Chapter 5.2

The Kernel and Range of a Linear Transformation

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The Kernel of a Linear Transformation

Definition: (***Kernel***)

Let $T: V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of T and is denoted by $\ker(T)$.

Remark:

Since $T(\mathbf{0}) = \mathbf{0}$ for any linear transformation T, then $\mathbf{0} \in \ker(T)$.

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Example 1:

Let $T: M_{3,2} \to M_{2,3}$ be the linear transformation that maps a 3×2 matrix A to its transpose. Find the $\ker(T)$.

Solution:

For this linear transformation, the 3×2 zero matrix is clearly the only matrix in $M_{3,2}$ whose transpose is the zero matrix in $M_{2,3}$. So, the kernel of T consists of a single element: the zero matrix in $M_{3,2}$.

Example 2:

1. Find the kernel of the zero transformation.
2. Find the kernel of the identity transformation.

Solution:

1. The kernel of the zero transformation $T: V \to W$ consists of all of V because $T(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v}$ in V. That is, $\ker(T) = V$.
2. The kernel of the identity transformation $T: V \to V$ consists of the single element $\mathbf{0}$. That is, $\ker(T) = \{\mathbf{0}\}$.

Example 3:

Find the kernel of the linear transformation:

(a) $T: R^3 \rightarrow R^3$ defined by $T(x, y, z) = (x, y, 0)$.

(b) T: $R^2 \rightarrow R^3$ defined by $T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1)$.

(c) $T: R^3 \rightarrow R^2$ defined by $T(\mathbf{x}) = A(\mathbf{x})$, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}.$$

Theorem 1:

The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

Example 4:

Define $T: R^5 \to R^4$ by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in R^5$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for $\ker(T)$ as a subspace of R^5.

Corollary 1:

Let $T: R^n \rightarrow R^m$ be the linear transformation given by

$$T(\mathbf{x}) = A\mathbf{x}.$$

Then the kernel of T is equal to the solution space of $A\mathbf{x} = \mathbf{0}$. In other words, the kernel of T is the nullspace of the matrix A.

The Range of a Linear Transformation

Definition: ***(Range)***

Let $T: V \to W$ be a linear transformation. The set of all vectors $\mathbf{w} \in W$ that are images of vectors in V are elements of the range of T. That is,

$$range(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}.$$

Theorem 2:

The range of a linear transformation $T: V \to W$ is a subspace of W.

Theorem 3:

Let $T: R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A(\mathbf{x})$. Then the column space of A is equal to the range of T.

Example 5:

Define $T: R^5 \to R^4$ by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in R^5$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for the range of T.

Definition: ***(Rank and Nullity)***

Let $T: V \to W$ be a linear transformation. The dimension of the kernel of T is called the **nullity** of T and is denoted by $\mathbf{nullity}(T)$. The dimension of the range of T is called the **rank** of T and is denoted by $\mathbf{rank}(T)$.

Remark:

If T is given by the matrix A, then the rank of T is equal to the rank of A, and the nullity of T is equal to the nullity of A.

Theorem 4:

Let $T: V \to W$ be a linear transformation from an n-dimensional vector space V into a vectors space W. Then the sum of the dimensions of the range and kernel of T is equal to the dimension of the domain.

$$\textbf{rank}(T) + \textbf{nullity}(T) = n$$

$$\dim\big(\textbf{range}(T)\big) + \dim(\textbf{kernel}(T)) = \dim\big(\textbf{domain}(T)\big)$$

Example 6:

Find the rank and nullity of the linear transformation $T: R^3 \to R^3$ defined by the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Remark:

The relationship between the rank and the nullity of a linear transformation provided by a matrix:

1. the number of leading 1’s determines the rank
2. the number of free variables (columns without leading 1’s) determines the nullity.
3. their sum must be the total number of columns in the matrix, which is the dimension of the domain.

Example 7: Let $T: R^5 \to R^7$ be a linear transformation.

1. Find the dimension of the kernel of T when the dimension of the range is 2.
2. Find the rank of T when the nullity of T is 4.
3. Find the rank of T when $\ker\{T\} = \{\mathbf{0}\}$.

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One-to-one and Onto Linear Transformation

Definition: ***(One-to-one)***

A function $T: V \to W$ is called **one-to-one** when the preimage of every vector $\mathbf{w}$ in the range of T consists of a single vector $\mathbf{v}$. Equivalently, T is one-to-one if and only if, for every $\mathbf{u}$ and $\mathbf{v} \in V$, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.

Theorem 5:

Let $T: V \to W$ be a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Example 8:

1. The linear transformation $T: M_{mn} \to M_{nm}$ represented by $T(A) = A^T$ is one-to-one since $\ker(T) = \{\mathbf{0}_{m\times n}\}$.
2. The zero transformation is not one-to-one since its kernel is entirely its domain.

Definition: **(*Onto*)**

A function $T: V \to W$ is called **onto** when every vector $\mathbf{w} \in W$ is in the range of T, or equivalently, when every vector $\mathbf{w} \in W$ has a preimage in V.

Remark:

Let $T: V \to W$ be a linear transformation. Then T is onto when $T(V) = W$.

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Theorem 6:

Let $T: V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if $\operatorname{rank}(T) = \dim(W)$.

Theorem 7:

Let $T: V \to W$ be a linear transformation with vector spaces V and W, both of dimension n. Then T is one-to-one if and only if T is onto.

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Example 9:

Consider the linear transformation $T: R^n \to R^m$ represented by $T(\mathbf{x}) = A(\mathbf{x})$. Find the nullity and rank of T and determine whether T is one-to-one, onto, or neither.

a. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

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Example 9: (Solution)

$T: R^n \to R^m$	Dim(domain)	Dim(range) Rank(T)	Dim(kernel) Nullity(T)	One-to-One	Onto
a. $T: R^3 \to R^3$	3	3	0	Yes	Yes
b. $T: R^2 \to R^3$	2	2	0	Yes	No
c. $T: R^3 \to R^2$	3	2	1	No	Yes
d. $T: R^3 \to R^3$	3	2	1	No	No

Isomorphisms of Vector Spaces

Definition: (***Isomorphism***)

A linear transformation $T: V \to W$ that is one-to-one and onto is called an **isomorphism**. In addition, if V and W are vector spaces such that there is an isomorphism T from V to W, then V and W are said to be **isomorphic**.

Theorem 8:

Two finite dimensional vectors spaces V and W are isomorphic if and only if they are of the same dimension.

Example 10:

The following vector spaces are isomorphic to each other.

1. R^4, the 4 – space
2. $M_{4,1}$, the space of all 4×1 matrices
3. $M_{2,2}$, the space of all 2×2 matrices
4. P_3, the space of all polynomials of degree at most 3
5. $V = \{\, (x_1, x_2, x_3, x_4, 0) \mid x_i$ is a real number for $i = 1,2,3,4\,\}$, a subspace of R^5.

Assignment

1. Find the kernel of the following linear transformations:

   (a) $T\colon P_3 \to R, T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_0$

   (b) $T\colon R^2 \to R^2, T(x, y) = (x + 2y, y - x)$

2. Define the linear transformation by $T(\mathbf{x}) = A\mathbf{x}$. For each given matrix A, find (i) $\ker T$, (ii) nullity(T), (iii) range(T), and (iv) rank(T).

   (a)
   $$A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (b)
   $$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix}$$

3. Consider the linear transformation $T: R^4 \to R^3$ represented by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the dimension of the domain.

(b) Find the dimension of the range.

(c) Find the dimension of the kernel.

(d) Is T one-to-one? Explain.

(e) Is T onto? Explain.

(f) Is T an isomorphism? Explain.