Chapter 4.3

Orthonormal Bases: Gram-Schmidt Process

Orthogonal and Orthonormal Sets

Definition (***Orthogonal and Orthonormal***)

A set S of vectors in an inner product space V is called **orthogonal** if every pair of vectors in S is orthogonal. In addition, if each vector in S is a unit vector, then S is called **orthonormal**

Remarks:

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then
   i. Orthogonal: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ when $i \neq j$
   ii. Orthonormal: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ when $i \neq j$ and $\|\mathbf{v}_i\| = 1$ for all $i = 1,2, \dots, n$
2. If S is a *basis*, then it is an *orthogonal basis* (i) or an *orthonormal basis* (ii), respectively.

Examples:

1. Show that $S = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right\}$ is an orthonormal basis for R^3.

2. Show that in P_3, with the inner product
$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$
the standard basis $B = \{1, x, x^2, x^3\}$ is orthonormal.

Theorem:

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

Corollary:

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

Example:

Show that

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$
$$= \{(2,3,2,-2), (1,0,0,1), (-1,0,2,1), (-1,2,-1,1)\}$$

is a basis for R^4.

Definition: ***Coordinate Representation Relative to a Basis***

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of x relative to the basis *B*.** The **coordinate matrix** (or **coordinate vector**) **of x relative to *B*** is the column matrix in R^n whose components are the coordinates of **x**.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example:

Find the coordinate matrix of $\mathbf{x} = (1,2,-1)$ in R^3 relative to the (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1,0,1), (0,-1,2)(2,3,-5)\}$$

Solution: Write $\mathbf{x}$ as a linear combination of the vectors in B', that is,

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

$$(1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$

Equating corresponding components produces the system

$$\begin{aligned} c_1 \quad\quad\quad + 2c_3 &= 1 \\ -c_2 + 3c_3 &= 2 \\ c_1 + 2c_2 - 5c_3 &= -1 \end{aligned}$$

The solution to this system is $c_1 = 5, c_2 = -8, c_3 = -2$.

So that $\mathbf{x} = 5(1,0,1) + (-8)(0,-1,2) + (-2)(2,3,-5)$, and the coordinate matrix of $\mathbf{x}$ relative to B' is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Theorem: Coordinates Relative to an Orthonormal Basis

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{u}$ relative to B is

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Remarks:

1. The coordinates of $\mathbf{u}$ relative to an orthonormal basis B are called **Fourier coefficients** of $\mathbf{u}$ relative to B.
2. The corresponding coordinate matrix of $\mathbf{u}$ relative to B is

$$\begin{aligned}[\mathbf{u}]_B &= [c_1 \quad c_2 \quad \cdots \quad c_n]^T \\ &= [\langle \mathbf{u}, \mathbf{v}_1 \rangle \quad \langle \mathbf{u}, \mathbf{v}_2 \rangle \quad \cdots \quad \langle \mathbf{u}, \mathbf{v}_n \rangle]^T\end{aligned}$$

Example:

The set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{\left(\frac{3}{5}, \frac{4}{5}, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0\right), (0,0,1)\right\}$ is an orthonormal basis for R^3. Find the coordinate matrix of $\mathbf{u} = (5, -5, 2)$ relative to B.

Gram – Schmidt Orthonormalization Process

Theorem: Gram – Schmidt Orthonormalization Process

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of an inner product space V.
2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ where

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_n = \mathbf{v}_n - \sum_{i=1}^{n-1} \frac{\langle \mathbf{v}_n, \mathbf{w}_i \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle} \mathbf{w}_i$$

Then B' is an orthogonal basis for V.

3. Let $\mathbf{u}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then the set $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal basis for V. In addition,

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$$

for all $k = 1, 2, \dots, n$.

Examples:

1. Give an orthonormal basis for R^2 using the basis
$$B = \{(1,1), (0,1)\}.$$
2. Give an orthonormal basis for R^3 using the basis
$$B = \{(1,1,0), (1,2,0), (0,1,2)\}.$$
3. The vectors $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (1,1,1)$ span a plane in R^3. Find an orthonormal basis for this subspace.