Chapter 1.3 Matrix Operations and their Properties

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Theorem

Properties of Matrix Addition and Scalar Multiplication

If A, B, and C are $m \times n$ matrices, and c and d are scalars, then the following properties are true.

1. $A + B = B + A$	Commutative property of addition
2. $A + (B + C) = (A + B) + C$	Associative property of addition
3. $(cd)A = c(dA)$	Associative property of multiplication
4. $1A = A$	Multiplicative identity
5. $c(A + B) = cA + cB$	Distributive property
6. $(c + d)A = cA + dA$	Distributive property

Theorem

Properties of Zero Matrices

If A is an $m \times n$ matrix and c is a scalar, then the following properties are true.

1. $A + O_{mn} = A$

2. $A + (-A) = O_{mn}$

3. If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$.

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Example: Solving a Matrix Equation
Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Algebra of Matrices

Theorem

Properties of Matrix Multiplication

If A, B, and C are matrices (with sizes such that the given matrix products are defined), and c is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$ — Associative property of multiplication
2. $A(B + C) = AB + AC$ — Distributive property
3. $(A + B)C = AC + BC$ — Distributive property
4. $c(AB) = (cA)B = A(cB)$

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Example: Show that $AC = BC$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Identity Matrix

The **identity matrix of order** n is a special type of *square* matrix that has 1's on the main diagonal and 0's elsewhere. It is usually denoted by I_n.

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Properties of the Identity Matrix

If A is a matrix of size $m \times n$, then the following properties are true.

1. $AI_n = A$ **2.** $I_mA = A$

Matrix Multiplication

For repeated multiplication of *square* matrices, use the same exponential notation used with real numbers. That is, $A^1 = A$, $A^2 = AA$, and for a positive integer k, A^k is

$$A^k = \underbrace{AA \cdot \cdot \cdot A}_{k \text{ factors}}.$$

It is convenient also to define $A^0 = I_n$ (where A is a square matrix of order n). These definitions allow you to establish the properties (1) $A^jA^k = A^{j+k}$ and (2) $(A^j)^k = A^{jk}$, where j and k are nonnegative integers.

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Example: For the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$. Find A^3.

The Transpose of a Matrix

The **transpose** of a matrix is formed by writing its columns as rows. For instance, if A is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

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Size: $m \times n$

then the transpose, denoted by A^T, is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

The Transpose of a Matrix

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Properties of Transposes

If A and B are matrices (with sizes such that the given matrix operations are defined) and c is a scalar, then the following properties are true.

1. $(A^T)^T = A$	Transpose of a transpose
2. $(A + B)^T = A^T + B^T$	Transpose of a sum
3. $(cA)^T = c(A^T)$	Transpose of a scalar multiple
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Note: If $A = [a_{ij}]$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all $i \neq j$.

Finding the Transpose of a Product

Example:

Show that $(AB)^T$ and B^TA^T are equal.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

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SOLUTION

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = B^TA^T$$

The Product of a Matrix and Its Transpose

Example:

For the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$, find the product AA^T and show that it is symmetric.

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Because $AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$,

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Note: For any matrix A the matrix AA^T is symmetric. The matrix $A^T A$ is also symmetric.