Assignment – 10

a)

We know,

$$f(y|p)\cdot g(p)$$

$$\Rightarrow \binom{n}{y} p^{y}(1-p)^{n-y}\cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot p^{\alpha-1}(1-p)^{\beta-1}$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot p^{y+\alpha-1}(1-p)^{n-y+\beta-1}$$

Now, $m(y)$ ..

$$\Rightarrow \int_0^1 \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot p^{y+\alpha-1}\cdot(1-p)^{n-y+\beta-1}\,dp$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \frac{\Gamma(y+\alpha+n-y+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\,p^{y+\alpha-1}\cdot(1-p)^{n-y+\beta-1}\,dp$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}\int_0^1 \frac{\Gamma(n+\alpha+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\cdot p^{y+\alpha-1}(1-p)^{n-y+\beta-1}\,dp$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}$$

Now,

$$g^*(p|y) = \frac{f(y|p)\cdot g(p)}{\int_0^1 f(y|p)\cdot g(p)}$$

$$= \frac{\binom{n}{y}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot p^{y+\alpha-1}(1-p)^{n-y+\beta-1}}{\binom{n}{y}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}}$$

$$= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\cdot p^{y+\alpha-1}\cdot(1-p)^{n-y+\beta-1}$$

$g^*(p|y) \sim \text{Beta}(y+\alpha,\ n+\beta-y)$

Proved

b) $$E[g(p|y)] = \frac{y+\alpha}{y+\alpha+n+\beta-y} = \boxed{\frac{y+\alpha}{n+\alpha+\beta}}$$