Chapter 1.5 Elementary Matrices

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Elementary Matrices and Elementary Row Operations

Recall: Three elementary row operations for matrices

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2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

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An $n \times n$ matrix is called an **elementary matrix** when it can be obtained from the identity matrix I_n by a single elementary row operation.

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Example: Which of the following matrices are elementary? For those that are, describe the corresponding elementary row operation.

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ **f.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

SOLUTION

a. This matrix *is* elementary. To obtain it from I_3, multiply the second row of I_3 by 3.

b. This matrix is *not* elementary because it is not square.

c. This matrix is *not* elementary because to obtain it from I_3, you must multiply the third row of I_3 by 0 (row multiplication must be by a nonzero constant).

d. This matrix *is* elementary. To obtain it from I_3, interchange the second and third rows of I_3.

e. This matrix *is* elementary. To obtain it from I_2, multiply the first row of I_2 by 2 and add the result to the second row.

f. This matrix is *not* elementary because it requires two elementary row operations to obtain from I_3.

Elementary Matrices and Elementary Row Operations

Theorem

Representing Elementary Row Operations

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

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Example:

In the matrix product below, E is the elementary matrix in which the first two rows of I_3 are interchanged.

$$\overset{E}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

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Note that the first two rows of A are interchanged by multiplying *on the left* by E.

Elementary Matrices and Elementary Row Operations

Definition

Row Equivalence

Let A and B be $m \times n$ matrices. Matrix B is **row-equivalent** to A when there exists a finite number of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A.$$

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Example:

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Elementary Matrices

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Elementary Matrices Are Invertible

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Examples: Find the inverses of the following elementary matrices

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$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

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A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Example:

Find a sequence of elementary matrices whose product is the nonsingular matrix

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}.$$

Elementary Matrices

Theorem

Equivalent Conditions

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.

The LU-Factorization

Definitions

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$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3×3 lower triangular matrix

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3×3 upper triangular matrix

Definition

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an LU-**factorization** of A.

Examples:

a. $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU$

is an *LU*-factorization of the matrix

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b. $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$

is an LU-factorization of the matrix A.

Finding an LU-Factorization of a Matrix

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SOLUTION

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Observe that the above matrix is upper triangular. We denote it by U so that we have $E_2 E_1 A = U$. This implies that $A = E_1^{-1} E_2^{-1} U$.

Finding an LU-Factorization of a Matrix

Moreover, notice that the product of lower triangular matrices

$$E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

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Remark:

If A row reduces to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then A has an LU-factorization.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U = LU$$

Here L is the product of the inverses of the elementary matrices used in the row reduction.

Remark:

Once you have obtained an LU-factorization of a matrix A, you can then solve the system of n linear equations in n variables $A\mathbf{x} = \mathbf{b}$ very efficiently in two steps.

1. Write $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

2. Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

Solving a Linear System Using LU-Factorization

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Example:

Solve the linear system.

$$\begin{aligned} x_1 - 3x_2 &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$