Chapter 3
Vectors and Vector Spaces

Chapter 3.1

Vectors in R^n

Learning Outcomes:

- Perform basic vector operations in R^2 and represent them graphically.
- Perform basic vector operations in R^n.
- Write a vector as a linear combination of other vectors.

Vectors in the Plane

A **vector in the plane**

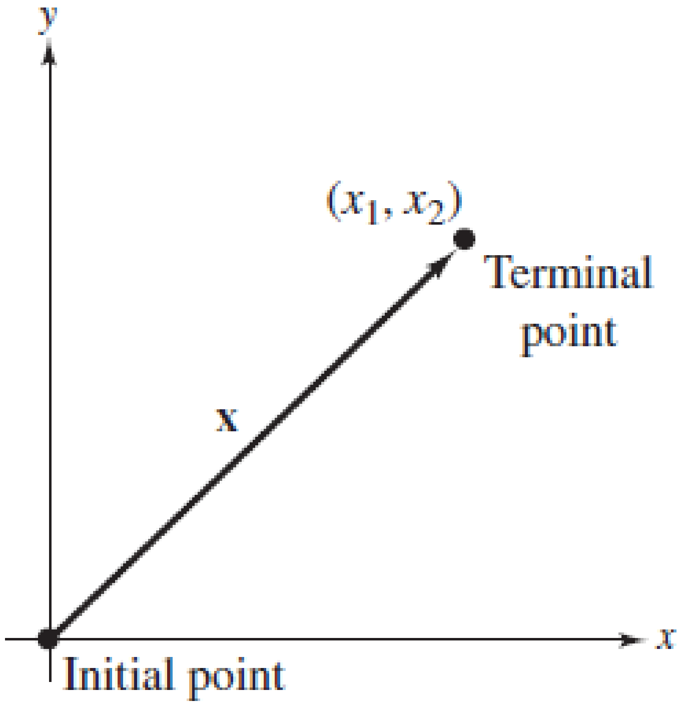
- is represented geometrically by a **directed line segment** whose **initial point** is the origin and whose **terminal point** is the point (x_1, x_2)

- represented by the same **ordered pair** used to represent its terminal point.

$$\mathbf{x} = (x_1, x_2)$$

The coordinates x_1 and x_2 are called the **components** of the vector **x.**

Note: Two vectors in the plane $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.



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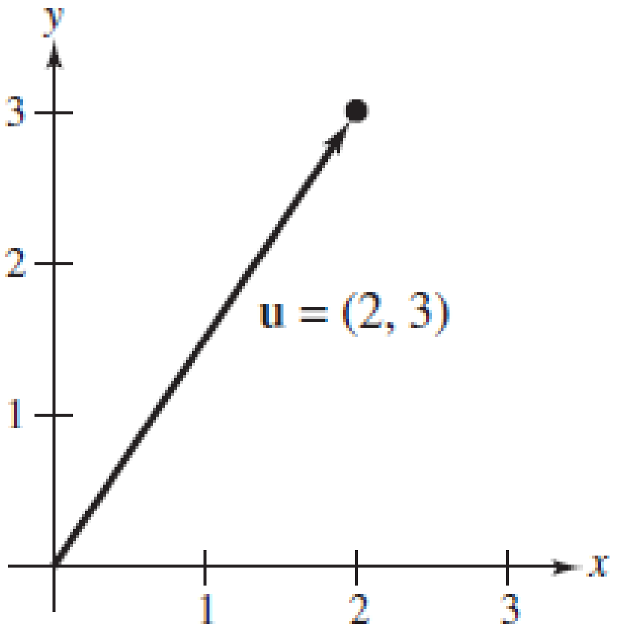
Vectors in the Plane

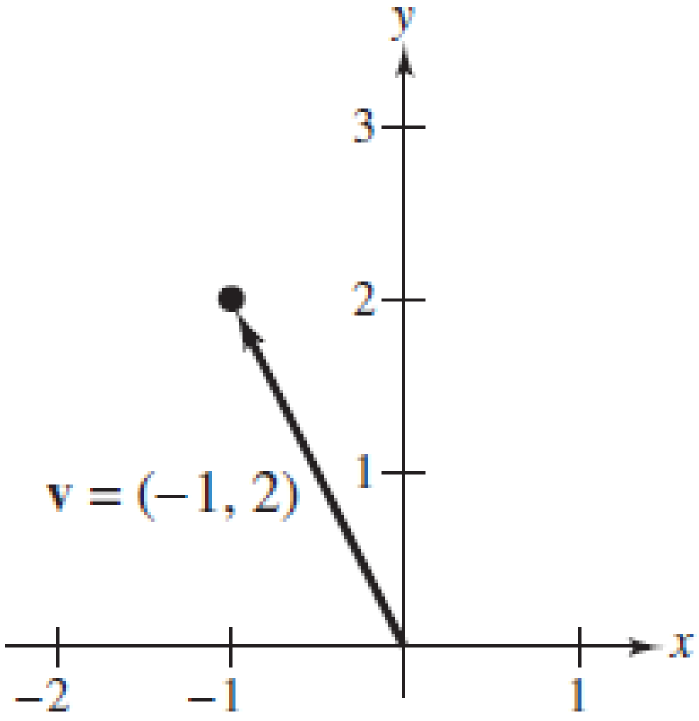
Representing vector in the plane:

(a) **u** = (2, 3)

(b) **v** = (-1, 2)



(a)

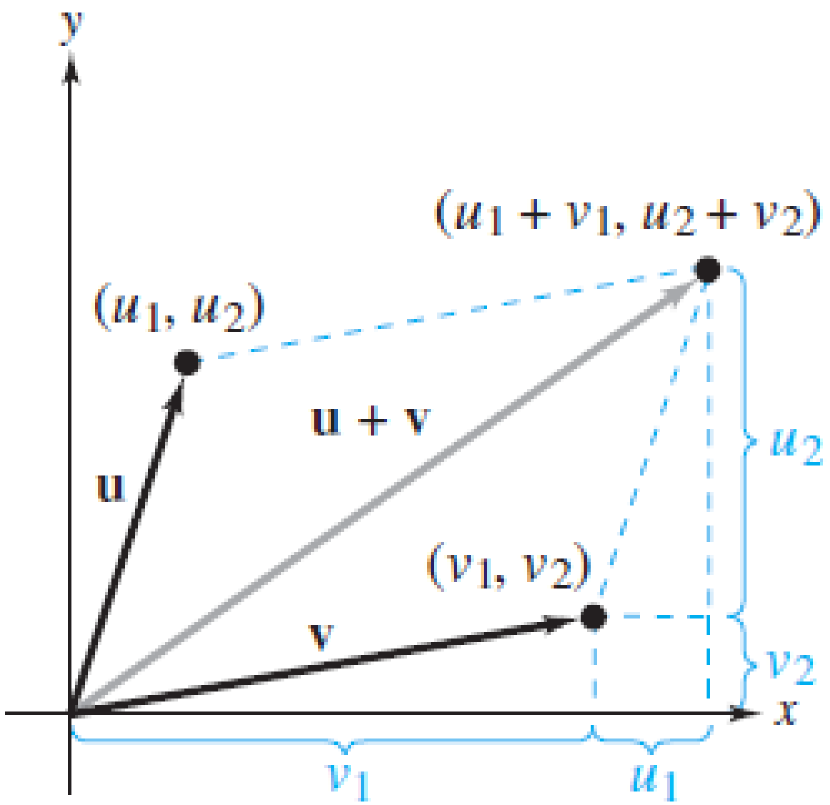


(b)

Vector Addition in the plane

The **sum** of **u** and **v** is the vector

$$\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$



Vector Addition

Examples: Find the sum of the vectors and graph.

(a) $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, -2)$

(b) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$

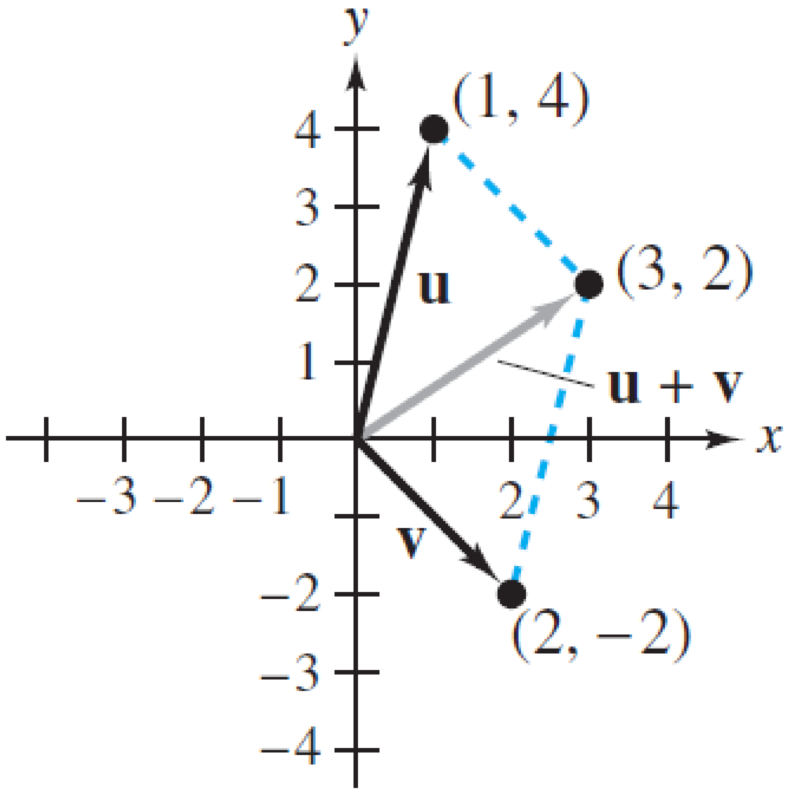
(c) $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

Vector Addition

(a) $\mathbf{u} = (1, 4), \mathbf{v} = (2, -2)$

Solution:

(a) $\mathbf{u} + \mathbf{v} = (1, 4) + (2, -2) = (3, 2)$



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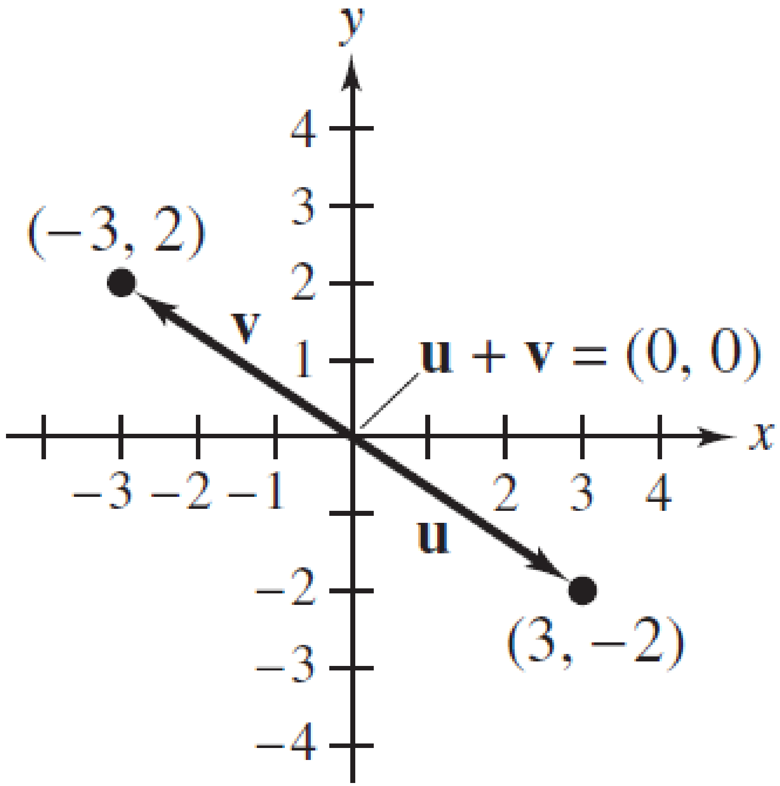
Vector Addition

(b) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$

Solution:

(b) $\mathbf{u} + \mathbf{v} = (3, -2) + (-3, 2) = (0, 0)$

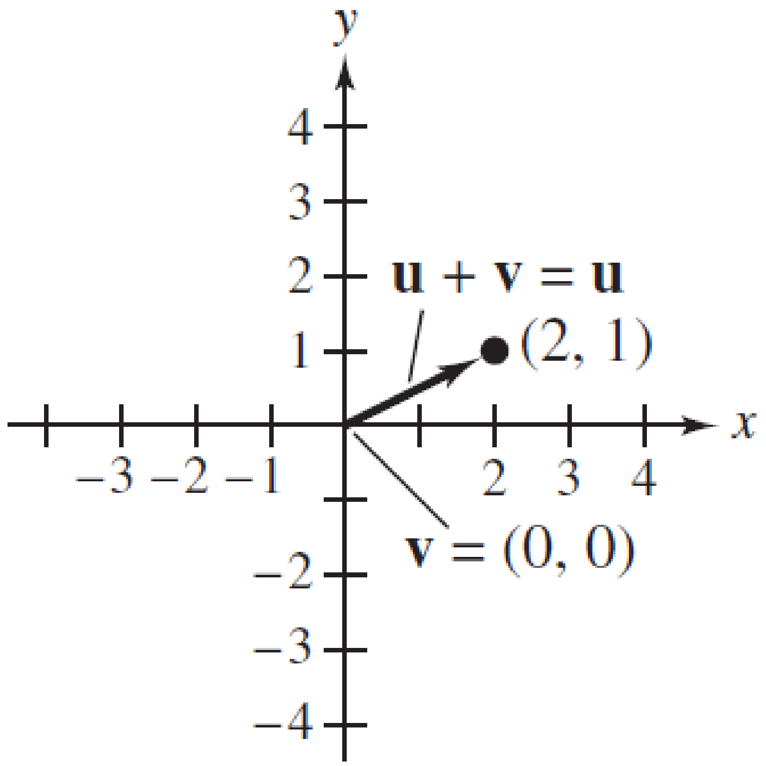


Vector Addition

(c) $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

Solution:

(c) $\mathbf{u} + \mathbf{v} = (2, 1) + (0, 0) = (2, 1)$

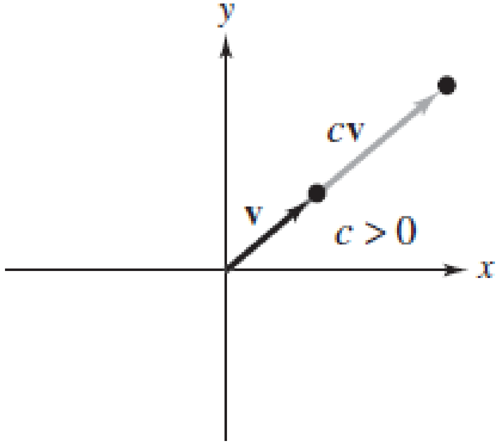


Scalar Multiplication in the Plane

To multiply a vector **v** by a scalar c, multiply each of the components of **v** by c.

$$c\boldsymbol{v} = c(v_1, v_2) = (cv_1, cv_2)$$

For a scalar *c*, the vector *c***v** will be |c| times as long as **v.** If *c* is positive, then *c***v** and **v** have the same direction, and if *c* is negative, then c**v** and **v** have opposite directions.

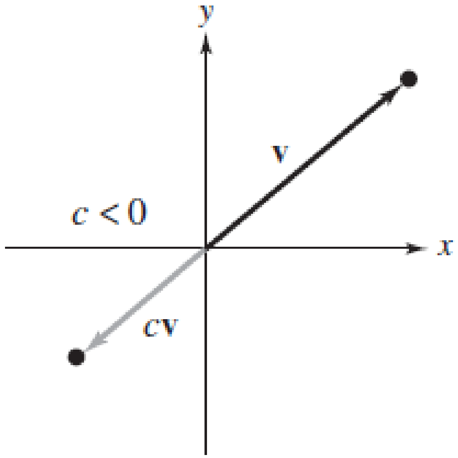


The vector $-\mathbf{v}$ is called the **negative** of $\mathbf{v}$.

The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}),$$

and you can say $\mathbf{v}$ is **subtracted** from $\mathbf{u}$.



Examples:

Provided with $\mathbf{v} = (-2, 5)$ and $\mathbf{u} = (3, 4)$, find each vector.

(a) $\frac{1}{2}\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\frac{1}{2}\mathbf{v} + \mathbf{u}$

Solutions:

(a) Because $\mathbf{v} = (-2, 5)$, you have

$$\tfrac{1}{2}\mathbf{v} = \left(\tfrac{1}{2}(-2), \tfrac{1}{2}(5)\right) = \left(-1, \tfrac{5}{2}\right).$$

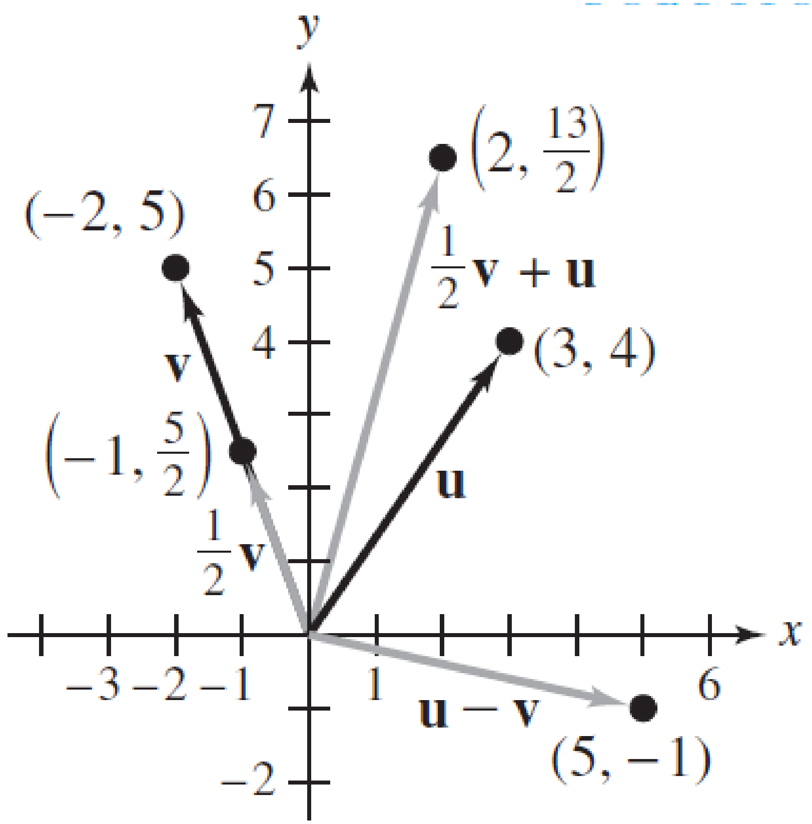
(b) By the definition of vector subtraction, you have

$$\mathbf{u} - \mathbf{v} = (3 - (-2), 4 - 5) = (5, -1).$$

(c) Using the result of part(a), you have

$$\tfrac{1}{2}\mathbf{v} + \mathbf{u} = \left(-1, \tfrac{5}{2}\right) + (3, 4) = \left(-1 + 3, \tfrac{5}{2} + 4\right) = \left(2, \tfrac{13}{2}\right).$$

Graph:



Properties of Vector Addition and Scalar Multiplication in the Plane

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Vectors in R^n

A vector in **n**-space is represented by an **ordered n-tuple.** For instance, an ordered triple has the form (x_1, x_2, x_3), an ordered quadruple has the form (x_1, x_2, x_3, x_4) and a general ordered n-tuple has the form $(x_1, x_2, x_3, \ldots, x_n)$. The set of all **n**-tuples is called **n-space** and is denoted by $\boldsymbol{R^n}$.

$$\begin{aligned}
R^1 &= \text{1-space} = \text{set of all real numbers} \\
R^2 &= \text{2-space} = \text{set of all ordered pairs of real numbers} \\
R^3 &= \text{3-space} = \text{set of all ordered triples of real numbers} \\
R^4 &= \text{4-space} = \text{set of all ordered quadruples of real numbers} \\
&\ \ \vdots \\
R^n &= n\text{-space} = \text{set of all ordered } n\text{-tuples of real numbers}
\end{aligned}$$

An **n**-tuple can be viewed as a **point** in $\boldsymbol{R^n}$ with the x_i's as its coordinates or as a **vector.**

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \qquad \text{Vector in } R^n$$

Vector Operations in R^n

The sum of two vectors in $\boldsymbol{R^n}$ and the scalar multiple of a vector in $\boldsymbol{R^n}$ are called the **standard operations in $\boldsymbol{R^n}$**.

Def'n. Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in $\boldsymbol{R^n}$ and let *c* be a real number. Then the sum of **u** and **v** is defined as the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n).$$

and the **scalar multiple** of **u** by *c* is defined as the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, \ldots, cu_n).$$

The **negative** of a vector in $\boldsymbol{R^n}$ is defined as

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

and the **difference** of two vectors in $\boldsymbol{R^n}$ is defined as

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n).$$

The **zero vector** in $\boldsymbol{R^n}$ is denoted by $\mathbf{0} = (0, 0, \ldots, 0)$.

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Properties of Vector Addition and Scalar Multiplication in R^n

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Examples:

Provided that $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in R^3, find each vector.

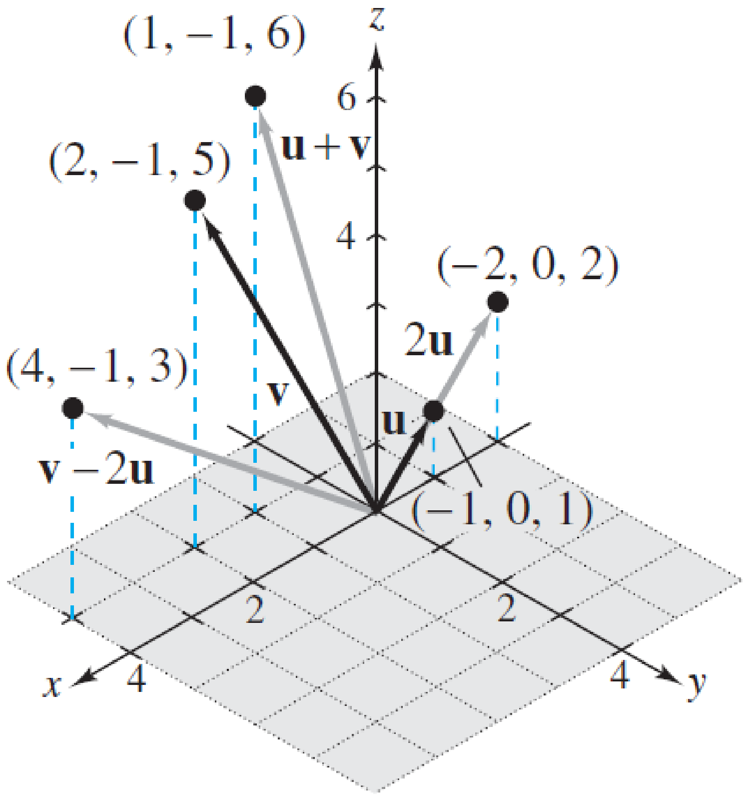
(a) $\mathbf{u} + \mathbf{v}$ (b) $2\mathbf{u}$ (c) $\mathbf{v} - 2\mathbf{u}$

Solutions:

a. $$\mathbf{u} + \mathbf{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$$

b. $$2\mathbf{u} = 2(-1, 0, 1) = (-2, 0, 2)$$

c. $$\mathbf{v} - 2\mathbf{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3).$$



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Practice Exercise

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in R^4. Solve for $\mathbf{x}$.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

Properties of Additive Identity and Additive Inverse

The zero vector **0** in $\boldsymbol{R^n}$ is called the **additive identity** in $\boldsymbol{R^n}$. The vector **-v** is called the **additive inverse** of **v.**

Let **v** be a vector in $\boldsymbol{R^n}$, and let *c* be a scalar. Then the following properties are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

A Vector as a Linear Combination of Other Vectors

In writing one vector $\mathbf{x}$ as the sum of scalar multiples of other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_n$, that is,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot\ \cdot\ \cdot + c_n\mathbf{v}_n$$

the vector $\mathbf{x}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_n$.

Example 1:

Provided that $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in R^3, find scalars a, b, and c such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

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Note:

It is often useful to represent a vector $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ in $\boldsymbol{R^n}$ as either a 1 x *n* row matrix (row vector),

$$\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_n],$$

or an *n* x 1 column matrix (column vector),

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Solution to Example 1:

Find a, b and c such that

$$\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 4a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 2b \end{bmatrix} + \begin{bmatrix} 3c \\ c \\ 2c \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -b+3c \\ a+b+c \\ 4a+2b+2c \end{bmatrix}$$

Hence, we want to solve the system of liner equation

$$\begin{aligned} -b+3c &= -1 \\ a+b+c &= -2 \\ 4a+2b+2c &= -2 \end{aligned}$$

(Note that you can use any of the methods that were introduced in our class discussions)

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By Gaussian elimination and back-substitution, we have

$$\begin{bmatrix} 0 & -1 & 3 & -1 \\ 1 & 1 & 1 & -2 \\ 4 & 2 & 2 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & -1 \\ 4 & 2 & 2 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & -2 & -2 & 6 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & -2 & 6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & -8 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$c = -1$$

$$b - 3c = 1 \implies b = -2$$

$$a + b + c = -2 \implies a = 1$$

Thus,

$$\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$$

In the given example, we can say that the vector $\mathbf{x}$ is a linear combination of the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

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Exercise:

(1) Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 2, -2)$ and $\mathbf{w} = (4, 0, -4)$.

(a) If $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$, find $\mathbf{z}$.

(b) If $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$, find $\mathbf{z}$.

(2) Write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ if possible, where

$\mathbf{u}_1 = (2, 3, 5)$, $\mathbf{u}_2 = (1, 2, 4)$, $\mathbf{u}_3 = (-2, 2, 3)$ and $\mathbf{v} = (10, 1, 4)$