Lecture 26: FDM for The Heat Equation Contd..

Rajen Kumar Sinha

Department of Mathematics
IIT Guwahati

Weighted Average Approximation: A more general finite-difference approximation to the heat equation $U_t = U_{xx}$ is given by

$$\begin{aligned} \frac{u_{i,j+1} - u_{i,j}}{k} &= \frac{1}{h^2}\Big[\theta(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) \\ &\quad +(1-\theta)(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})\Big], \qquad (1) \end{aligned}$$

where $0 \le \theta \le 1$. Observe that

$$\begin{aligned} \theta = 0 \quad &\Longrightarrow \quad \text{the explicit scheme (Schmidt's scheme)} \\ \theta = 1 \quad &\Longrightarrow \quad \text{the Euler's implicit scheme} \\ \theta = \frac{1}{2} \quad &\Longrightarrow \quad \text{the Crank-Nicolson scheme} \end{aligned}$$

FACT:

- The scheme (1) is unconditionally stable for $\frac{1}{2} \le \theta \le 1$.
- The scheme (1) is conditionally stable for $0 \le \theta < \frac{1}{2}$. The condition for stability is

$$r = \frac{k}{h^2} \le \frac{1}{2(1-2\theta)}.$$

Define the central difference operators δ_x and δ_t as:

$$\begin{aligned} \delta_x \phi_{i,j} &= \phi_{i+\frac{1}{2},j} - \phi_{i-\frac{1}{2},j} \\ \delta_t \phi_{i,j} &= \phi_{i,j+\frac{1}{2}} - \phi_{i,j-\frac{1}{2}} \end{aligned}$$

Using this, the explicit scheme can be written as

$$\frac{1}{k}\delta_t u_{i,j+\frac{1}{2}} = \frac{1}{h^2}\delta_x^2 u_{i,j}, \quad \text{where}$$

$$\begin{aligned} \delta_t u_{i,j+\frac{1}{2}} &= u_{i,j+1} - u_{i,j} \\ \delta_x^2 u_{i,j} &= \delta_x(\delta_x u_{i,j}) = \delta_x(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}) \\ &= u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \end{aligned}$$

The Euler's implicit scheme:

$$\frac{1}{k}\delta_t u_{i,j-\frac{1}{2}} = \frac{1}{h^2}\delta_x^2 u_{i,j}.$$

The Crank-Nicolson scheme:

$$\frac{1}{k}\delta_t u_{i,j+\frac{1}{2}} = \frac{1}{2h^2}\left[\delta_x^2 u_{i,j+1} + \delta_x^2 u_{i,j}\right]$$

The weighted average scheme:

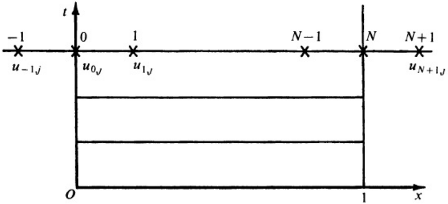
$$\frac{1}{k}\delta_t u_{i,j+\frac{1}{2}} = \frac{1}{h^2}\left[\theta\,\delta_x^2 u_{i,j+1} + (1-\theta)\,\delta_x^2 u_{i,j}\right]$$

Derivative Boundary Conditions:

$$\frac{\partial U}{\partial x} = C(U - V) \text{ at } x = 0,\ t > 0$$

$$\frac{\partial U}{\partial x} = -C(U - V) \text{ at } x = 1,\ t > 0$$

V – the temperature of the surrounding and it is assumed to be constant.



Using forward difference approximation, we have at $x = 0$

$$
\begin{aligned}
& \frac{u_{1,j} - u_{0,j}}{h} = C(u_{0,j} - V) \\
\implies \qquad & u_{1,j} = u_{0,j} + Ch(u_{0,j} - V) \\
\implies \qquad & u_{1,j} = (1 + Ch)u_{0,j} - ChV
\end{aligned}
$$

The truncation error (T.E.)$=O(h)$, which leads to a loss of accuracy in h. To obtain a better approximation (of $O(h^2)$), use central difference scheme to have

$$
\begin{aligned}
& \frac{u_{1,j} - u_{-1,j}}{2h} = C(u_{0,j} - V) \\
\implies \qquad & u_{1,j} = u_{-1,j} + 2hC(u_{0,j} - V). \qquad (2)
\end{aligned}
$$

Recall the explicit scheme

$$
u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}. \qquad (3)
$$

For $i = 0$,

$$
u_{0,j+1} = ru_{-1,j} + (1 - 2r)u_{0,j} + ru_{1,j}, \quad j = 0, 1, 2, \ldots. \qquad (4)
$$

Now, substituting the value of $u_{-1,j}$ from (2) in (4), it follows that

$$\begin{aligned} u_{0,j+1} &= r\{u_{1,j} - 2hC(u_{0,j} - V)\} + (1-2r)u_{0,j} + ru_{1,j} \\ &= 2ru_{1,j} + \{(1-2r) - 2rhC\}u_{0,j} + 2rhCV \\ &= \{1 - 2r(1+hc)\}u_{0,j} + 2ru_{1,j} + 2rhCV, \quad j = 0, 1, 2, \ldots \end{aligned}$$

At the other end $x = 1$, That is, with $i = N$, we have

$$\left(\frac{\partial U}{\partial x}\right)_{N,J} = C(V - U)_{N,j}$$

Use central difference approximation to obtain

$$\begin{aligned} & \frac{u_{N+1,j} - u_{N-1,j}}{2h} = C(V - u_{N,j}) \\ \Longrightarrow \qquad & u_{N+1,j} = u_{N-1,j} + 2hC(V - u_{N,j}) \end{aligned}$$

From the explicit scheme (3), we have for $i = N$,

$$\begin{aligned} u_{N,j+1} &= ru_{N+1,j} + (1-2r)u_{N,j} + ru_{N-1,j} \\ &= r\{u_{N-1,j} + 2hC(V - u_{N,j})\} + (1-2r)u_{N,j} + ru_{N-1,j} \\ &= 2ru_{N-1,j} + \{(1-2r) - 2rhC\}u_{N,j} + 2rhCV \\ &= 2ru_{N-1,j} + \{1 - 2r(1+hc)\}u_{N,j} + 2rhCV, \quad j = 0, 1, 2, \ldots \end{aligned}$$

For $i = 1, \ldots N-1$,

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}, \quad j = 0, 1, \ldots.$$

*** Ends ***