Chapter 3.5- Rank of a Matrix

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Learning Outcomes:

- Find a basis for the row space, a basis for the column space, and the rank of a matrix.
- Find the nullspace of a matrix.

Rank of a Matrix

For an $m \times n$ matrix the n–tuples corresponding to the rows of A are called the **row vectors** of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row Vectors of A

$$\begin{gathered} (a_{11}, a_{12}, \ldots, a_{1n}) \\ (a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{gathered}$$

The columns of A are called the **column vectors** of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column Vectors of A

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Definition:

Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace of R^n spanned by the row vectors of A.
2. The **column space** of A is the subspace of R^m spanned by the column vectors of A.

Theorem: (Row-Equivalent Matrices Have the Same Row Space)

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

Theorem: (Basis for the Row Space of a Matrix)

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

Examples:

1. Find the basis for the row space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}. \quad B = \begin{array}{cc} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \end{matrix} \end{array}$$

$\mathbf{w}_1 = (1, 3, 1, 3), \mathbf{w}_2 = (0, 1, 1, 0)$ and $\mathbf{w}_3 = (0, 0, 0, 1)$ form a basis for the row space of A.

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2. Find a basis for the subspace of R^3 spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5)}, \overset{\mathbf{v}_2}{(3, 0, 3)}, \overset{\mathbf{v}_3}{(5, 1, 8)}\}.$$

Use $\mathbf{v}_1, \mathbf{v}_2$,and $\mathbf{v}_3$ to form the rows of a matrix A.

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \longrightarrow B = \begin{array}{c} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \phantom{\mathbf{w}} \end{array}$$

$\mathbf{w}_1 = (1, -2, -5)$ and $\mathbf{w}_2 = (0, 1, 3)$, form a basis for the row space of A.

Therefore, $\mathbf{w}_1 = (1, -2, -5)$ and $\mathbf{w}_2 = (0, 1, 3)$ form a basis for the subspace of R^3 spanned by

$$S = \{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}.$$

3. Find the basis for the column space of matrix A,

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \end{array}$$

So, $\mathbf{w}_1 = (1, 0, -3, 3, 2), \mathbf{w}_2 = (0, 1, 9, -5, -6)$ and $\mathbf{w}_3 = (0, 0, 1, -1, -1)$ form a basis for the row space of A^T.

This is equivalent to saying that the column vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

form a basis for the column space of A.

3. Find the basis for the column space of matrix A,

Method 2:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array} \end{array} \qquad B = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{array} \end{array}$$

Since $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2$,

and so $\mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$

Similarly, $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_4$ of matrix B are linearly independent, and so are the corresponding columns of matrix A.

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$$
\begin{array}{cccc}
 & & & \\
A = \left[\begin{array}{rrrr} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{array}\right] & & B = \left[\begin{array}{rrrr} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
\quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 & & \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4
\end{array}
$$

Hence, the corresponding columns of matrix A form a basis for the column space of A.

$$
\left[\begin{array}{r} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{array}\right], \quad \left[\begin{array}{r} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{array}\right], \quad \text{and} \quad \left[\begin{array}{r} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{array}\right]
$$

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Theorem:

If A is an $m \times n$ matrix, then the row space and column space of A have the same dimension.

Definition:

The dimension of the row (or column) space of a matrix A is called the **rank of $\boldsymbol{A}$** and is denoted by $\text{rank}(A)$.

Finding the Rank of a Matrix

Example:

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Because B has three non-zero rows, the rank of A is 3.

The Nullspace of a Matrix

Theorem:

If A is an $m \times n$ matrix, then the set of all solutions of the homogenous system of linear equations

$$A\mathbf{x} = 0$$

is a subspace of R^n called the ***nullspace*** of A and is denoted by $N(A)$. So,

$$N(A) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = 0\}.$$

The dimension of the nullspace of A is called the **nullity** of A.

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Example:

Find the nullspace of the matrix and its nullity.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 \qquad + 3x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

$$x_1 = -2s - 3t, x_2 = s, x_3 = -t, x_4 = t$$

Therefore, the nullspace of A is the set of all solution vectors of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

A basis for the nullspace of A consists of the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, the nullity of A is 2.

Theorem: (Dimension of the Solution Space)

If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\mathbf{x} = 0$ is $n - r$. That is,

$$n = \text{rank}(A) + \text{nullity}(A).$$

Example:

Let the column vectors of the matrix A be denoted by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{a}_5$.

$$
\begin{array}{cc}
A = & \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\
 & \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array}
\end{array}
$$

1. Find the rank and nullity of A.
2. Find a subset of the column vectors of A that forms the basis for the column space of A.

$$B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. Because B has three nonzero rows, the rank of A is 3. Also, the number of columns of A is $n = 5$, which implies that the nullity of A is $n - \text{rank} = 5 - 3 = 2$.

b. Because the first, second, and fourth column vectors of B are linearly independent, the corresponding column vectors of A,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the column space of A.

Theorem:

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Summary of Equivalent Conditions for Square Matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. $|A| \neq 0$
6. $\text{Rank}(A) = n$
7. The n row vectors of A are linearly independent.
8. The n column vectors of A are linearly independent.

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