Lecture 25: Finite Difference Methods for The Heat Equation

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Model Problem

Consider one-dimensional heat equation of the form

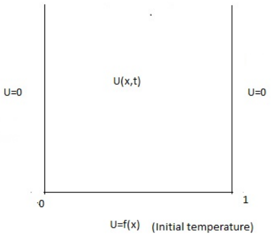
$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial x^2} \quad x \in (0,1),\ t>0 \\ U(0,t) &= U(1,t) = 0, \quad t>0 \ \ \text{(Boundary conditions)} \\ U(x,0) &= f(x), \ \ 0 \le x \le 1. \ \ \text{(Initial condition)}\end{aligned}$$

Set $x_i = ih$, $(i = 0, 1, 2, \ldots)$ and $t_j = jk$, $(j = 0, 1, 2, \ldots)$.

Let U_{ij} be the true value of the solution at the grid-point (x_i, t_j).

Let u_{ij} denote finite difference approximation to the true solution at (x_i, t_j).

$u_{ij} \approx U_{i,j} = U(x_i, t_j)$.



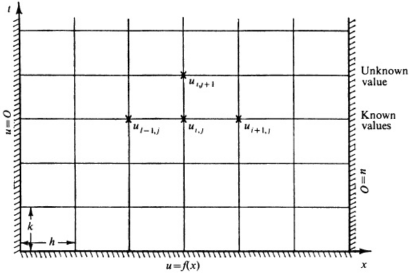
Schmidt's explicit scheme: Using forward time and central space (FTCS) schemes to approximate $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ at the grid-point (x_i, t_j).

$$\left(\frac{\partial U}{\partial t}\right)_{(x_i,t_j)} \approx \frac{u_{i,j+1} - u_{i,j}}{k}, \quad \left(\frac{\partial^2 U}{\partial x^2}\right)_{(x_i,t_j)} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2},$$

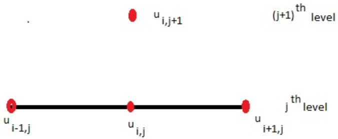
$$\begin{aligned}
& \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \\
\implies \quad & u_{i,j+1} = u_{i,j} + \frac{k}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \\
\implies \quad & u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}), \quad r = \frac{k}{h^2}, \\
\implies \quad & u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j},
\end{aligned}$$

FACT:

- It is a two-level explicit scheme.
- This scheme is conditionally stable $(0 < r \leq \frac{1}{2})$.
- The local truncation error is $O(h^2) + O(k)$.



(Schmidt's explicit scheme)



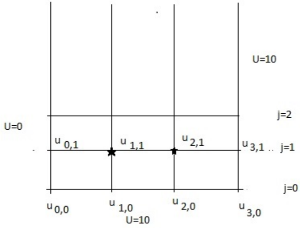
(Computational stencil)

Example:

$$\begin{aligned} U_t &= U_{xx}, \quad 0 < x < 1, \ t > 0 \\ U(0,t) &= 0, \quad U(1,t) = 10, \quad t > 0 \\ U(x,0) &= 10, \quad 0 \leq x \leq 1 \end{aligned}$$

Choose h and k such that $r = 1/2$.

$$\begin{aligned} u_{1,1} &= \frac{1}{2}(u_{0,0} + u_{2,0}) = 10 \\ u_{2,1} &= \frac{1}{2}(u_{1,0} + u_{3,0}) = 10 \\ u_{1,2} &= \frac{1}{2}(u_{0,1} + u_{2,1}) = 5 \\ u_{2,2} &= \frac{1}{2}(u_{1,1} + u_{3,1}) = 10 \end{aligned}$$



Euler's implicit scheme: Use backward in time and central in space (BTCS) schemes at the point (x_i, t_j) to have

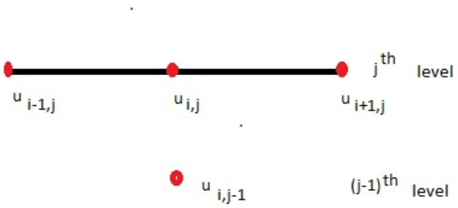
$$\left(\frac{\partial U}{\partial t}\right)_{(x_i,t_j)} \approx \frac{u_{i,j}-u_{i,j-1}}{k}, \quad \left(\frac{\partial^2 U}{\partial x^2}\right)_{(x_i,t_j)} \approx \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2},$$

to obtain

$$\begin{aligned} & \frac{u_{i,j}-u_{i,j-1}}{k} = \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2} \\ \Longrightarrow \quad & u_{i,j} = u_{i,j-1} + r(u_{i-1,j}-2u_{i,j}+u_{i+1,j}), \quad r = \frac{k}{h^2}, \\ \Longrightarrow \quad & -ru_{i-1,j} + (1+2r)u_{i,j} - ru_{i+1,j} = u_{i,j-1} \end{aligned}$$

FACT:

- It is a two-level implicit scheme.
- At each time level, we are required to solve a linear system.
- This scheme is unconditionally stable (no restriction on r).
- The local truncation error is $O(h^2) + O(k)$.



(Computational stencil)

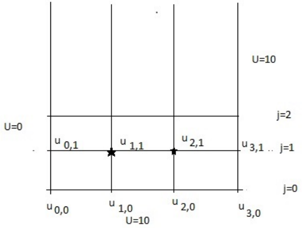
Example:

$$\begin{aligned} U_t &= U_{xx}, \ \ 0 < x < 1, \ \ t > 0 \\ U(0,t) &= 0, \ \ U(1,t) = 10, \ \ \ t > 0 \\ U(x,0) &= 10, \ \ 0 \le x \le 1 \end{aligned}$$

Choose h and k such that $r = 1/2$.

$$-\frac{1}{2}u_{0,1} + 2u_{1,1} - \frac{1}{2}u_{2,1} = u_{1,0}$$

$$\implies 2u_{1,1} - \frac{1}{2}u_{2,1} = 10$$



$$\begin{aligned} & -\frac{1}{2}u_{1,1} + 2u_{2,1} - \frac{1}{2}u_{3,1} = u_{2,0} \\ \implies \quad & -\frac{1}{2}u_{1,1} + 2u_{2,1} = 15 \end{aligned}$$

Crank-Nicolson scheme:

$$
\begin{aligned}
\left(\frac{\partial U}{\partial t}\right)_{(x_i, t_{j+1/2})} &\approx \frac{u_{i,j+1} - u_{i,j}}{k}, \\
\left(\frac{\partial^2 U}{\partial x^2}\right)_{(x_i, t_{j+1/2})} &\approx \frac{1}{2}\left[\frac{(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})}{h^2}\right. \\
&\qquad \left. + \frac{(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})}{h^2}\right],
\end{aligned}
$$

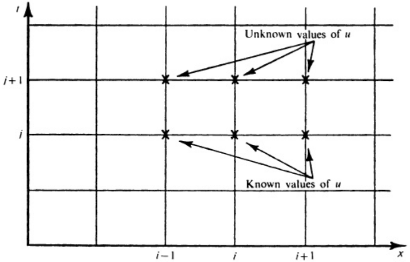
$$
\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right].
$$

$$
\implies -ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j},
$$

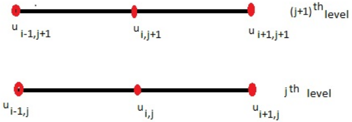
where $r = \frac{k}{h^2}$.

FACT:

- It is a two-level implicit scheme.
- At each time level, we are required to solve a linear system.
- This scheme is unconditionally stable (no restriction on r).
- The local truncation error is $O(h^2) + O(k^2)$.



(Crank-Nicolson scheme)



(Computational stencil)

Richardson's scheme: An application of central in time and central in space (CTCS) approximation i.e.,

$$\left(\frac{\partial U}{\partial t}\right)_{(x_i,t_j)} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2k}, \quad \left(\frac{\partial^2 U}{\partial x^2}\right)_{(x_i,t_j)} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2},$$

to obtain the resulting scheme

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \tag{1}$$

FACT:

- It is a three-level explicit scheme.
- This scheme is unstable (hence not recommended).
- The local truncation error is $O(h^2) + O(k^2)$.

DuFort-Frankel explicit scheme: A modification of (1) is as follows:

$$u_{i,j} = \frac{u_{i,j-1} + u_{i,j+1}}{2}.$$

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i-1,j} - 2\frac{(u_{i,j-1}+u_{i,j+1})}{2} + u_{i+1,j}}{h^2}$$

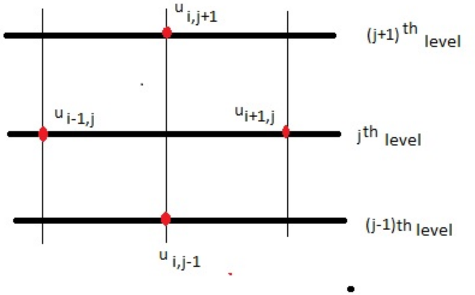
$$\implies \quad u_{i,j+1} - u_{i,j-1} = 2r\{u_{i-1,j} - (u_{i,j-1} + u_{i,j+1}) + u_{i+1,j}\}, \quad r = \frac{k}{h^2}$$

$$\implies \quad (1+2r)u_{i,j+1} = (1-2r)u_{i,j-1} + 2r(u_{i-1,j} + u_{i+1,j})$$

$$\implies \quad u_{i,j+1} = \frac{(1-2r)}{(1+2r)}u_{i,j-1} + \frac{2r}{(1+2r)}(u_{i-1,j} + u_{i+1,j})$$

FACT:

- It is a three-level explicit scheme.
- This scheme is unconditionally stable.
- The local truncation error is $O(h^2) + O(k^2)$.



(Computational stencil)
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