Chapter 1.4 The Inverse of a Matrix

IRISH C. SIDAYA

The Inverse of a Matrix

Definition

Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) **inverse** of A. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Definition

Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) **inverse** of A. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Note: *Nonsquare matrices do not have inverses.*

The Inverse of a Matrix

Theorem

Uniqueness of an Inverse Matrix

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

Theorem

Uniqueness of an Inverse Matrix

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

PROOF

Because A is invertible, you know it has at least one inverse B such that

$$AB = I = BA.$$

Suppose A has another inverse C such that

$$AC = I = CA.$$

Theorem

Uniqueness of an Inverse Matrix

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

PROOF

Because A is invertible, you know it has at least one inverse B such that

$$AB = I = BA.$$

Suppose A has another inverse C such that

$$AC = I = CA.$$

$$\begin{aligned} AB &= I \\ C(AB) &= CI \\ (CA)B &= C \\ IB &= C \\ B &= C \end{aligned}$$

Example:

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Finding the Inverse of a Matrix

Example: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

Example: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \quad I]$. This process is called **adjoining** matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A \quad I]$. The result will be the matrix $[I \quad A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Finding the Inverse of a Matrix

Example: Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$.

A Singular Matrix

Example: Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

A Singular Matrix

Example: Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

SOLUTION

Adjoin the identity matrix to A to form

$$[A \quad I] = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{bmatrix}$$

and apply Gauss-Jordan elimination to obtain the following.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Note that the "A portion" of the matrix has a row of zeros. So it is not possible to rewrite the matrix $\begin{bmatrix} A & I \end{bmatrix}$ in the form $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. This means that A has no inverse, or is noninvertible (or singular).

Finding the Inverse of a 2×2 Matrix

If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by

Finding the Inverse of a 2×2 Matrix

If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Finding the Inverse of a 2×2 Matrix

If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example: Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

Properties of Inverses

Theorem

Properties of Inverse Matrices

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $(A^{-1})^{-1} = A$

2. $(A^k)^{-1} = \underbrace{A^{-1}A^{-1} \cdot \cdot \cdot A^{-1}}_{k \text{ factors}} = (A^{-1})^k$

3. $(cA)^{-1} = \dfrac{1}{c}A^{-1}$

4. $(A^T)^{-1} = (A^{-1})^T$

The Inverse of a Product

Theorem

The Inverse of a Product

If A and B are invertible matrices of order n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The Inverse of a Product

Theorem

The Inverse of a Product

If A and B are invertible matrices of order n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

PROOF

To show that $B^{-1}A^{-1}$ is the inverse of AB, you need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

In a similar way, $(B^{-1}A^{-1})(AB) = I$. So, AB is invertible and has the indicated inverse.

Theorem

The Inverse of a Product

If A and B are invertible matrices of order n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

PROOF

To show that $B^{-1}A^{-1}$ is the inverse of AB, you need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

In a similar way, $(B^{-1}A^{-1})(AB) = I$. So, AB is invertible and has the indicated inverse.

Note: This theorem can be generalized to include the product of several invertible matrices:

$$(A_1A_2...A_n)^{-1} = A_n^{-1}...A_2^{-1}A_1^{-1}$$

Finding the Inverse of a Matrix Product

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$.

Find $(AB)^{-1}$.

Cancellation Properties

Theorem

Cancellation Properties

If C is an invertible matrix, then the following properties hold.

1. If $AC = BC$, then $A = B$. Right cancellation property
2. If $CA = CB$, then $A = B$. Left cancellation property

Theorem

Cancellation Properties

If C is an invertible matrix, then the following properties hold.

1. If $AC = BC$, then $A = B$. Right cancellation property
2. If $CA = CB$, then $A = B$. Left cancellation property

PROOF

To prove Property 1, use the fact that C is invertible and write

$$\begin{aligned} AC &= BC \\ (AC)C^{-1} &= (BC)C^{-1} \\ A(CC^{-1}) &= B(CC^{-1}) \\ AI &= BI \\ A &= B. \end{aligned}$$

The second property can be proved in a similar way.

Theorem

Systems of Equations with Unique Solutions

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem

Systems of Equations with Unique Solutions

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.

PROOF

Because A is nonsingular, the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This solution is unique because if $\mathbf{x}_1$ and $\mathbf{x}_2$ were two solutions, then you could apply the cancellation property to the equation $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$ to conclude that $\mathbf{x}_1 = \mathbf{x}_2$.

Solving a System of Equations Using an Inverse Matrix

Example: Use an inverse matrix to solve each system.

a. $$\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$$

b. $$\begin{aligned} 2x + 3y + z &= 4 \\ 3x + 3y + z &= 8 \\ 2x + 4y + z &= 5 \end{aligned}$$

c. $$\begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$$