Chapter 3.4 - Basis and Dimension

Learning Outcomes:

- Recognize bases in the vector spaces R^n, P_n, and $M_{m,n}$.
- Find the dimension of a vector space.

Definition of Basis

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a **basis** for V if the following conditions are true:

i. S spans V.

ii. S is linearly independent.

Standard Basis for n- space

The vectors

$$
\begin{aligned}
\mathbf{e}_1 &= (1, 0, \ldots, 0) \\
\mathbf{e}_2 &= (0, 1, \ldots, 0) \\
&\vdots \\
\mathbf{e}_n &= (0, 0, \ldots, 1)
\end{aligned}
$$

form a basis for R^n called the **standard basis** for R^n.

Standard Basis for P_n

$$S = \{1, x, x^2, \ . \ . \ . \ , x^n\}$$

Standard Basis for vector space M_{mn}

-consists of the mn distinct $m \times n$ matrices having a single 1 and all the other entries equal to zero.

Examples:

1. Show that the following set is a basis for R^3.

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

i. Since any vector in R^3 can be written as a linear combination of vectors in *S,* that is,

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

Then, *S* spans R^3.

ii. The vector equation

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$.

Thus, *S* is linearly independent.

Therefore, S is a basis for R^3, and is called the standard basis for R^3.

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Examples:

2. Show that the set

$$S = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(1, -1)}\}$$

is a basis for R^2.

i. Show that S spans R^2.

Let $\mathbf{x} = (x_1, x_2)$ be an arbitrary vector in R^2.

Show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}$$

$$c_1(1, 1) + c_2(1, -1) = (x_1, x_2)$$

$$(c_1 + c_2, c_1 - c_2) = (x_1, x_2)$$

$$c_1 + c_2 = x_1$$

$$c_1 - c_2 = x_2$$

Since the system has nonzero determinant, then it has a unique solution. Hence, S spans R^2.

Examples:

ii. Show that S is linearly independent.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$
$$c_1(1, 1) + c_2(1, -1) = (0, 0)$$
$$(c_1 + c_2, c_1 - c_2) = (0, 0).$$

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

$$c_1 = c_2 = 0$$

Thus, S is linearly independent.

Therefore, S is a basis for R^2.

3. The set

$$S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

is a basis for $M_{2,2}$ and is called the **standard basis** for $M_{2,2}$.

Theorem: (Uniqueness of Basis Representation)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Example:

Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in R^3. Show that the equation $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ has a unique solution for the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$

Solution:

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1)\\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3),\end{aligned}$$

$$\begin{aligned} c_1 \qquad\qquad - 2c_3 &= u_1\\ 2c_1 + \quad c_2 \qquad\quad &= u_2\\ 3c_1 + 2c_2 + \quad c_3 &= u_3\end{aligned}$$

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Theorem: (Bases and Linear Dependence)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Theorem: (Number of Vectors in a Basis)

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Examples:

(a) The set $S_1 = \{(3, 2, 1), (7, -1, 4)\}$ is not a basis for R^3.

(b) The set

$$S_2 = \{x + 2, x^2, x^3 - 1, 3x + 1, x^2 - 2x + 3\}$$

is not a basis for P_3.

Definition of Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then the number n is called the **dimension** of V, denoted by $\dim(V) = n$. If V consists of the zero vector alone, the dimension of V is defined as zero.

1. The dimension of R^n with the standard operations is n.
2. The dimension of P_n with the standard operations is $n + 1$.
3. The dimension of $M_{m,n}$ with the standard operations is mn.

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Note:

If W is a subspace of an n-dimensional vector space, then it can be shown that W is finite dimensional and the dimension of W is less than or equal to n.

Determining of the dimension of W is done by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace, and the dimension of the subspace is the number of vectors in the basis.

Finding the Dimension of a Subspace

Examples:

1. Determine the dimension of each subspace of R^3.

   (a) $W = \{(d, c-d, c) \mid c \text{ and } d \text{ are real numbers}\}$

   (b) $W = \{(2b, b, 0) \mid b \text{ is a real number}\}$

2. Find the dimension of the subspace W of R^4 spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5, 0)}, \overset{\mathbf{v}_2}{(3, 0, 1, -2)}, \overset{\mathbf{v}_3}{(-5, 4, 9, 2)}\}$$

3. Let W be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of W?

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Theorem: (Basis Tests in an n-Dimensional Space)

Let V be a vector space of dimension n.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then S is a basis for V.

Example: Show that the set of vectors is a basis for $M_{5,1}$.

$$\begin{array}{ccccccccccc} & & \mathbf{v}_1 & & \mathbf{v}_2 & & \mathbf{v}_3 & & \mathbf{v}_4 & & \mathbf{v}_5 & \\ S = & \Bigg\{ & \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, & & \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}, & & \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}, & & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}, & & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} & \Bigg\} \end{array}$$

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Exercise:

1. Show whether $S = \{x^3 - 1, 2x^2, x + 3, 5 - 2x + 2x^2 + x^3\}$ is a basis for P_3.
2. Find a basis for the vector space of all symmetric 3×3 matrices. What is the dimension of this vector space?