Chapter 6

Eigenvalues and Eigenvectors

Chapter 6.1

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Eigenvalue and Eigenvector

The Eigenvalue Problem:

Given any $n \times n$ matrix A, are there any nonzero vectors $\mathbf{x} \in R^n$ for which $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

Definition: (***Eigenvalue and Eigenvector***)

Let A be an $n \times n$ matrix. The scalar λ (lambda) is called an **eigenvalue** of A when there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

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Remarks:

1. In our discussion, we will only consider real-valued eigenvalues.
2. The zero vector $\mathbf{0}$ is not an eigenvector for any λ.
3. The scalar 0 could be an eigenvalue.

Note:

A matrix can have more than one eigenvalue.

Example 1:

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Show that $\mathbf{x}_1 = (1,0)$ and $\mathbf{x}_2 = (0,1)$ are eigenvectors of A. Identify their corresponding eigenvalues.

Example 2:

Verify that $\mathbf{x}_1 = (1,0,0)$ and $\mathbf{x}_2 = (-3,-1,1)$ are eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and find their corresponding eigenvalues.

Remarks:

1. If A is an $\text{n} \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A.
2. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to the *same* eigenvalue λ then their sum is also an eigenvector corresponding to λ.

*Theorem 1: (**Eigenspace**)*

If A is an $n \times n$ matrix with eigenvalue λ, then the set S of all eigenvectors associated to λ, together with the zero vector $\mathbf{0}$

$$S = \{\mathbf{x} \mid \mathbf{x} \text{ is an eigenvector of } \lambda\} \cup \{\mathbf{0}\}.$$

is a subspace of R^n. This subspace of R^n is called the **eigenspace** of λ.

Finding Eigenvalues and Eigenvectors

Theorem 2:

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$, where I is the $n \times n$ identity matrix.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of the $(\lambda I - A)\mathbf{x} = 0$.

Remarks:

1. The equation $|\lambda I - A| = 0$ is called the **characteristic equation** of A.
2. The polynomial
$$|\lambda I - A| = \lambda^n + c_{n-1}\,\lambda^{n-1} + \cdots + c_1\,\lambda + c_0$$
is called the **characteristic polynomial** of A.
3. The eigenvalues of A are the zeros of the characteristic polynomial of A.
4. The Fundamental Theorem of Algebra implies that A has at most n eigenvalues.

Example 4:

Find the eigenvalues and the corresponding eigenvectors of
$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

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Steps in finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $\det(\lambda I - A) = 0$. This will give the degree n characteristic polynomial.
2. Find the real roots of the characteristic polynomial. These will be the eigenvalues of A.
3. For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. By row-reducing the coefficient matrix to RREF, it will have at least one zero row.

Example 5:

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is the dimension of the eigenspace for each eigenvalue?

Remarks:

1. If an eigenvalue λ is a repeated root of the characteristic polynomial for k times, then λ has **multiplicity** k.

2. The corresponding dimension of the eigenspace of λ is at most its multiplicity.

Example 6:

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each corresponding eigenspace.

Theorem 3:

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

Example 7:

Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Eigenvalues and Eigenvectors of Linear Transformation

Remarks:

1. A scalar λ is an **eigenvalue** of the linear transformation $T: V \rightarrow V$ where there is a nonzero vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$.
2. The nonzero vector **v** is called the **eigenvector** associated to the eigenvalue λ.
3. The set of all eigenvectors of λ is called the **eigenspace** of λ.

Example 8:

Find the eigenvalues and the basis for each corresponding eigenspace of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$