Lecture 20: Numerical Solutions to IVPs for ODEs (One-Step Methods) Contd...

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Runge-Kutta Method

Seek a formula of the form

$$y_{n+1} = y_n + ak_1 + bk_2, \quad \text{where} \tag{1}$$
$$k_1 = hf(x_n, y_n), \quad k_2 = hf(x_n + \alpha h, y_n + \beta k_1),$$

and the constants a, b, α, β are to be determined so that (1) will agree with the Taylor algorithm of as high an order as possible.

Expanding $y(x_{n+1})$ in a Taylor's series

$$\begin{aligned} y(x_{n+1}) = y(x_n + h) &= y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + \cdots \\ &= y(x_n) + hf(x_n, y_n) + \frac{h^2}{2}(f_x + f_y f)_n \qquad (2) \\ &\quad + \frac{h^3}{6}(f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f)_n + O(h^4), \end{aligned}$$

where the subscript n means that all functions involved are to be evaluated at (x_n, y_n).

On the other hand, using Taylor's expansion for functions of two variables, we have

$$\begin{aligned}\frac{k_2}{h} &= f(x_n+\alpha h+y_n+\beta k_1) = f(x_n,y_n)+\alpha h f_x+\beta k_1 f_y \\ &\quad +\frac{\alpha^2h^2}{2}f_{xx}+(\alpha h\beta k_1)f_{xy}+\frac{\beta^2k_1^2}{2}f_{yy}+O(h^3),\end{aligned}$$

where all derivatives are evaluated at (x_n, y_n). Substituting the expression for k_2 into (1) and using $k_1 = h\,f(x_n, y_n)$, we find upon rearrangement in powers of h that

$$\begin{aligned}y_{n+1} &= y_n+(a+b)hf+bh^2(\alpha f_x+\beta ff_y) \\ &\quad +bh^3\left(\frac{\alpha^2}{2}f_{xx}+\alpha\beta ff_{xy}+\frac{\beta^2}{2}f^2f_{yy}\right)+O(h^4). \qquad (3)\end{aligned}$$

Comparing the corresponding powers of h and h^2 from (2) and (3), we must have

$$a+b=1$$
$$b\alpha=b\beta=\frac{1}{2}$$

There are many solutions to the above system, the simplest perhaps being

$$a = b = \frac{1}{2}, \quad \alpha = \beta = 1.$$

Algorithm(Runge-Kutta method of order 2): For the equation

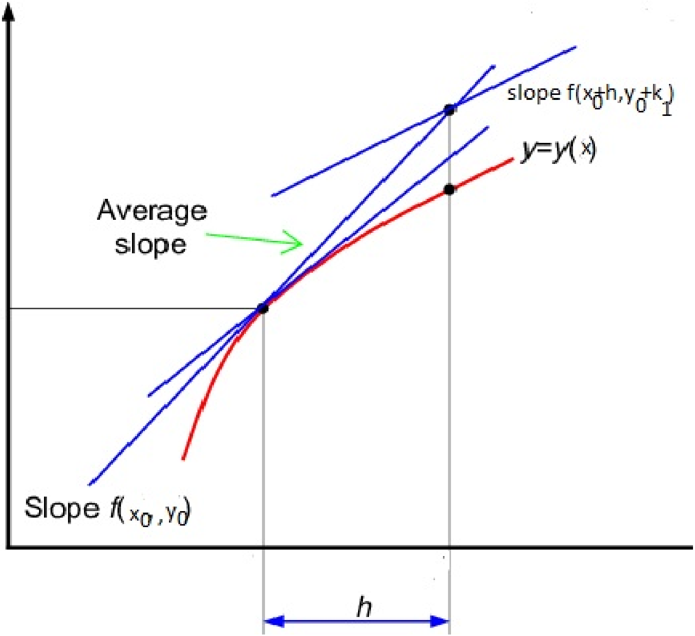
$$y' = f(x, y), \;\; y(x_0) = y_0$$

generate approximations y_n to the exact solution $y(x_0 + nh)$, for h fixed and $n = 0, 1, \ldots$, using the recursion formula

$$\begin{aligned} y_{n+1} \;\; &= \;\; y_n + \frac{1}{2}(k_1 + k_2), \;\; \text{with} \\ & \;\;\;\;\; k_1 = h\, f(x_n, y_n), \\ & \;\;\;\;\; k_2 = h\, f(x_n + h, y_n + k_1). \end{aligned}$$

The local error is of the form

$$\begin{aligned} y(x_{n+1}) - y_{n+1} \;\; &= \;\; \frac{h^3}{12}\left(f_{xx} + 2ff_{xy} + f^2 f_{yy} - 2f_x f_y - 2ff_y^2\right) + O(h^4) \\ &= O(h^3). \end{aligned}$$



(Second-order Runge-Kutta Method)

Example. Consider the IVP:

$$y' = y - x, \quad y(0) = 2.$$

Find $y(0.1)$ and $y(0.2)$ using second-order Runge-Kutta method.
With $x_0 = 0, y_0 = 2$ and $h = 0.1$, compute

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0) = h(y_0 - x_0) = 0.2 \\
k_2 &= h\,f(x_0 + h, y_0 + k_1) = 0.1f(0.1, 2 + 0.2) = 0.21 \\
y(0.1) &\approx y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + (0.41)/2 = 2.2050
\end{aligned}$$

To determine $y_2 \approx y(0.2)$, set $x_1 = 0.1\ y_1 = 2.2050$. Compute

$$\begin{aligned}
k_1 &= 0.1(2.105) = 0.2105, \quad k_2 = 0.1(2.4155 - 0.2) = 0.22155. \\
y_2 &= y_1 + \frac{1}{2}(k_1 + k_2) = 2.2050 + \frac{1}{2}(0.2105 + 0.22155) = 2.4210.
\end{aligned}$$

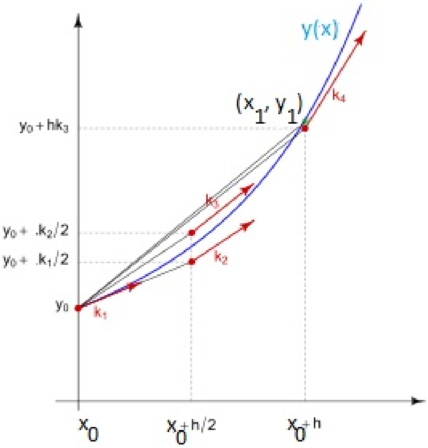
Algorithm(Runge-Kutta method of order 4): For the equation

$$y' = f(x, y), \quad y(x_0) = y_0$$

generate approximations y_n to the exact solution $y(x_0 + nh)$, for h fixed and $n = 0, 1, \ldots$, using the recursion formula

$$\begin{aligned} y_{n+1} \quad &= \quad y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \text{ with} \\ &\quad\;\; k_1 = h\, f(x_n, y_n), \\ &\quad\;\; k_2 = h\, f(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1) \\ &\quad\;\; k_3 = h\, f(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2) \\ &\quad\;\; k_4 = h\, f(x_n + h, y_n + k_3) \end{aligned}$$

The local discretization error is $= O(h^5)$.



(Fourth-order Runge-Kutta Method)

Systems of Differential Equations

An Nth-order equation of the form

$$y^{(N)}(x) = f(x, y(x), y'(x), \ldots, y^{(N-1)}(x)) \tag{4}$$

can be written as a system of N first-order equations as follows. With $y_1 = y$, set

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= y_4 \\
&\vdots \\
y_{N-1}' &= y_N \\
y_N' &= f(x, y_1, y_2, \ldots, y_N).
\end{aligned}$$

More generally, a system of N first-order equations will have the form

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, \ldots, y_N) \\ y_2' &= f_2(x, y_1, y_2, \ldots, y_N) \\ &\vdots \\ y_N' &= f_N(x, y_1, y_2, \ldots, y_N). \end{aligned}$$

For simplicity, we now illustrate fourth-order Runge-Kutta method for the system of two equations of the form

$$\begin{aligned} y' &= f(x, y, z) \\ z' &= g(x, y, z) \end{aligned}$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

Algorithm. Set $x_n = x_0 + nh$, $n = 0, 1, \ldots$. Generate approximations y_n and z_n to the exact solutions $y(x_n)$ and $z(x_n)$, using the recursion formula

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\
z_{n+1} &= z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \quad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= h\, f(x_n, y_n, z_n), \\
l_1 &= h\, g(x_n, y_n, z_n), \\
k_2 &= h\, f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}) \\
l_2 &= h\, g(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}) \\
k_3 &= h\, f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}) \\
l_3 &= h\, g(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}) \\
k_4 &= h\, f(x_n + h, y_n + k_3, z_n + l_3) \\
l_4 &= h\, g(x_n + h, y_n + k_3, z_n + l_3)
\end{aligned}
$$

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