A linear difference equation (LDE) of order $N$ is of the form

$$y_{n+N} + a_{N-1}\, y_{n+N-1} + a_{N-2}\, y_{n+N-2} + \cdots + a_0\, y_n = b_n, \tag{1}$$

where $a_{N-1}$, $a_{N-2}$, ..., $a_0$ are constants. If $b_n = 0$, equation (1) is called a homogeneous LDE, i.e.,

$$y_{n+N} + a_{N-1}\, y_{n+N-1} + a_{N-2}\, y_{n+N-2} + \cdots + a_0\, y_n = 0. \tag{2}$$

Example.

$$\begin{aligned} y_{n+1} - y_n &= 1,\ \forall n; & y_n &= n + c,\ \forall n \\ y_{n+1} - y_n &= n,\ \forall n \geq 0; & y_n &= \frac{n(n-1)}{2} + c,\ \forall n \geq 0 \\ y_{n+l} - (n+1)y_n &= 0,\ \forall n \geq 0; & y_n &= c\, n!, \end{aligned}$$

where $c$ is an arbitrary constant.

Solutions to homogeneous LDE. Seek solutions of the form $y_n = \beta^n$, $\forall n$. Substituting

$$y_{n+N} = \beta^{n+N},\ y_{n+N-1} = \beta^{n+N-1},\ \cdots, y_n = \beta^n$$

into (2) to obtain

$$\beta^{n+N} + a_{N-1}\,\beta^{n+N-1} + \cdots + a_0\,\beta^n = 0.$$

Dividing by $\beta^n$, we obtain

$$p(\beta) = \beta^N + a_{N-1}\,\beta^{N-1} + \cdots + a_0 = 0, \tag{3}$$

which is known as characteristic equation.

**Case I:** Assume that (3) has $N$ distinct zeros $\beta_1,\ \beta_2,\ \ldots,\ \beta_N$. Then

$$\beta_1^n,\ \beta_2^n,\ \ldots,\ \beta_N^n$$

are all solutions of (2). By linearity, we write the general solution (GS) as

$$y_n = c_1\beta_1^n + c_2\beta_2^n + \cdots + c_N\beta_N^n, \quad \forall n,$$

where $c_1$, $c_2$, ..., $c_N$ are arbitrary constants.

Note. If the first $N-1$ values of $y_n$ are given, the resulting initial-value difference equation can be solved explicitly for all succeeding values of $n$.

Example. Consider

$$y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0.$$

Note that it is a third order difference equation. Its characteristic equation is

$$\beta^3 - 2\beta^2 - \beta + 2 = 0,$$

whose roots are $1, -1, 2$. Thus, the GS is

$$y_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n = c_1 + (-1)^n c_2 + 2^n c_3.$$

If $y_0 = 0$, $y_1 = 1$, $y_2 = 1$, then imposing the initial conditions for $n = 0, 1, 2$, we obtain the following system of equations for $c_1, c_2, c_3$:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - c_2 + 2c_3 &= 1 \\ c_1 + c_2 + 4c_3 &= 1 \end{aligned}$$

Its solution is $c_1 = 0$, $c_2 = -1/3$, $c_3 = 1/3$. The solution of the initial-value problem is

$$y_n = -\frac{1}{3}(-1)^n + \frac{2^n}{3}.$$

**Case II**: If the characteristic polynomial in (3) has a pair of conjugate-complex roots, the solution can still be expressed in real form. Thus, if $\beta_1 = \alpha + i\beta$ and $\beta_2 = \alpha - i\beta$, then write

$$\beta_1 = re^{i\theta}, \quad \beta_2 = re^{-i\theta},$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$. The solution corresponding to this pair

$$\begin{aligned} c_1\beta_1^n + c_2\beta_2^n &= c_1 r^n e^{in\theta} + c_2 r^n e^{-in\theta} \\ &= r^n[c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)] \\ &= r^n(C_1\cos n\theta + C_2\sin n\theta), \end{aligned}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.
Example. Consider

$$y_{n+2} - 2y_{n+1} + 2y_n = 0.$$

Its characteristic equation is $\beta^2 - 2\beta + 2 = 0$, and its roots are $\beta_1 = 1 + i$ and $\beta_2 = 1 - i$. Here $r = \sqrt{2}$ and $\theta = \pi/4$. The GS is

$$y_n = (\sqrt{2})^n\left(C_1\cos\frac{n\pi}{4} + C_2\sin\frac{n\pi}{4}\right).$$

**Case III**: If $\beta_1$ is a double root of the characteristic equation (3), then a second solution of (2) is $n\beta_1^n$. Since $\beta_1$ is a double zero of $p(\beta) \implies p(\beta_1) = 0$ and $p'(\beta_1) = 0$. Substituting $y_n = n\beta_1^n$ in (2), we find that

$$\begin{aligned}(n+N)\beta_1^{n+N} &+ a_{n-1}(n+N-1)\beta_1^{n+N-1} + \cdots + a_0 n\beta_1^n \\ &= \beta_1^n\Big\{n(\beta_1^N + a_{N-1}\beta_1^{N-1} + \cdots + a_0) \\ &\quad +\beta_1(N\beta_1^{N-1} + a_{N-1}(N-1)\beta_1^{N-2} + \cdots + a_1)\Big\} \\ &= \beta_1^n[n\,p(\beta_1) + \beta_1 p'(\beta_1)] = 0.\end{aligned}$$

Example. Consider the difference equation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0.$$

The roots of the characteristic equation are $2, 2, 1$, and the GS is

$$y_n = 2^n(c_1 + n\,c_2) + c_3.$$

The nonhomogeneous LDE with constant coefficients. The GS of the equation

$$y_{n+N} + a_{N-1}y_{n+N-1} + \cdots + a_0 y_n = b_n \tag{4}$$

can be written in the form

$$y_n = y_n^G + y_n^P, \quad \text{where}$$

$$y_n^G - \text{the GS of the corresponding homogeneous equation (2)}$$
$$y_n^P - \text{a particular solution(PS) of (4)}$$

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Consider the special case when $b_n = b$ (constant). A PS is obtained by setting $y_n^P = A$ (a constant) in (4). Substitution of $y_n^P = A$ in (4) leads to

$$A = \frac{b}{1 + a_{N-1} + \cdots + a_0}, \quad \text{provided } 1 + a_{N-1} + \cdots + a_0 \neq 0.$$

Example. For $y_{n+2} - 2y_{n+l} + 2y_n = 1$,

$$y_n^G = (\sqrt{2})^n \left( C_1 \cos \frac{n\pi}{4} + C_2 \sin \frac{n\pi}{4} \right), \quad y_n^P = 1.$$

The GS is $y_n = (\sqrt{2})^n \left( C_1 \cos \frac{n\pi}{4} + C_2 \sin \frac{n\pi}{4} \right) + 1.$

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