Chapter 3.3- Spanning Sets and Linear Independence

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Learning Outcomes:

- Write a linear combination of a set of vectors in a vector space V.
- Determine whether a set S of vectors in a vector space V is a spanning set of V.
- Determine whether a set of vectors in a vector space V is linearly independent.

Definition of Linear Combination of Vectors

A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

where $c_1, c_2, \dots, c_k$ are scalars.

Examples:

a. For the set of vectors in $\boldsymbol{R}^3$,

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ since

$$\begin{aligned}\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 &= 3(0, 1, 2) + (1, 0, -5) \\ &= (1, 3, 1).\end{aligned}$$

Examples:

b. For the set of vectors in $M_{2,2}$,

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}} \right\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4 \\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

Finding a Linear Combination

1. Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set S.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)}\}$$

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + c_2 \qquad\; &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

$$c_1 = 1 + t, \qquad c_2 = -1 - 2t, \qquad c_3 = t.$$

where t is any real number.

Thus, $\mathbf{w}$ can be written as

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Finding a Linear Combination

2. Write the vector $\mathbf{w} = (1, -2, 2)$ as a linear combination of vectors in the set S.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)}\}$$

$$\begin{aligned}(1, -2, 2) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1) \\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1 \\ 2c_1 + c_2 \qquad\quad &= -2 \\ 3c_1 + 2c_2 + c_3 &= 2. \end{aligned}$$

The system has no solution.

Therefore, $\mathbf{w}$ cannot be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

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Spanning Sets

Definition:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is called a **spanning set** of V if *every* vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V.

Examples:

1. The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3 since any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as a linear combination of vectors in S, that is,

$$\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1) = (u_1, u_2, u_3)$$

Spanning Sets

Examples:

2. The set $S = \{1, x, x^2\}$ spans P_2 (set of polynomials of degree 2 or less).

   This is because any polynomial function

   $p(x) = a + bx + cx^2$ in P_2 can be written as

$$\begin{aligned} p(x) &= a(1) + b(x) + c(x^2) \\ &= a + bx + cx^2. \end{aligned}$$

Note: The spanning sets in examples 1 and 2 are called the standard spanning sets of R^3 and P_2, respectively.

More examples:

1. Show that set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3.

Solution: Let $\mathbf{u} = (u_1, u_2, u_3)$ be *any* vector in R^3. For S to span R^3, there must be scalars c_1, c_2 and c_3 such that

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1)\\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1\\ 2c_1 + \ \ c_2 \qquad\quad &= u_2\\ 3c_1 + 2c_2 + \ \ c_3 &= u_3.\end{aligned}$$

It can be verified that the determinant of the coefficient matrix is nonzero, so the system has a unique solution. Thus, any vector in R^3 can be written as a linear combination of the vectors in *S.*

Therefore, the set *S* spans R^3.

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2. Does the set $S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$ span R^3?

Solution: For S to span R^3, it must be that for any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3, there must be scalars c_1, c_2 and c_3 such that

$$(u_1, u_2, u_3) = c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)$$

$$= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).$$

$$\begin{aligned} c_1 \qquad\quad - c_3 &= u_1 \\ 2c_1 + c_2 \qquad\quad &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3. \end{aligned}$$

$\mathbf{w} = (1, 1, 2)$ is in R^3 and cannot be written as a linear combination of the vectors in S.

Therefore, the set S does not span R^3.

Definition of the Span of a Set

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid c_1, c_2, \ldots, c_k \text{ are real numbers}\}$$

The span of S is denoted by

$$\text{span}(S) \text{ or } \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

If $\text{span}(S) = V$, it is said that V is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, or that S **spans** V.

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Theorem:

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S).

Proof:

Note that the zero vector is contained in span(S), so span(S) is nonempty. Now, we show that span(S) is closed under addition and scalar multiplication.

Let $\mathbf{u}$ and $\mathbf{v}$ be in span(S), that is,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$
$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k$$

where $c_1, c_2, \ldots, c_k$ and $d_1, d_2, \ldots, d_k$ are scalars.

Proof: (continuation)

Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_k + d_k)\mathbf{v}_k$$

and for any scalar c,

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_k)\mathbf{v}_k$$

These show that $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ can be written as linear combinations of vectors in S, hence, $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are in span(S). Thus, span(S) is a subspace of V.

Linear Dependence and Linear Independence

Definition:

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution, $c_1 = 0, c_2 = 0, \dots, c_k = 0.$

If there are also nontrivial solutions, then S is called **linearly dependent**.

Examples:

(a) The set $S = \{(1, 2), (2, 4)\}$ in R^2 is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

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Linear Dependence and Linear Independence

Examples:

(b) The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in R^2 is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

(c) The set $S = \{(0, 0), (1, 2)\}$ in R^2 is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

Testing for Linear Dependence and Linear Independence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, perform the following steps.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0$ write a homogeneous system of linear equations in the variables c_1, c_2,…, and c_k.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

Examples:

1. Determine whether the set of vectors in R^3 is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

Find c_1, c_2, and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

$$c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) = (0, 0, 0)$$

$$(c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) = (0, 0, 0)$$

$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + c_2 \qquad\quad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

By Gaussian elimination, we can see that

$$c_1 = c_2 = c_3 = 0.$$

So, S is linearly independent.

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Examples:

2. Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

Solution: $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

$$c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) = 0 + 0x + 0x^2$$

$$(c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 = 0 + 0x + 0x^2$$

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned} \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the system has infinitely many solutions. So, S is linearly dependent.

Examples:

3. Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

Solution:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned} \quad \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, S is linearly independent.

Theorem:

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

Examples:

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

$$\begin{aligned} c_1 + 2c_2 \phantom{{}+c_3} &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

$$\mathbf{v}_2 = 2\mathbf{v}_1 + 3\mathbf{v}_3$$

Corollary:

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

Examples:

1. The set

$$S = \{\underbrace{(1, 2, 0)}_{\mathbf{v}_1}, \underbrace{(-2, 2, 1)}_{\mathbf{v}_2}\}$$

is linearly independent since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other.

2. The set

$$S = \{\underbrace{(4, -4, -2)}_{\mathbf{v}_1}, \underbrace{(-2, 2, 1)}_{\mathbf{v}_2}\}$$

is linearly dependent since $\mathbf{v}_1 = -2\mathbf{v}_2$.

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Assignment

1. Determine whether or not the set $S = \{(1,-2,0), (0,0,1), (-1,2,0)\}$ spans R^3. Justify your answer.

2. Given the matrices $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$. Show that the matrix $C = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$ is a linear combination of matrices A and B.

3. For which value/s of t is the set $S = \{(t,1,1), (1,0,1), (1,1,3t)\}$ linearly independent?