Analysis II - 2014.04.14

After horribly singing an awful song we finally get on with the lecture. Only a single question remains in my head to this point. How am I still typing when I'm already long dead? Though even while I haunt this classroom for what feels like an eternity of pain and suffering, there never seems to be an answer. When will this curse finally end? When will I finally be free? When will

Polarkoordinaten

$dxdy = rdrd\varphi$

Kugelkoordinaten

$$\begin{pmatrix} x = r\cos\vartheta\cos\varphi \\ y = r\cos\vartheta\sin\varphi \\ z = r\sin\vartheta \end{pmatrix} \quad dxdydz = r^2\cos\vartheta\ drd\vartheta d\varphi$$

Beispiel: $B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right| \leq R \right\} \quad \mathrm{vol}(B) = \int\int\int_R 1\ dxdydz = \int\limits_0^R \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_0^{2\pi} 1r^2\cos\vartheta\ d\varphi d\vartheta dr$

$= \int\limits_0^R \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi r^2\cos\vartheta\ d\vartheta dr = \int\limits_0^R (2\pi r^2\sin\vartheta|_{\vartheta=-\frac{\pi}{2}}^{\vartheta=\frac{\pi}{2}})\ dr = \int\limits_0^R 4\pi r^3\ dr = \frac{4\pi r^3}{3}\Big|_0^R = \frac{4\pi R^3}{3}$

Beispiel: $B := \left\{ \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \in \mathbb{R}^4 \middle| \text{Betrag} \leq R \right\}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \quad \begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} \rho\cos\psi \\ \rho\sin\psi \end{pmatrix}$

$\left| \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \right| = \sqrt{r^2+\rho^2} \leq R, \quad \varphi,\psi \in [0,2\pi] \quad dxdydzdu = r\rho drd\varphi d\rho d\psi$

$\mathrm{vol}(B) = \int_{r,rho\geq 0}\int_{r^2+\rho^2\leq R^2}\int_0^{2\pi}\int_0^{2\pi} 1r\rho\ d\psi d\varphi d\rho dr = \int_0^R\int_0^{\sqrt{R^2-r^2}} (2\pi)^2 r\rho\ d\rho dr$

$\int_0^R \left((2\pi)^2\frac{r\rho^2}{2}\Big|_{\rho=0}^{\rho=\sqrt{R^2-r^2}} \right)\ dr = \int_0^R (2\pi)^2\frac{r}{2}(R^2-r^2)\ dr = \frac{4\pi^2}{2}(\frac{r^2}{2}R^2 - \frac{r^4}{4})\Big|_{r=0}^R = 2\pi^2(\frac{R^4}{2} - \frac{R^4}{4})$

$= \frac{pi^2R^4}{2}$

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Rotationskörper

Zylinderkoordinaten

$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z \quad X = \left\{ \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix} \middle| \begin{pmatrix} \rho \\ z \end{pmatrix} \in B,\ \varphi \in [0, 2\pi] \right\}$ für $B \subset \mathbb{R}^{\geq 0} \times \mathbb{R}$

$\Rightarrow \int_X f \begin{pmatrix} x \\ y \\ z \end{pmatrix} \, d\,\mathrm{vol}_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \int_B \left(\int_0^{2\pi} f \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix} \, d\varphi \right) \rho \, d\,\mathrm{vol}_2 \begin{pmatrix} \rho \\ z \end{pmatrix}$

Falls f unabhängig von φ: $= \int_B 2\pi f \begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix} \rho \, d\,\mathrm{vol} \begin{pmatrix} \rho \\ z \end{pmatrix}$

Beispiel: Torus $R > r > 0 \quad B = \left\{ \begin{pmatrix} \rho \\ z \end{pmatrix} \in \mathbb{R}^2 \middle| (\rho - R)^2 + z^2 \leq r^2 \right\}$

$\mathrm{vol}(X) = \int_X 1 \, d\,\mathrm{vol} = \int_B 2\pi\rho \, d\,\mathrm{vol} \begin{pmatrix} \rho \\ z \end{pmatrix} = \int\limits_{-r}^{r} \left(\int\limits_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} 2\pi\rho \, d\rho \right) dz = \int\limits_{-r}^{r} \pi\rho^2 \Big|_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} dz$

$= \int\limits_{-r}^{r} \pi 4R\sqrt{r^2 - z^2} \, dz$ Apparently this is a dumb way to do things and you should've instead used polar coordinates. So instead he just gives us the result directly since we've long gone past break time: $= 2\pi R r^2$

Spezialfall: Sei $g : [a, b] \to \mathbb{R}^{\geq 0} \quad X = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \substack{z \in [a,b] \\ \sqrt{x^2+y^2} \leq g(z)} \right\}$ $\quad f$ Rotationsinvariant

$\Rightarrow \int_X f \, d\,\mathrm{vol}_3 = \int_a^b \left(\int_0^{g(z)} f \begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix} 2\pi\rho \, d\rho \right) dz \overset{\text{falls } f \text{ nur von } z \text{ abhängt}}{=} \int_a^b f \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \pi\rho^2 \,\Big|_0^{g(z)} \, dz$

$= \int_a^b f \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \pi g(z)^2 \, dz$

Zum Beispiel: $\mathrm{vol}(X) = \int_a^b \pi g(z)^2 \, dz$

Beispiel: $g : [0, h] \to \mathbb{R}^{\geq 0}$ $z \mapsto \frac{Rz}{h} \Rightarrow X =$ Kegel der Höhe h und Radius der Basis R.
$\Rightarrow \mathrm{vol}(X) = \int_0^h \pi(\frac{Rz}{h})^2 \, dz = \frac{\pi R^2}{h^2} \frac{z^3}{3} \,\Big|_0^h = \frac{\pi R^2 h}{3}$

Physikalische Grössen

Masse: Sei $\mu : X \to \mathbb{R}^{\geq 0}$ die Massendichte-Funktion, dann ist die Gesamtmasse $= \int_X \mu \, d\,\mathrm{vol}$. Insbesondere ist X homogen mit konstanter Massendichte, so ist die Gesamtmasse $= \mu \,\mathrm{vol}(X)$.

Schwerpunkt: Das gewichtete Mittel der Ortsvektoren aller Massen. $\rightsquigarrow S = \frac{\sum m_i x_i}{\sum m_i}$
$S = \frac{1}{\text{Masse von } X} \int_X \mu(x) x \, d\,\mathrm{vol}(x)$

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Beispiel: Schwerpunkt einer homogenen Halbkreisscheibe mit Gesamtmasse $\mu\frac{\pi R^2}{2}$.

$$X = \left\{ \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \middle| \, {\scriptstyle 0\leq r\leq R \atop 0\leq\varphi\leq\pi} \right\} \Rightarrow S = \frac{2}{\mu\pi R^2}\int_X \mu \begin{pmatrix} x \\ y \end{pmatrix} \, d\,\mathrm{vol}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2}{\pi R^2}\int_0^R\int_0^\pi \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} r \, d\varphi dr$$

$$= \frac{2}{\pi R^2}(\int_0^R r^2 \, dr)\left(\int_0^\pi \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \, d\varphi\right) = \frac{2}{\pi R^2}(\frac{R^3}{3})\left(\left.\begin{pmatrix} \sin\varphi \\ -\cos\varphi \end{pmatrix}\right|_{\varphi=0}^{\varphi=\pi}\right) = \frac{2R}{3\pi}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4R}{3\pi} \end{pmatrix}$$