Analysis II - 2014.04.10

Erinnerung: Satz von Fubini: $X \subset \mathbb{R}^n$ kompakt, $\varphi, \psi : X \to \mathbb{R}$ stetig,
$\forall x \in X : \varphi(x) \leq \psi(x) Z := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+1} \middle| \substack{x \in X \\ \varphi(x) \leq y \leq \psi(x)} \right\}$
$\Rightarrow \int\limits_Z f \begin{pmatrix} x \\ y \end{pmatrix} \, d\,\mathrm{vol}_{n+1} \begin{pmatrix} x \\ y \end{pmatrix} = \int\limits_X \left(\int\limits_{\varphi(x)}^{\psi(x)} f \begin{pmatrix} x \\ y \end{pmatrix} \, dy \right) \, d\,\mathrm{vol}_n(x)$

Beispiel: $\int\limits_{[1,a]\times[1,b]} \frac{1}{(x+y)^2} \, d\,\mathrm{vol} \begin{pmatrix} x \\ y \end{pmatrix} \overset{\text{Fubini}}{=} \int\limits_1^b (\int\limits_1^a \frac{dx}{(x+y)^2}) \, dy = \int\limits_1^b (\frac{-1}{x+y}|_{x=1}^{x=a}) \, dy = \int_1^b \frac{1}{1+y} - \frac{1}{a+y} \, dy$
$= (\log|1+y| - \log|a+y|)|_{y=1}^{y=b} = \log|1+b| - \log|a+b| - \log|2| + \log|a+1| = \log \frac{(1+a)(1+b)}{2(a+b)}$

Beispiel: $\int\limits_{[0,\pi]^3} \sin(x+y+z) \, d\,\mathrm{vol} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
$= \int\limits_0^\pi \int\limits_0^\pi \int\limits_0^\pi \sin(x+y+z) \, dzdydx = \int\limits_0^\pi \int\limits_0^\pi 2\cos(x+y) \, dydz = \int\limits_0^\pi 4 \sin x \, dx = -8$

Beispiel: Umrechnen des Integrationsbereichs
$\int\limits_0^2 \int\limits_0^{4\sqrt{2y}} f(x,y) \, dxdy \quad B = \left\{ (x,y) \in \mathbb{R}^2 \mid \substack{0 \leq y \leq 2 \\ 0 \leq x \leq 4\sqrt{2y}} \iff \substack{0 \leq y \leq 2 \\ 0 \leq x \\ x^2 \leq 32y} \iff \substack{x \geq 0 \\ \frac{x^2}{32} \leq y \leq 2} \iff \substack{0 \leq x \leq 8 \\ \frac{x^2}{32} \leq y \leq 2} \right\}$
$\Rightarrow = \int\limits_0^8 \int\limits_{\frac{x^2}{32}}^2 f(x,y) \, dydx$

Beispiel: $\int\limits_{-1}^2 \int\limits_{-x}^{2-x^2} f(x,y) \, dydx \quad$ Bereich : $\substack{-1 \leq x \leq 2 \\ -x \leq y \leq 2 - x^2} \iff \substack{-1 \leq x \leq 2 \\ -y \leq x \\ x^2 \leq 2 - y} \iff \substack{-2 \leq y \leq 2 \\ -1, -y \leq x \leq 2 \\ \sqrt{2-y} \leq x \leq \sqrt{-y}} \iff \cdots$
At this point we give up on this approach and instead draw a fancy graph.
$\Rightarrow = \int\limits_{-2}^1 \int\limits_{-y}^{\sqrt{2-y}} f(x,y) \, dxdy + \int\limits_1^2 \int\limits_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y) \, dxdy$

Substitution

Seien $X, \widetilde{X} \subset \mathbb{R}^n$ kompakt, sei $\varphi : \widetilde{X} \to X$ stetig diff'bar bijektiv ausserhalb gewisser Teilmengen von X, $\widetilde{X}$ der Dimension $< n$. $\rightsquigarrow |\det \nabla\varphi| =$ lokaler Volumenfaktor bei X. $A \subset X$ klein nahe $\xi \rightsquigarrow \mathrm{vol}(\varphi(A)) \approx |\det \nabla\varphi(\xi)| \, \mathrm{vol}(A)$

Fakt: f, g Riemannintegrierbar auf X kompakt, so dass $f = g$ ausserhalb einer Teilmenge der Dimension $< n \Rightarrow \int\limits_X f \, d\,\mathrm{vol} = \int\limits_X g \, d\,\mathrm{vol}$.

Satz: Für jede integrierbare Funktion f auf X ist $f \circ \varphi$ integrierbar und
$\int\limits_X f \, d\,\mathrm{vol}_n = \int\limits_{\widetilde{X}} (f \circ \varphi) \cdot |\det \nabla\varphi| \, d\,\mathrm{vol}_n$

Fall $n = 1$: $X = [a, b]$ $\quad \widetilde{X} = [\widetilde{a}, \widetilde{b}]$ $\quad \varphi : \widetilde{X} \to X$ bijektiv.

$\varphi \nearrow \iff \varphi' \geq 0 \iff |\det \nabla\varphi| = \varphi'$

$\varphi \searrow \iff \varphi' \leq 0 \iff |\det \nabla\varphi| = -\varphi'$

$$\int_a^b f(x)\, dx = \int_{\widetilde{a}}^{\widetilde{b}} f(\varphi(\widetilde{x})) \left\{ \begin{array}{ll} +\varphi'(\widetilde{x}) & \text{falls } \varphi \nearrow \\ -\varphi'(\widetilde{x}) & \text{falls } \varphi \searrow \end{array} \right\} d\widetilde{x}$$

$$\int_a^b f(x)\, dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(\widetilde{x}))\varphi'(\widetilde{x})\, d\widetilde{x}$$

Spezialfall: $\varphi(\widetilde{x}) := \nu \in \mathbb{R}^n + \overbrace{A}^{n\times n\text{-Matrix}} \widetilde{x} \Rightarrow \nabla\varphi = A$

$$\int_{\varphi(\widetilde{X})} f(x)\, d\operatorname{vol}_n(x) = \int_{\widetilde{X}} f(\nu + A\widetilde{x})|\det A|\, d\operatorname{vol}_n(\widetilde{x})$$

Spezialfall: Polarkoordinaten

$f : [0, \infty[\times [0, 2\pi] \to \mathbb{R}$, $\begin{pmatrix} r \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$ $\quad \det \nabla f = \nu \Rightarrow \forall \widetilde{X} \subset [0, \infty[\times [0, 2\pi]$ kompakt

$$\Rightarrow \int_{f(\widetilde{X})} g \begin{pmatrix} x \\ y \end{pmatrix} d\operatorname{vol} \begin{pmatrix} x \\ y \end{pmatrix} = \int_{\widetilde{X}} g \left(f \begin{pmatrix} r \\ \varphi \end{pmatrix} \right) r\, d\operatorname{vol} \begin{pmatrix} r \\ \varphi \end{pmatrix}$$

d.h. $\displaystyle\int_{f(\widetilde{X})} \int g \begin{pmatrix} x \\ y \end{pmatrix} dxdy = \int_{\widetilde{X}} \int g \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} r\, drd\varphi$ $\qquad$ $\underline{\text{“}dxdy\text{”=”}rdrd\varphi\text{”}}$

Beispiel: $\widetilde{X} = [0, R] \times [0, 2\pi] \Rightarrow f(\widetilde{X}) =$ Kreisscheibe vom Radius R

$$\Rightarrow \operatorname{vol}(f(\widetilde{X})) = \int_{f(\widetilde{X})} 1\, dxdy = \int_0^R \int_0^{2\pi} 1r\, d\varphi dr = 2\pi \int_0^R dr = 2\pi \tfrac{R^2}{2} = \pi R^2$$