Chapter 3.2- Vectors Spaces and Subspaces

Learning Outcomes:

- Define a vector space and recognize some important vector spaces.
- Show that a given set is not a vector space.
- Determine whether a subset W of a vector space V is a subspace of V.

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Vector Spaces

Definition. Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d then V is called a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	Additive identity
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	Additive inverse

Scalar Multiplication:

6. $c\mathbf{u}$ is in V.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property
10. $1(\mathbf{u}) = \mathbf{u}$	Scalar identity

Note:

Any set that satisfies these properties (or **axioms**) is called a **vector space,** and the objects in the set are called **vectors.**

Examples:

1. $\boldsymbol{R^n}$ with the Standard Operations

The set of all ordered n-tuples of real numbers $\boldsymbol{R^n}$ with the standard operations is a vector space.

Vectors in this space are of the form

$$\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n).$$

This is verified by the Properties of Vector Addition and Scalar Multiplication in $\boldsymbol{R^n}$ given in Chapter 3.1.

Examples:

2. The set of all 2×3 matrices

The set of all 2×3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

Vectors in this space are of the form

$$\mathbf{a} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Note that the sum of 2×3 matrices is a 2×3 matrix and so it is closed under addition. Also, for every scalar c, when multiplied to a 2×3 matrix A, cA is also a 2×3 matrix. Hence, the given set is closed under scalar multiplication. The remaining 8 vector space axioms are verified by the properties of matrix operations (given in Chapter 1.3).

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Examples:

Show if the given set is a vector space or not.

1. **The set of all polynomials of degree 2 or less**

Let P_2 be the set of all polynomials of the form

$$p(x) = a_2x^2 + a_1x + a_0,$$

where a_0, a_1, and a_2 are real numbers.

The *sum* of two polynomials

$$p(x) = a_2x^2 + a_1x + a_0 \text{ and } q(x) = b_2x^2 + b_1x + b_0$$

is defined in the usual way by

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

and the *scalar multiple* of $p(x)$ by the scalar c is defined by

$$cp(x) = ca_2x^2 + ca_1x + ca_0$$

Solution:

1. Let $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0 \in P_2$. Then

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

   Since $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$, then $a_2 + b_2$, $a_1 + b_1$, $a_0 + b_0 \in \mathbb{R}$ (because the set of real numbers is closed under addition) and so $p(x) + q(x) \in P_2$ (since it is a polynomial of degree 2 or less). Thus, P_2 is closed under addition.

2. Let $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0 \in P_2$. Then

$$\begin{aligned} p(x) + q(x) &= (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) \\ &= (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \\ &= (b_2 + a_2)x^2 + (b_1 + a_1)x + (b_0 + a_0) \\ &= (b_2x^2 + b_1x + b_0) + (a_2x^2 + a_1x + a_0) \\ &= q(x) + p(x) \end{aligned}$$

   Note that in line 3, we apply the commutative property of addition of real numbers.

Solution:

4. The zero vector in this space is the zero polynomial given by $\mathbf{0}(x) = 0x^2 + 0x + x$ for all x. Note that if $p(x) = a_2x^2 + a_1x + a_0 \in P_2$, then

$$\begin{aligned} p(x) + \mathbf{0}(x) &= (a_2x^2 + a_1x + a_0) + (0x^2 + 0x + x) \\ &= (a_2 + 0)x^2 + (a_1 + 0)x + (a_0 + 0) \\ &= a_2x^2 + a_1x + a_0 \\ &= p(x) \end{aligned}$$

7. Let $p(x) = a_2x^2 + a_1x + a_0$, $q(x) = b_2x^2 + b_1x + b_0 \in P_2$ and c be a scalar. Then

$$\begin{aligned} c\big(p(x) + q(x)\big) &= c(a_2 + b_2)x^2 + c(a_1 + b_1)x + c(a_0 + b_0) \\ &= (ca_2 + cb_2)x^2 + (ca_1 + cb_1)x + (ca_0 + cb_0) \\ &= (ca_2x^2 + ca_1x + ca_0) + (cb_2x^2 + cb_1x + cb_0) \\ &= c(a_2x^2 + a_1x + a_0) + c(b_2x^2 + b_1x + b_0) \\ &= cp(x) + cq(x) \end{aligned}$$

Note that in line 2, we apply the distributive property of real numbers.

(*The verifications of the remaining vector space axioms are left as exercise.*)

Examples:

2. The Set of Integers

Solution:

The set of all integers (with the standard operations) does not form a vector space because it is not closed under scalar multiplication.

For example, given an integer 3 and scalar $\frac{1}{2}$,

$$\frac{1}{2}(3) = \frac{3}{2} \text{ which is not an integer.}$$

Examples:

3. The Set of Second-Degree Polynomials

$$p(x) = a_2x^2 + a_1x + a_0,$$

Solution:

The set of all second-degree polynomials is not a vector space because it is not closed under addition. For instance, consider the second-degree polynomials

$$p(x) = 2x^2 + 3 \text{ and } q(x) = -2x^2 + x - 1.$$

The sum $p(x) + q(x) = x + 2$ which is not a second-degree polynomial.

Examples:

4. Let the set of all ordered pairs of real numbers, with the standard operation of addition and the *nonstandard* definition of scalar multiplication

$$c(x_1, x_2) = (cx_1, 0)$$

Solution:

This actually satisfies the first nine axioms of the definition of a vector space (left as an exercise). However, for the tenth axiom (scalar identity), the nonstandard definition of scalar multiplication gives us

$$1(x_1, x_2) = (x_1, 0) \neq (x_1, x_2)$$

The tenth axiom is not verified and so the given set (together with the two operations) is not a vector space.

Properties of Scalar Multiplication

Let $\mathbf{v}$ be any element of a vector space and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} = \mathbf{0}$
2. $c\mathbf{0} = \mathbf{0}$
3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$

Subspaces of Vector Spaces

Definition of Subspace of a Vector Space

A nonempty subset W of a vector space V is called a **subspace** of V if W is a vector space under the operations of addition and scalar multiplication defined in V.

Example: Show that

The set $W = \{(x_1, 0, x_3) | x_1$ and x_3 are real numbers$\}$ is a subspace of R^3 with the standard operations.

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Solution:

We need to show that W is nonempty and it is a vector space with the standard operations.

Note that W is non-empty since it contains the zero vector $(0, 0, 0)$. Also, if $(x_1, 0, x_3)$ and $(y_1, 0, y_3)$ are in W, then the sum $(x_1 + y_1, 0, x_3 + y_3)$ is in W. Hence, W is closed under addition. Moreover, if $(x_1, 0, x_3) \in W$ and c is a scalar, then $c(x_1, 0, x_3) = (cx_1, 0, cx_3)$ is in W (since the first and third components are real numbers and the second component is zero). Thus, W is closed under scalar multiplication.

The other 8 axioms can also be verified (left as exercise).

Theorem: Test for a Subspace

If W is a nonempty subset of a vector space V then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

(This theorem states that it is sufficient to test for closure in order to establish that a nonempty subset of a vector space is a subspace.)

Note: Every vector space has at least two subspaces, itself and the subspace $\{\mathbf{0}\}$ called the **zero subspace**, consisting only of the zero vector. These subspaces are called the **trivial** subspaces.

Example:

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$ (the set of 2×2 matrices), with the standard operations of matrix addition and scalar multiplication.

Solution:

Recall: A square matrix is *symmetric* when it is equal to its own transpose.

W is nonempty since it contains I_2 (2×2 identity matrix). Now, we need to show that W (with the standard operations of matrix addition and scalar multiplication) satisfies the closure conditions.

Let $A_1, A_2 \in W$. Then $A_1 = A_1^T$ and $A_2 = A_2^T$. We have

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2$$

which implies that $A_1 + A_2$ is symmetric of order 2 and so it is in W. Moreover, if $A \in W$, then $A^T = A$. Let c be any scalar, so we have

$$(cA)^T = cA^T = cA.$$

Hence, cA is symmetric of order 2.

Therefore, W is closed under matrix addition and scalar multiplication.

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Example:

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2,2}$ with the standard operations.

Note: To show that a set W is not a subspace of a larger set, you may show that W is empty, W is not closed under addition, or W is not closed under scalar multiplication

Solution:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Observe that A and B are singular matrices of order 2.

However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is a nonsingular matrix. Hence, W is not closed under matrix addition. Thus, W is not a subspace of $M_{2,2}$.

The Intersection of Two Subspaces Is a Subspace

Theorem:

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

Assignment

1. Show that the set of 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$

is a vector space with the standard operations.

2. Let $V = \boldsymbol{R}^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the *nonstandard* definition of scalar multiplication

$$c(x, y) = (cx, y)$$

Show that V is not a vector space.

3. Show that $W = \{(x, y, 2x - 3y) | x \text{ and } y \text{ are real numbers}\}$ is a subspace of $\boldsymbol{R}^3$ with the standard operations.

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