Chapter 4.2
Inner Products

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Vector Length and Unit Vectors

Definition (***Inner Product***)

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms.

i. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

ii. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

iii. $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$

iv. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Remark:

A vector space with an inner product is called an **inner product space**.

Example 1: ***Euclidean inner product for R^n***

The dot product in R^n satisfies the four axioms of an inner product by the properties of the dot product discussed in Chapter 4.1.

Example 2:

Show that the following function defines an inner product in R^2, where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, and

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$$

Remark:

The function

$$\langle \mathbf{u}, \mathbf{v} \rangle = c_1 u_1 v_1 + c_2 u_2 v_2 + \cdots + c_n u_n v_n$$

where $c_i > 0$ for all $i = 1,2, \dots, n$, is an inner product in R^n. The positive scalar $c_i's$ are called **weights**. (*Proof is left as an exercise)*

Example 3:

Determine whether the function defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

is an inner product on R^3.

Example 4:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be matrices in the vector space M_{22}. Determine whether the function defined by

$$\langle A, B \rangle = a_{11} b_{11} + a_{21} b_{21} + a_{12} b_{12} + a_{22} b_{22}$$

is an inner product on M_{22}.

Theorem: Properties of Inner Products

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in an inner product space V, and let c be any real number.

1. $\langle \mathbf{u}, \mathbf{0} \rangle = 0 = \langle \mathbf{0}, \mathbf{u} \rangle$
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

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Definitions: ***Length, Distance and Angle***

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. The **length** (or **norm**) of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$.
2. The **distance** between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
3. The **angle** between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|},$$

   where $0 \le \theta \le \pi$.

4. $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** when $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

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Remarks:

1. If $\|\mathbf{u}\|=1$, then **u** is called a **unit vector**.
2. If **v** is any nonzero vector in an inner product space V, then the vector $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector called **the unit vector in the direction of v**.

Example:

Let $p = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $q = b_0 + b_1x + b_2x^2 + \cdots b_nx^n$ be polynomials in the vector space P_n. Determine whether the function

$$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

is an inner product on P_n.

Example:

Using the inner product on P_n from the previous example, solve for the following

1. $\langle p, q \rangle$
2. $\langle q, r \rangle$
3. $\|q\|$
4. $d(p, q)$ (the distance of p and q)

when $p(x) = 1 - 2x^2$, $q(x) = 4 - 2x + x^2$, and $r(x) = x + 2x^2$.

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Theorem:

Let **u** and **v** be vectors in an inner product space V.

1. Cauchy-Schwarz inequality: $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$
2. Triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
3. Pythagorean Theorem: **u** and **v** are orthogonal if and only if
$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2.$$