Systems of Linear Equations

IRISH C. SIDAYA

Introduction to Systems of Linear Equations

Definition

A **linear equation in** n **variables** $x_1, x_2, x_3, ..., x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

The **coefficients** $a_1, a_2, a_3, ..., a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient**, and x_1 is the **leading variable**.

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Examples (Nonlinear Equations)

$xy + z = 2$; $e^x - 2y = 4$; $\sin x_1 + 2x_2 - 3x_3 = 0$

Solutions and Solution Sets

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A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, s_3, ..., s_n$ arranged to satisfy the equation when you substitute the values

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Note: To describe the entire solution set of a linear equation, use a **parametric representation**

Parametric Representation of a Solution Set

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Let $x_2 = t$, then the solution set can be represented as

$$x_1 = 4 - 2t,\ x_2 = t,\ t \text{ is any real number}$$

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$$x_1 = s,\ x_2 = 2 - \frac{1}{2}s,\ s \text{ is any real number}$$

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Letting $y = s$ and $z = t$, we obtain the parametric representation

$$x = 1 - \frac{2}{3}s + \frac{1}{3}t, \; y = s, \; z = t$$

where s and t are any real number.

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where s and t are any real number.

Two particular solutions are

$$x = 1, y = 0, z = 0 \text{ and } x = 1, y = 1, z = 2$$

Systems of Linear Equations

Definition

A **system of m linear equations in n variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
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A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, s_3, ..., s_n$ that is a solution of each of the linear equations in the system.

Example

The system

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\begin{aligned}
3x_1 + 2x_2 &= 3 \\
-x_1 + x_2 &= 4
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has $x_1 = -1$ and $x_2 = 3$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy both equations.

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Exercise

Graph the two lines

$$3x - y = 1$$
$$2x - y = 0$$

in the xy- plane.

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in the xy- plane. Where do they intersect? How many solutions does this system of linear equations have?

Systems of Linear Equations

Continuation

Do the same for the systems:

$$\begin{aligned} 3x - y &= 1 \\ 3x - y &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 3x - y &= 1 \\ 6x - 2y &= 0 \end{aligned}$$

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3. The system has no solution (inconsistent system) - graphs do not have a point in common

Solving a System of Linear Equations

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \quad \text{or} \quad \begin{array}{cccc} x & -2y & +3z & = 9 \\ & y & +3z & = 5 \\ & & z & = 2 \end{array}$$

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Note: Two systems of linear equations are **equivalent** when they have the same solution set

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Writing Systems of Equations in Row-echelon Form

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Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, each of which is obtained by using one the three basic operations. This process is called **Gaussian elimination** (named after the German mathematician Carl Friedrich Gauss (1777-1855))

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Writing Systems of Equations in Row-echelon Form

Exercise

Rewrite the following systems in row-echelon form and use back-substitution to solve the system:

1. $$\begin{array}{rrrcr} x_1 & -3x_2 & +x_3 & = & 1 \\ 2x_1 & -x_2 & -2x_3 & = & 2 \\ x_1 & +2x_2 & -3x_3 & = & -1 \end{array}$$

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2. $$\begin{array}{rrrcr} & x_2 & -x_3 & = & 0 \\ x_1 & & -3x_3 & = & -1 \\ -x_1 & +3x_2 & & = & 1 \end{array}$$

3. $$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ x & +y & +z & = & 6 \\ 3x & & -z & = & 0 \end{array}$$