Chapter 4
Inner Product Spaces

Chapter 4.1

Length and Dot Product in R^n

Vector Length and Unit Vectors

Definition (***Length of a Vector in R^n***)

The **length**, or **norm**, of a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{(v_1^2 + v_2^2 + \cdots + v_n^2)}$$

Remarks:

1. The length of a vector is a nonnegative number, that is,
$$\|\mathbf{v}\| \geq 0.$$
2. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = (0,0,\dots,0) = \mathbf{0}$.
3. The length of a vector is also referred to as the **magnitude** of the vector.
4. If $\|\mathbf{v}\| = 1$, then the vector $\mathbf{v}$ is called a **unit vector**.

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Examples:

1. Find the length of the vector $\mathbf{v} = (0, -2, 1, 4, -2)$ in R^5.
2. Find the length of the vector $\mathbf{v} = \left(\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ in R^3.

Remarks:

1. Each vector in the standard basis of R^n has length 1, and is called the **standard unit vector** in R^n.
2. Two nonzero vectors in R^n are **parallel** when one is a scalar multiple of the other.
   - If the scalar is greater than 0, then the vectors are on the same direction.
   - If the scalar is less than 0, then the vectors are on the opposite directions.

Theorem:

Let $\mathbf{v}$ be a vector in R^n and let c be any scalar. Then

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

where $|c|$ is taken as the absolute value of c.

Theorem: ***(Unit vector in the direction of v)***

If $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^n$, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

has length 1 and has the same direction as $\mathbf{v}$. The vector $\mathbf{u}$ is called the **unit vector in the direction of v**.

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Remark:

The process of finding a unit vector in the direction of the given vector is called **normalizing** the given vector.

Example:

Find a unit vector in the direction of $\mathbf{v} = (3, -1, 2)$.

Distance Between Two Vectors in R^n

Definition

The distance between two vectors **u** and **v** in R^n is

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$$

Remark:

The distance has the following properties:

1. $d(\mathbf{u},\mathbf{v}) \geq 0$
2. $d(\mathbf{u},\mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$

Example: Find the distance between the following pairs of vectors in R^n.

1. $\mathbf{u} = (-1,-4)$ and $\mathbf{v} = (2,3)$
2. $\mathbf{u} = (3,-1,0,-3)$ and $\mathbf{v} = (4,0,1,2)$

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Dot Product and the Angle Between Two Vectors

Definition (***Dot Product in R^n***)

The **dot product** of the vectors $\mathbf{u} = (u_1, u_2, \ldots u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_2)$ in R^n is the scalar quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example:

Find the dot product of $\mathbf{u} = (1,2,0,-3)$ and $\mathbf{v} = (3,-2,4,2)$.

Theorem: ***(Properties of the Dot Product)***

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ vectors in R^n and c be any scalar. Then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
5. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Examples:

1. Let $\mathbf{u} = (2, -2)$, $\mathbf{v} = (5,8)$, and $\mathbf{w} = (-4,3)$. Solve for the following:

   a. $\mathbf{u} \cdot \mathbf{v}$

   b. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

   c. $\mathbf{u} \cdot 2\mathbf{v}$

   d. $\|\mathbf{w}\|^2$

   e. $\mathbf{u} \cdot (\mathbf{v} - 2\mathbf{w})$

2. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n. If $\mathbf{u} \cdot \mathbf{u} = 39$, $\mathbf{u} \cdot \mathbf{v} = -3$, and $\mathbf{v} \cdot \mathbf{v} = 79$, evaluate $(\mathbf{u} + 2\mathbf{v}) \cdot (3\mathbf{u} + \mathbf{v})$.

Theorem: ***(Cauchy – Schwarz Inequality)***

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

where $|\mathbf{u} \cdot \mathbf{v}|$ is the absolute value of the dot product of $\mathbf{u}$ and $\mathbf{v}$.

Example:

Verify the Cauchy – Schwarz Inequality for $\mathbf{u} = (1, -1, 3)$ and $\mathbf{v} = (2, 0, -1)$.

*Definition: (**The Angle Between Two Vectors**)*

The **angle** θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

where $0 \le \theta \le \pi$.

Example:

Find the angle between the vectors $\mathbf{u} = (-4,0,2,-2)$ and $\mathbf{v} = (2,0,-1,1)$ in R^4.

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*Definition: (**Orthogonal Vectors**)*

Two vectors **u** and **v** in R^n are **orthogonal** when

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Remark:

The vector **0** is orthogonal to every vector.

Examples:

1. Show that the following pairs of vectors are orthogonal.

   a. $\mathbf{u} = (1,0,0)$ and $\mathbf{v} = (0,0,1)$

   b. $\mathbf{u} = (3,2,-1,4)$ and $\mathbf{v} = (1,-1,1,0)$

2. Determine all vectors in R^2 that are orthogonal to $\mathbf{u} = (4,2)$.

*Theorem: (**Triangle Inequality**)*

If $\mathbf{u}$ and $\mathbf{v}$ in R^n, then

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

*Theorem: (**The Pythagorean Theorem**)*

If $\mathbf{u}$ and $\mathbf{v}$ in R^n, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

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Dot Product and Matrix Multiplication

Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_\mathrm{n})$ be vectors in R^n. When **u** and **v** are written as $n \times 1$ vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

then, the dot product can be represented as the matrix product of the transpose of **u** multiplied by **v**, that is,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\mathrm{T}\mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n].$$

Example:

Find the dot product of the following pairs of vectors.

1. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$