Spacial Economic Lecture2 Note

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Abstract

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This lecture introduces the Dixit-Stiglitz Model of monopolistic competition, in which firms have freedom to choose products from many potentially differentiated product varieties. The cost structure is the same regardless of which commodity is selected, which means the fixed cost and the marginal cost are constant. On the other hand, the consumer preferences for products are diverse and symmetrical. Based on these two assumptions, firms specialize in one product which is different from other firms. In this lecture, the author gives the model under the CES utility function and there is no subsidy to firms.

1 Utility Function and Competition

We assume that products can be divided into agricultural products and industrial products. All the agricultural products are homogeneous and industrial products are made up of many differentiated products. Consumer's preference can be described as following functions.

$$U = M^{\mu} A^{1-\mu}, 0 < \mu < 1 \tag{1}$$

$$M = [\int_0^n q(i)^{\rho} di]^{\frac{1}{\rho}} \tag{2}$$

Here, A is the consumption of agricultural products. $q(i)$ is the consumption of industrial product i. M is the sub-utility defined by all the industrial products. In other words, upper utility function (1) is Cobb-Douglas function and lower sub-utility function (2) is the CES(constant elasticity of substitution) function. Besides, if we assume $\sigma \equiv 1/(1-\rho)$, then σ is the elasticity of substitution of any two industrial products. Here, σ is constant under the assumption.

1.1 CES function

Here we give the mathematical explanation of constant elasticity of substitution function. If we rewrite (2) into discrete form, then we have

$$M = [\sum_{i=1}^{n} q(i)^{\rho}]^{\frac{1}{\rho}}$$

Let us consider when there are two industrial products what will happen to the utility function. If there is a constant number $\alpha \in (0,1), \rho < 1$, consumer's utility function of these two products is

$$u(x) = [\alpha x_1^{\rho} + (1-\alpha) x_2^{\rho}]^{\frac{1}{\rho}}$$

Combined with the first order condition of minimizing $p_1 x_1 + p_2 x_2$, we have

$$\frac{x_1}{x_2} = (\frac{1-\alpha}{\alpha}\frac{p_1}{p_2})^{\frac{1}{\rho-1}}$$

$$\ln\frac{x_1}{x_2} = \frac{1}{\rho-1}(\ln\frac{1-\alpha}{\alpha} + \ln\frac{p_1}{p_2})$$

Thus, the elasticity of substitution is

$$-\frac{\frac{p_1}{p_2}}{\frac{\partial(\frac{x_1}{x_2})}{\partial(\frac{p_1}{p_2})}} = \frac{1}{1-\rho} = \sigma$$

When $\rho \to 1, \sigma \to \infty$, the differentiated product tends to be substituted good, while when $\rho \to -\infty, \sigma \to 0$, the differentiated product tends to be complementary good.

As special examples, CES function contains common following functions in economics.

When $\rho = 1$, the CES function turns into linear function $u(x) = \alpha x_1 + \alpha x_2$

When $\rho \to 0$, because

$$\begin{aligned}
\lim_{\rho\to 0} \ln u(x) &= \lim_{\rho\to 0} \frac{ln[\alpha x_1^\rho + (1-\alpha)x_2^\rho}{\rho} \\
&= \lim_{\rho\to 0} \frac{\alpha x_1^\rho ln x_1 + (1-\alpha)x_2^\rho ln x_2}{\alpha x_1^\rho + (1-\alpha)x_2^\rho} \\
&= \frac{\alpha ln x_1 + (1-\alpha) ln x_2}{\alpha + 1 - \alpha} = ln x_1^\alpha x_2^{1-\alpha}
\end{aligned}$$

Thus, we have

$$\lim_{\rho\to 0} u(x) = x_1^\alpha x_2^{1-\alpha}$$

It is called Cobb-Douglas Function.

1.2 Two-stage approach

Here, Income y, agricultural product price p^a, and all the industrial products' price $p(i)$ are given. Consumer's budget constraint is

$$p^a A + \int_0^n p(i)q(i)di = y$$

In order to maximize the utility function (1), we use two-stage approach.

In the first stage, we assume the sub-utility M is given and solve the minimum cost problem.

$$\begin{aligned}
&min \int_0^n p(i)q(i)di \\
s.t. \quad &[\int_0^n q(i)^\rho di]^{\frac{1}{\rho}} = M
\end{aligned} \tag{3}$$

The FOC of this problem is that the ratio of elasticity of substitution of any two industrial products i, j equals to the ratio of the price, which is

$$\frac{q(i)^{\rho-1}}{q(j)^{\rho-1}} = \frac{p(i)}{p(j)}$$

Recall the Lagrangian Function

$$L = \int_0^n p(i)q(i)di + \lambda(M^\rho - \int_0^n q(i)^\rho di)$$

FOC of it is

$$\frac{\partial L}{\partial q(i)} = 0,\text{for} \quad i = 1, \ldots, n$$

Then we have

$$q(i) = q(j)[\frac{p(j)}{p(i)}]^{\frac{1}{1-\rho}}$$

Take this line into the constraint, we have

$$q(j) = \frac{p(j)^{\frac{1}{1-\rho}}}{[\int_0^n p(i)^{\frac{\rho}{\rho-1}} di]^{\frac{1}{\rho}}} \tag{4}$$

Take (4) into the target function, we have the minimum cost of consumer

$$\int_0^n p(j)q(j)dj = [\int_0^n p(i)^{\frac{\rho}{\rho-1}} di]^{\frac{\rho}{1-\rho}} M$$

Rewrite the right-side part of this equation,

$$P \equiv [\int_0^n p(i)^{\frac{\rho}{\rho-1}} di]^{\frac{\rho}{1-\rho}} = [\int_0^n p(i)^{1-\sigma} di]^{\frac{1}{1-\sigma}} \tag{5}$$

Here P stands for the minimum cost to get one unit sub-utility. Thus, we can regard P as the **price index** of industrial products. Last, take (5) back into (4), we can rewrite the compensation demand function into

$$q(j) = [\frac{p(j)}{P}]^{\frac{1}{\rho-1}} M \tag{6}$$

In the second stage, we assume that the income y is given and maximize the consumer's utility.

$$\max U = M^{\mu} A^{1-\mu}$$
$$s.t. \quad PM + p^a A = y$$

FOC:

$$M = \mu \frac{y}{P} \qquad A = (1-\mu)\frac{y}{p^a}$$

Bring it back to (6), we can get

$$q(j) = \frac{p(j)^{-\sigma}}{P^{1-\sigma}} \mu y \tag{7}$$

From (1), we can get the indirect utility function

$$V = \mu^{\mu}(1-\mu)^{1-\mu} y P^{-\mu} (p^a)^{-(1-\mu)} \tag{8}$$

To show the Competition Effect, which means the more competitive the industry is, the smaller the demand for each product will be, let all the differentiated products' price be p, with (5) the Price Index can be rewrote into:

$$P = (np^{1-\sigma})^{\frac{1}{1-\sigma}} = n^{\frac{1}{1-\sigma}} p$$

Thus, we know the decrease of price p and the increase of the mass n will cause the decrease of the Price Index.

2 Introduce Transport Costs

2.1 Iceberg Transport Cost Model

Recording to the Samuelson(1952)'s Model, we use Iceberg model to calculate the transport cost. Here we assume that if 1 unit of product is transported from region 1 to region 2, only $1/\tau$ units actually arrive, which means $1 - 1/\tau$ unit of products are disappeared during the transportation. In this condition, $\tau > 1$.

2.2 Application

Based on Iceberg model, we have

$$\begin{aligned}\text{supply} &= \tau \times \text{foreign demand} + \text{domestic demand}\\ \phi &\equiv \tau^{1-\sigma}\\ p_i &: \text{mill price in } i\\ p_{ij} &= p_i\tau \text{if } j \neq i \text{ from } i \text{ to } j\end{aligned}$$

Two Price Indices:

$$\begin{aligned}P_1 &= [n_1 p_1^{1-\sigma} + n_2(p_2\tau)^{1-\sigma}]^{\frac{1}{1-\sigma}}\\ P_2 &= [n_1(p_1\tau)^{1-\sigma} + n_2 p_2^{1-\sigma}]^{\frac{1}{1-\sigma}}\end{aligned} \tag{9}$$

Besides, with (7), we have country j's demand on country i products

$$q_i j = \mu Y_j (p_\tau)^{-\sigma} P_j^{\sigma-1}$$

With Iceberg transport cost, we have the total demand of country i

$$q_i = q_{ii} + \tau q_{ij} = \mu p_i^{-\sigma}(\frac{Y_i}{P_i^{1-\sigma}} + \frac{\phi Y_j}{P_j^{1-\sigma}}) \tag{10}$$

Here ϕ is called trade freeness or trade openness and $\phi \in (0,1)$ as definition.

3 Firm production and allocation

We assume that the industrial production has the technology of increasing returns to scale. For simplicity, we also assume that all the firms have the same production technology. Its fixed input C^f and its marginal input C^m are constant. If the mill price is p_i, the profit of a firm is

$$\Pi_i = p_i q_i - (C^f + C^m q_i) \tag{11}$$

where

$$q = \frac{p^{-sigma}}{P^{1-\sigma}}\mu y$$

Firm chooses optimal price p to maximize the profit. FOC of it

$$\frac{d\pi}{dp} = q + (p - C^m)\frac{dq}{dp}$$

Consider the monopolistic competition, a single price p does not impact on the price index. So the P is independent on p_i.

$$\frac{dq}{dp} = -\sigma\mu p_i^{-\sigma-1}\frac{Y_i}{P_i^{1-\sigma}} - \sigma\mu p_i^{-\sigma-1}\frac{\phi Y_j}{P_j^{1-\sigma}} = -\sigma\frac{q_i}{p_i}$$

So we have the optimal price

$$p_i = \frac{\sigma}{\sigma-1}C^m = \frac{1}{\rho}C^m \tag{12}$$

Define the markup of firm i as $(p_i - C^m)/p_i$. With the elasticity of substitution of demand, we can know the markup is constant, which is $\frac{1}{\sigma}$.

Bring it back to the profit function, we have

$$\Pi_i = \frac{C^m}{\sigma-1}q_i - C^f \tag{13}$$

Due to the free entry condition, so the profit equals to zero. Then we can get the free entry condition

$$C^f = \frac{C^m q_i}{\sigma-1} = \frac{1}{\sigma}p_i q_i \tag{14}$$

Now consider that there is only one production factor which is labor. Denote wage as ω, fixed labor input as F, marginal labor input as m. Then we have $C^f = F\omega, C^m = m\omega$. Take forms into (14),we can get the output

$$q_i = \frac{F(\sigma - 1)}{m} \tag{15}$$

Also, entering firms have labor input $F + mq = F\sigma$. Thus, if the total labor input on industrial production is L, then the number of firms which enter the market gives by

$$n = \frac{L}{F\sigma}$$

4 Homework

Q: Assume that households and firms are homogeneous. Help a government to find the (nonconstrained) optimal number of firms n_0 and production of a firm q_0 to maximize the following welfare function

$$\max_{n,q} M^\mu A^{1-\mu} = (qn^{\frac{1}{\rho}})^\mu [L - n(F + mq)]^{1-\mu}$$