ETH Lecture 401-0663-00L Numerical Methods for CSE

Numerical Methods for Computational Science and Engineering

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URL: http://www.sam.math.ethz.ch/~hiptmair/tmp/NumCSE/NumCSE15.pdf

IV. Approximation of functions in 1D

Approximation of functions: Generic view

Given: function $\mathbf{f} : D \subset \mathbb{R}^n \mapsto \mathbb{R}^d$ (often in procedural form `y=feval(x)`, Rem. 3.1.4)

Goal: Find a "simple"$^{(*)}$ function $\widetilde{\mathbf{f}} : D \mapsto \mathbb{R}^d$ such that the approximation error $\mathbf{f} - \widetilde{\mathbf{f}}$ is "small"$^{(\clubsuit)}$

Focus: $n = d = 1$

simple : (piecewise) polynomial

accurate : $\|f - \widehat{f}\|$ small, $\|\cdot\| \hat{=}$ norm on $C^0(D)$,

e.g. supremum norm* $\|f\|_{L^\infty(D)} := \sup_{x \in D} |f(x)|$ $\|$
[uniform approximation]

L^2-norm $\|f\|^2_{L^2(D)} := \int_D |f(x)|^2 \, dx$

Approximation scheme : $A : X \longrightarrow V$

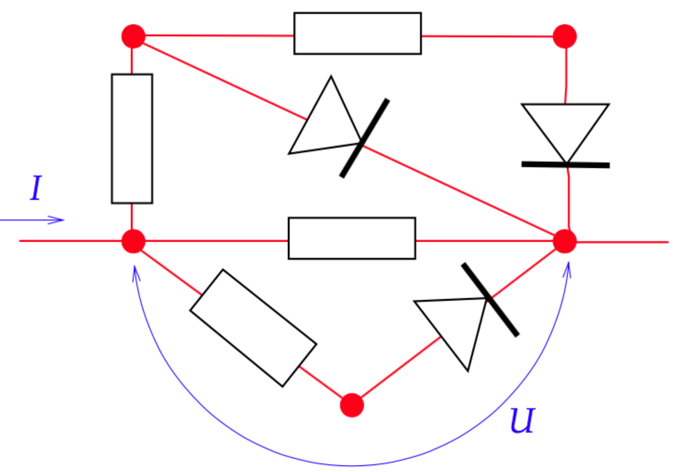
spaces of functions

e.g.: $X = C^0(I)$, $I \subset \mathbb{R}$

$V = \mathcal{P}_n$ or $V = \mathcal{S}_{3,M}$

Application: Model reduction

Two-port circuit $\triangleright$



Goal: Model this as a single circuit element described $\underbrace{I = I(U)}$

↳ computable

[$\hat{=}$ non-linear solve]

M.R.: Replace $I(u)$ with $\tilde{I}(u)$, based on a few evaluations of $I(u)$, that can be evaluated fast

Approximation by interpolation

Interpolation scheme + sampling $\rightarrow$ approximation scheme

$$f : I \subset \mathbb{R} \to \mathbb{K} \xrightarrow{\text{sampling}} (t_i, y_i := f(t_i))_{i=0}^{m} \xrightarrow{\text{interpolation}} \tilde{f} := \mathsf{I}_T \mathbf{y} \quad (\tilde{f}(t_i) = y_i)\,.$$

free choice of nodes $t_i \in I$

interpolation scheme

4.1 Approximation by global polynomial

Example from analysis: Taylor approximation

$$f(t) = \underbrace{\sum_{j=0}^{n} \frac{f^{(j)}(t_0)}{j!}(t-t_0)^j}_{\in \mathcal{P}_n} + \underbrace{f^{(n+1)}(\xi)\frac{(t-t_0)^{n+1}}{(n+1)!}}_{\text{remainder, small for } |t-t_0| \ll 1}$$

4.1.1 Polynomial Approximation: Theory

$\rightarrow$ polynomials can uniformly approximate any $f \in C^k(I)$, $I \subset \mathbb{R}$ closed, $k \geq 0$, along with their first k derivatives

Theorem 4.1.6. Uniform approximation by polynomials

For $f \in C^0([0,1])$, define the n-th Bernstein approximant as

$$p_n(t) = \sum_{j=0}^{n} f(j/n)\binom{n}{j} t^j (1-t)^{n-j}\,, \quad p_n \in \mathcal{P}_n\,. \tag{4.1.7}$$

It satisfies $\|f - p_n\|_\infty \to 0$ for $n \to \infty$. If $f \in C^m([0,1])$, then even $\left\|f^{(k)} - p_n^{(k)}\right\|_\infty \to 0$ for $n \to \infty$ and all $0 \leq k \leq m$.

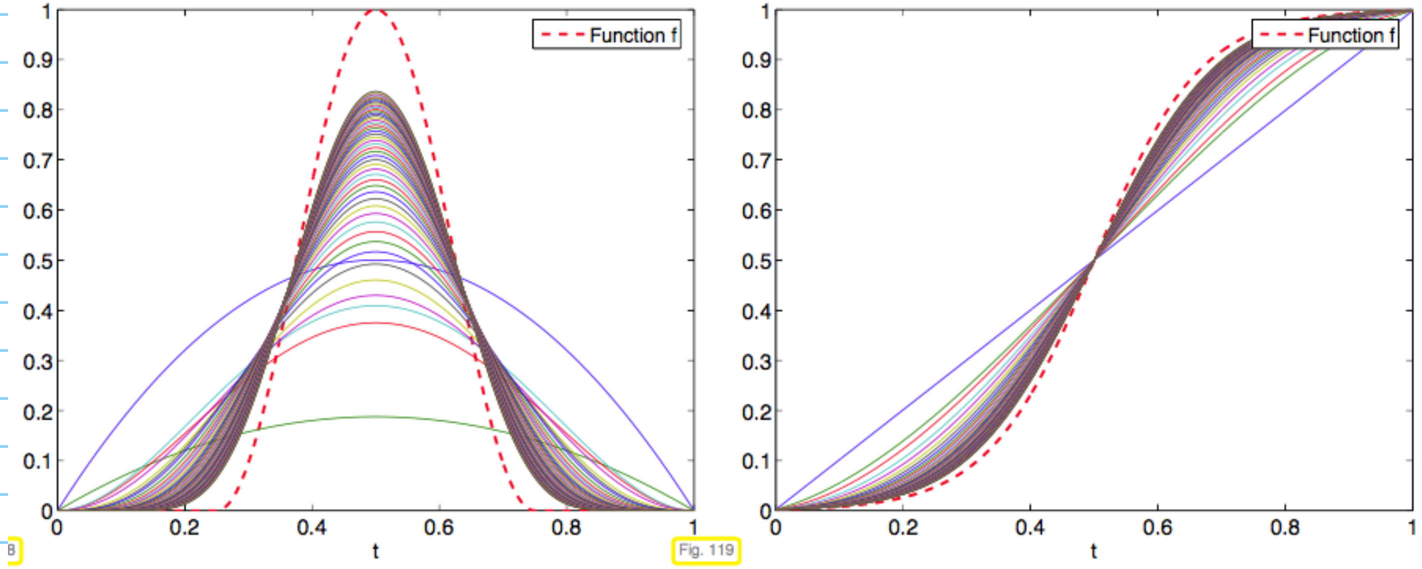


Fig. 119

Best approximation:

Definition 5.1.10. (Size of) best approximaton error

Let $\|\cdot\|$ be a (semi-)norm on a space X of functions $I \mapsto \mathbb{K}$, $I \subset \mathbb{R}$ an interval. The (size of the) best approximation error of $f \in X$ in the space $\mathcal{P}_k$ of polynomials of degree $\leq k$ with respect to $\|\cdot\|$ is

$$\mathrm{dist}_{\|\cdot\|}(f, \mathcal{P}_k) := \inf_{p \in \mathcal{P}_k} \|f - p\| .$$

Theorem 4.1.11. L^∞ polynomial best approximation estimate

If $f \in C^r([-1,1])$ (r times continuously differentiable), $r \in \mathbb{N}$, then

$$\inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty([-1,1])} \leq \underbrace{(1+\pi^2/2)^r \frac{(n-r)!}{n!}}_{\text{quantitative bound}} \left\|f^{(r)}\right\|_{L^\infty([-1,1])} .$$

crucial role of smoothness

Asymptotics for degree $n \to \infty$: $O(n^{-r})$

▷ Algebraic cvg. with rate r in the polynomial degree

In general: No quantitative cvg. bounds without smoothness.

Generalization to $[a,b]$:

Tool: Affine trf. $\phi : \begin{cases} [-1,1] \longrightarrow [a,b] \\ \hat{t} \longrightarrow a + \frac{1}{2}(b-a)(\hat{t}+1) \end{cases}$

Pullback $\phi^* : \begin{cases} C^0([a,b]) \longrightarrow C^0([-1,1]) \\ f \longrightarrow \{\hat{t} \to f(\phi(\hat{t}))\} \end{cases}$

Note:
- $\phi^* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ bijective (ii)
- $\|\phi^* f\|_{L^\infty([-1,1])} = \|f\|_{L^\infty([a,b])}$ (i)
- ⟹ $\|(\phi^* f)^{(r)}\|_{L^\infty([-1,1])} = (\frac{1}{2}(b-a))^r \|f^{(r)}\|_{L^\infty([a,b])}$

(Chain rule $(\phi^* f)' = (f \circ \phi)' = (f' \circ \phi)\phi'$)

$$\begin{aligned}\inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty([a,b])} &\overset{(i)}{=} \inf_{p \in \mathcal{P}_n} \|\phi^*(f-p)\|_{L^\infty([-1,1])} \\ &= \inf_{p \in \mathcal{P}_n} \|\phi^* f - \phi^* p\|_{L^\infty([-1,1])} \\ &\overset{(ii)}{=} \inf_{p \in \mathcal{P}_n} \|\phi^* f - p\|_{L^\infty([-1,1])} \\ &\overset{\text{Thm. 4.1.11}}{\leq} C n^{-r} \|(\phi^* f)^{(r)}\|_{L^\infty([-1,1])} \\ &\leq C n^{-r} \left(\frac{b-a}{2}\right)^r \|f^{(r)}\|_{L^\infty([a,b])}\end{aligned}$$

$\phi^*(\mathcal{P}_n) = \mathcal{P}_n$!

4.1.2. Error estimates for polynomial interpolation

Recall: interpolation scheme $\Longrightarrow$ approximation scheme

given on on $[-1,1]$, based on nodes $\{\hat{t}_0,\dots,\hat{t}_n\}$

Generalization to $[a,b]$: use nodes $t_j = \Phi(\hat{t}_j)$, where $\Phi:[-1,1]\to[a,b]$ affine mapping from above.

[Error estimates on $[a,b]$ by transformation techniques]

Definition 4.1.25. Lagrangian (interpolation polynomial) approximation scheme

Given an interval $I\subset\mathbb{R}$, $n\in\mathbb{N}$, a node set $\mathcal{T}=\{t_0,\dots,t_n\}\subset I$, the Lagrangian (interpolation polynomial) approximation scheme $\mathsf{L}_{\mathcal{T}}: C^0(I)\to\mathcal{P}_n$ is defined by

$$\underbrace{\mathsf{L}_{\mathcal{T}}(f) := \mathsf{I}_{\mathcal{T}}(\mathbf{y})}_{\text{Lagrangian interpolation}} \in\mathcal{P}_n \quad\text{with}\quad \underbrace{\mathbf{y} := (f(t_0),\dots,f(t_n))^T}_{\text{sampling}}\in\mathbb{K}^{n+1}.$$

New: Given families of node sets $(\mathcal{T}_n)_{n\in\mathbb{N}}$ *, $\#\mathcal{T}_n = n+1$

$\Rightarrow$ family of polynomial A.S. $L_{\mathcal{T}_n}: C^0(I)\to\mathcal{P}_n$

* Example: equispace nodes $\mathcal{T}_n := \{t_j^{(n)} := a+(b-a)\frac{j}{n} : j=0,\dots,n\}\subset I$

New aspect: Convergence $\|f - L_{\mathcal{T}_n} f\| \leq T(n)$

asymptotically for $n\to\infty$.

[n ≙ polynomial degree]

Example:

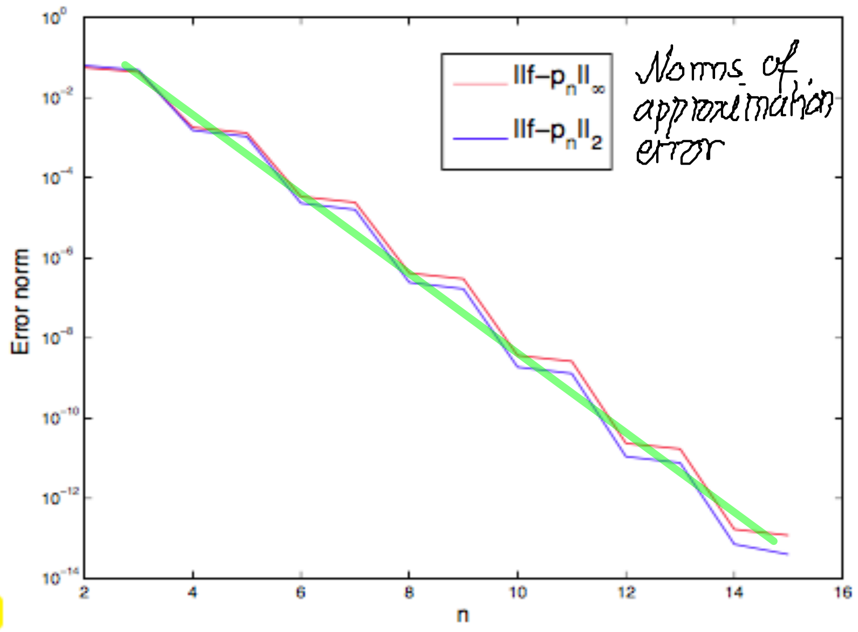
$f(t) = \sin t$

$I = [0,\pi]$

equispaced nodes

$\epsilon_n := \|f - L_{\mathcal{T}_n} f\|$

semi-logarithmic scale ▷



From plot: $\log\epsilon_n \approx C - Kn$, $K>0$

$$\epsilon_n \approx C(\underbrace{e^{-K}}_{<1})^n$$

Empiric asymptotics: $\epsilon_n = O(q^n)$ with $0<q<1$

≙ exponential cvg.

$$\exists\, C \neq C(n) > 0:\quad \|f - \mathsf{L}_{\mathcal{T}_n} f\| \leq C\,T(n) \quad\text{for } n\to\infty\,. \tag{4.1.30}$$

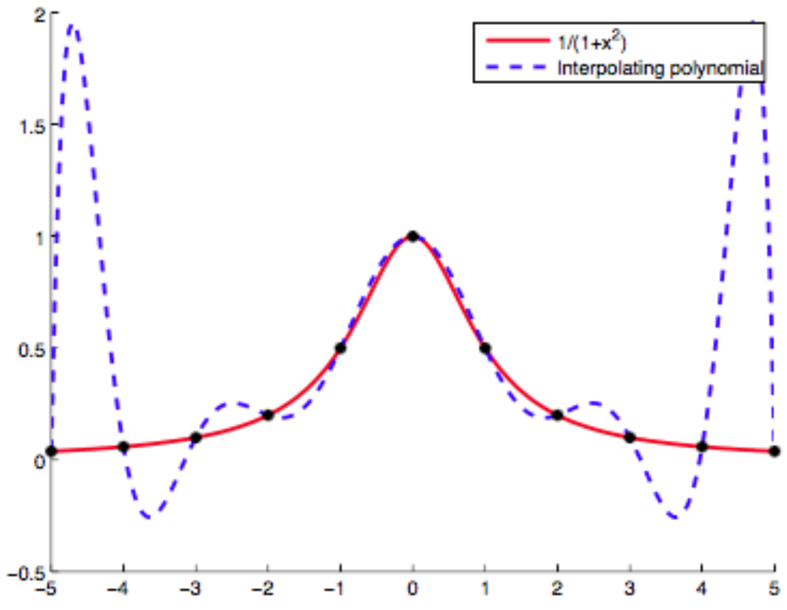
Definition 4.1.31. Types of asymptotic convergence of approximation schemes

Writing $T(n)$ for the bound of the norm of the interpolation error according to (4.1.30) we distinguish the following *types of asymptotic behavior*: → see Thm. 4.1.11.

$\exists\, p>0$: $T(n)\leq n^{-p}$: algebraic convergence, with rate $p>0$,
$\exists\, 0<q<1$: $T(n)\leq q^n$: exponential convergence, $\forall n\in\mathbb{N}$.

The bounds are assumed to be sharp in the sense, that no bounds with larger rate p (for algebraic convergence) or smaller q (for exponential convergence) can be found.

⑤ Will Lagrangian interpolation always converge?



$f(t) = \frac{1}{1+t^2}$ on $[-5,5]$:

$\|f - L_{\mathcal{T}_n} f\|_\infty \to \infty$

for equidistant nodes

Warning + promise

Theory ahead !

The main theorem:

Theorem 4.1.37. Representation of interpolation error [7, Thm. 8.22], [19, Thm. 37.4]

We consider $f \in C^{n+1}(I)$ and the Lagrangian interpolation approximation scheme (→ Def. 4.1.25) for a node set $\mathcal{T} := \{t_0, \dots, t_n\}$. Then, for every $t \in I$ there exists a $\tau_t \in]\min\{t, t_0, \dots, t_n\}, \max\{t, t_0, \dots, t_n\}[$ such that

$$\underbrace{f(t) - \mathsf{L}_{\mathcal{T}}(f)(t)}_{\text{approximation error}} = \frac{f^{(n+1)}(\tau_t)}{(n+1)!} \cdot \prod_{j=0}^{n} (t - t_j) . \tag{4.1.38}$$

Proof: $w_{\mathcal{T}}(t) := \prod_{j=0}^{n} (t - t_j)$ ["nodal polynomial"] $\in \mathcal{P}_{n+1}$

Fix $t \in I \setminus \mathcal{T}$

$c \in \mathbb{R}$: $f(t) - (L_{\mathcal{T}} f)(t) = c \cdot \underbrace{w_{\mathcal{T}}(t)}_{\neq 0}$

Auxiliary function: $\varphi(x) := f(x) - (L_{\mathcal{T}} f)(x) - c\, w_{\mathcal{T}}(x)$

$\Rightarrow$ $\varphi(t_j) = 0$:
$\varphi(t) = 0$ by def. of c $\Rightarrow$ φ has $n+2$ distinct zeros

$\Rightarrow$ [Rolle's theorem] φ' has $n+1$ distinct zeros

$\vdots$

$\varphi^{(n+1)}$ has at least one zero

$\exists \tau_t$: $\varphi^{(n+1)}(\tau) = f^{(n+1)}(\tau) - c\,(n+1)! = 0$

$c = \frac{1}{(n+1)!} f^{(n+1)}(\tau)$ □

Theorem 4.1.37. Representation of interpolation error [7, Thm. 8.22], [19, Thm. 37.4]

We consider $f \in C^{n+1}(I)$ *and the Lagrangian interpolation approximation scheme* (→ Def. 4.1.25) *for a node set* $\mathcal{T} := \{t_0,\dots,t_n\}$. *Then, for every* $t \in I$ *there exists a* $\tau_t \in \,]\min\{t,t_0,\dots,t_n\},\max\{t,t_0,\dots,t_n\}[$ *such that*

$$f(t) - \mathsf{L}_{\mathcal{T}}(f)(t) = \frac{f^{(n+1)}(\tau_t)}{(n+1)!}\cdot\prod_{j=0}^{n}(t-t_j)\,. \tag{4.1.38}$$

Estimates from Thm 4.1.37 by taking maximum over t

$$\Rightarrow \quad \|f - L_{\mathcal{T}} f\|_\infty \le \max_{\tau\in I}|f^{(n+1)}(\tau)|\,\frac{1}{(n+1)!}\,\|w_\tau\|_\infty$$

$$\boxed{\|f - \mathsf{L}_{\mathcal{T}} f\|_{L^\infty(I)} \le \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!}\max_{t\in I}|(t-t_0)\cdot\dots\cdot(t-t_n)|} \quad (*)$$

smoothness of f is essential!

• Back to $f(t) = \sin t$, $f \in C^\infty$ $\Rightarrow$ $\|f^{(n)}\|_\infty \le 1 \quad \forall n$

For equidistant nodes in $[0,\pi]$

$$(*) \Rightarrow \|\sin - L_{\mathcal{T}_n}\sin\|_\infty \le \frac{1}{(n+1)!}\max_{0\le t\le\pi}\underbrace{\prod_{j=0}^{n}\left(t-\frac{\pi}{n}j\right)}_{\text{attains max for } t=\frac{\pi}{2n}}$$

$$\le \frac{1}{(n+1)}\left(\frac{\pi}{n}\right)^{n+1}$$

→ more than exponential cvg.

• $f(t) = \frac{1}{1+t^2}$ $\Rightarrow$ $\|f^{(n)}\|_{\infty,[-5,5]} \ge 2^n n!$

Bound from $(*)$ can blow up as $n\to\infty$

Interpolation error estimates for analytic functions

Definition 5.1.54. Analyticity of a complex valued function

let $D \subset \mathbb{C}$ be an *open* set in the complex plane. A function $f : D \to \mathbb{C}$ is called analytic/holomorphic in D, if $f \in C^\infty(D)$ and it possesses a convergent Taylor series at every point $z \in D$.

Extension to $\mathbb{C}$: Reinterpret $f : I \subset \mathbb{R} \to \mathbb{R}$ as a restriction $f : D \subset \mathbb{C} \to \mathbb{C}$ to $I \subset D$

["just use the same formula"]

Example: $f(t) = \frac{1}{1+t^2} \Rightarrow f(z) = \frac{1}{1+z^2}$

→ a rational function, analytic in $\mathbb{C}\setminus\{\pm i\}$

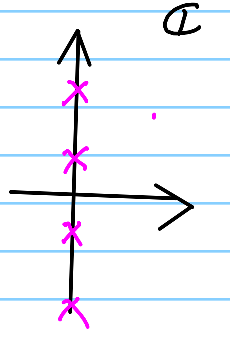
Entire functions: e^z, $\sin z$, $\cos z$: analytic in $\mathbb{C}$

Example: $f(z) = \frac{1}{1+e^z} = \frac{1}{1+e^x\cdot e^{iy}}$, $z = x+iy$, $x,y\in\mathbb{R}$

$e^z = e^x\cdot e^{iy} \overset{!}{=} -1$: $|e^x| = 1 \Rightarrow x = 0$

$e^{iy} = -1 \Leftrightarrow y = (2k+1)\pi$, $k\in\mathbb{Z}$

$\Rightarrow$ analytic in $\mathbb{C}\setminus\{(2k+1)\pi i,\, k\in\mathbb{Z}\}$



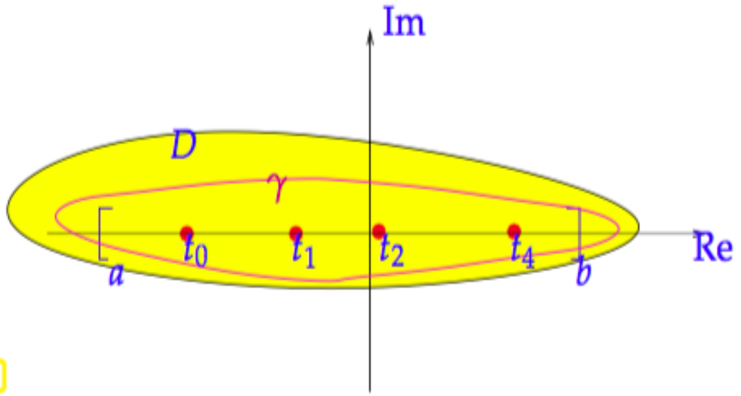
Example : $f(z) = u^T (A - zI)^{-1} b$, $z \in \mathbb{C}$, $A \in \mathbb{R}^{n,n}$, $u, b \in \mathbb{R}^n$

Analytic in $\mathbb{C} \setminus \{z : (A - zI)$ not invertible$\}$

$= \mathbb{C} \setminus \{z$ is an eigenvalue of $A\}$

Example : $f(z) = \sqrt{z+1}$ is analytic $\mathbb{C} \setminus]-\infty, -1[$

Note : $\sqrt{\cdot} : \mathbb{C} \setminus]-\infty, 0[\to \mathbb{C}$ analytic



Assumption 4.1.57. Analyticity of interpoland

We assume that the interpoland $f : I \to \mathbb{C}$ *can be extended* to a function $f : D \subset \mathbb{C} \to \mathbb{C}$, which is analytic ($\to$ Def. 4.1.54) on an open set $D \subset \mathbb{C}$ with $[a, b] \subset D$.

Theorem 4.1.55. Residue theorem [30, Ch. 13]

Let $D \subset \mathbb{C}$ be an open set, $\gamma \subset D$ a simple closed smooth curve (in the complex plane), contractible in D, and $\Pi \subset D$ a finite set. *Then for each function f that is analytic in $D \setminus \Pi$ holds*

* enclosed by γ

$$\frac{1}{2\pi\imath} \int_\gamma f(z)\,dz = \sum_{p \in \Pi} \operatorname{res}_p f\,,$$

contour integral

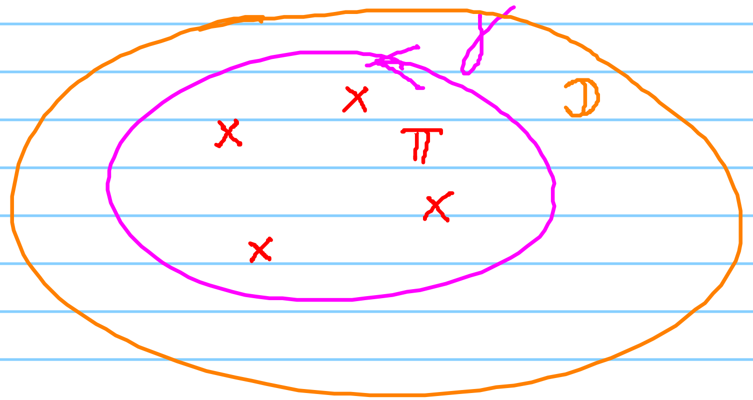
where $\operatorname{res}_p f$ is the residual of f in $p \in \mathbb{C}$.

Contour integral $\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(\tau)) \cdot \frac{d\gamma}{d\tau}\,d\tau$

$\gamma : [\alpha, \beta] \to \mathbb{C}$

$\to \left| \int_\gamma f(z)\,dz \right| \le |\gamma| \max_{z \in \gamma} |f(z)|$

length



Lemma 4.1.56. Rsidual formula for quotients

let g and h be complex valued functions that are both analytic in a neighborhood of $p \in \mathbb{C}$, and satisfy $h(p) = 0$, $h'(p) \neq 0$. Then

$$\operatorname{res}_p \frac{g}{h} = \frac{g(p)}{h'(p)}\,.$$

[residual in a simple pole]

$$L_j(t) = \prod_{k=0, k\neq j}^{n} \frac{t - t_k}{t_j - t_k} = \frac{w(t)}{(t - t_j) \prod_{k=0, k\neq j}^{n} (t_j - t_k)} = \frac{w(t)}{(t - t_j) w'(t_j)}$$

Lagrange polynomial for node set $\{t_0, \dots, t_n\}$

$$L_T f = \sum_{j=0}^{n} f(t_j) L_j$$

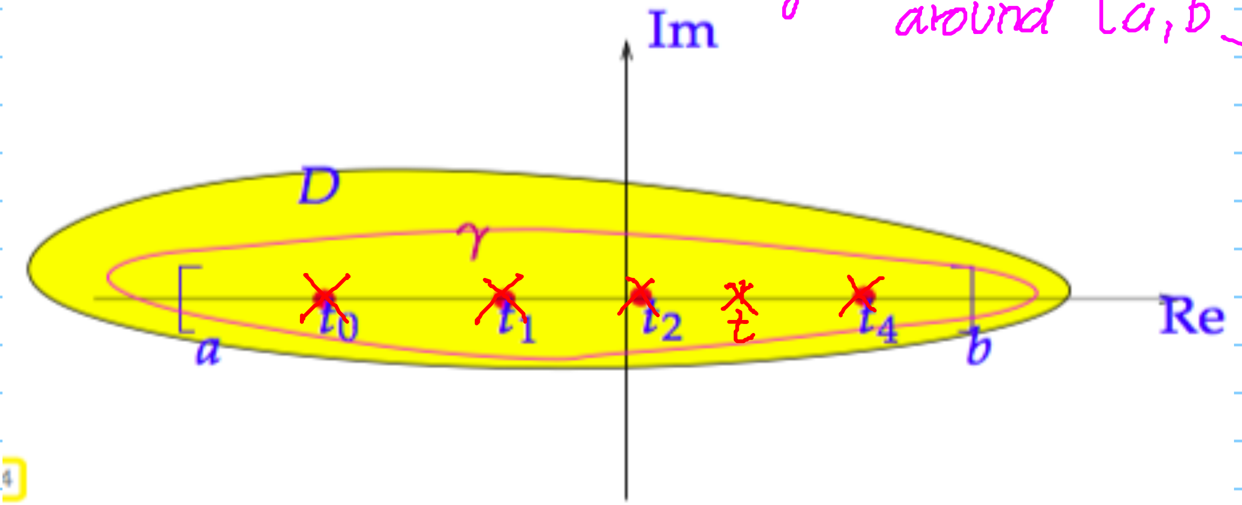
(8) $w(z) = (z-t_0)\cdot\ldots\cdot(z-t_n) \in \mathcal{P}_{n+1}$

Fix $t \in [a,b]\setminus\mathcal{J}$:

Auxiliary function :

$$g_t(z) := \frac{f(z)}{(z-t)\,w(z)}\,,\quad z\in D\setminus\overbrace{\{t,t_0,\ldots,t_n\}}^{\text{simple poles there !}}$$

$\gamma \mathrel{\hat=}$ simple closed contour around $[a,b]$



$$\frac{1}{2\pi i}\int_\gamma g_t(z)\,dz = \mathrm{res}_t\, g_t + \sum_{j=0}^{n}\mathrm{res}_{t_j} g_t$$

$$= \frac{f(t)}{w(t)} + \sum_{j=0}^{n}\frac{f(t_j)}{(t_j-t)\,w'(t_j)} \qquad \Big|\cdot w(t)$$

$$L_j(t) = \prod_{k=0,k\neq j}^{n}\frac{t-t_k}{t_j-t_k} = \frac{w(t)}{(t-t_j)\prod_{k=0,k\neq j}^{n}(t_j-t_k)} = \frac{w(t)}{(t-t_j)\,w'(t_j)}$$

$$w(t)\,\frac{1}{2\pi i}\int_\gamma g_t(z)\,dz = \underbrace{f(t) - \sum_{j=0}^{n} f(t_j)\,L_j(t)}_{\text{interpolation error}}$$

$$\boxed{f(t) = \underbrace{-\sum_{j=1}^{n} f(t_j)\underbrace{\frac{w(t)}{(t_j-t)\,w'(t_j)}}_{-\text{Lagrange polynomial !}}}_{\text{polynomial interpolant !}} + \underbrace{\frac{w(t)}{2\pi\imath}\int_\gamma g_t(z)\,dz}_{\text{interpolation error !}}\,.}$$

A representation formula !

$$|\cdot| \le \|w\|_\infty\,\frac{|\gamma|}{2\pi}\max_{a\le t\le b}\|g_t\|_{\infty,\gamma}$$

Towards estimates :

Use $g_t :=$

$$\boxed{|f(t) - \mathsf{L}_{\mathcal{T}}f(t)| \le \left|\frac{w(t)}{2\pi\imath}\int_\gamma \frac{f(z)}{(z-t)\,w(z)}\,dz\right| \le \frac{|\gamma|}{2\pi}\,\frac{\max_{a\le\tau\le b}|w(\tau)|}{\min_{z\in\gamma}|w(z)|}\cdot\frac{\max_{z\in\gamma}|f(z)|}{\mathrm{dist}([a,b],\gamma)}}$$

$$\mathrm{dist}([a,b],\gamma) = \inf_{z\in\gamma,\,a\le t\le b}|z-t|$$

The only term depending on n, and $\{t_0,\ldots,t_n\}$

only this depends on f

"derivative free bound"

9 4.1.3. Chebychev Interpolation

Goal: Optimal placement* of nodes t_j for Lagrange interpolation

Theory:
$$\|f - \mathsf{L}_{\mathcal{T}} f\|_{L^\infty(I)} \leq \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{L^\infty(I)} \|w\|_{L^\infty(I)} ,$$
with $w(t) := (t - t_0) \cdot \dots \cdot (t - t_n)$.

nodes enter only here

* make $\|w\|_{L^\infty(I)}$ as small as possible → f-independent optimal nodes

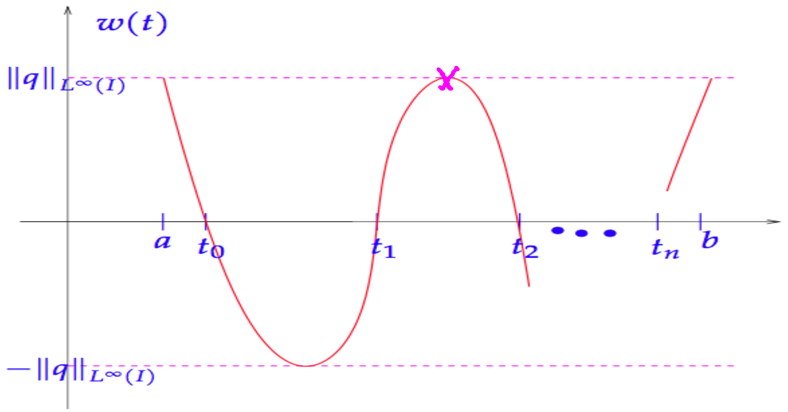
Optimal choice of interpolation nodes independent of interpoland

Idea: choose nodes $t_0, \dots, t_n$ such that $\|w\|_{L^\infty(I)}$ is minimal!

This is equivalent to finding a polynomial $q \in \mathcal{P}_{n+1}$
- with leading coefficient $= 1$,
- such that it minimizes the norm $\|q\|_{L^\infty(I)}$.

▶ Then choose nodes $t_0, \dots, t_n$ as zeros of q (caution: t_j must belong to I).

Heuristics:
- All zeros of w in $[a,b]$ [→ meaningful intp. nodes]
- Same extremal value (modulus) in all extrema



The solution

Definition 4.1.66. Chebychev polynomials → [19, Ch. 32]

The n^{th} Chebychev polynomial is $T_n(t) := \cos(n \arccos t)$, $-1 \leq t \leq 1, n \in \mathbb{N}$.

$\Rightarrow \quad |T_n(t)| \leq 1$

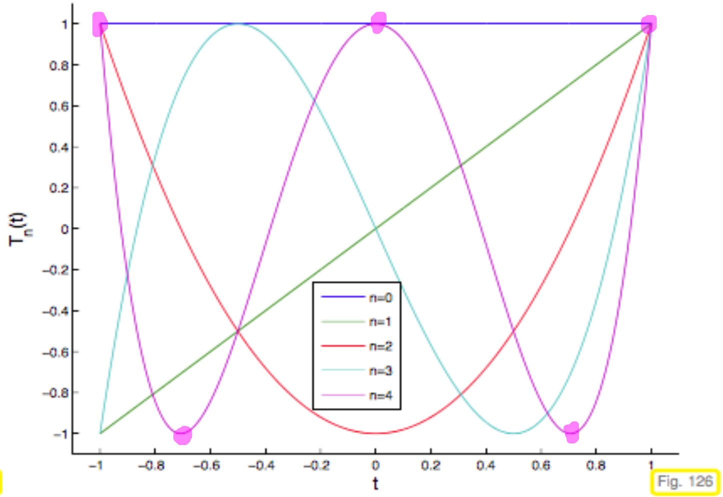
Theorem 4.1.67. 3-term recursion for Chebychev polynomials → [19, (32.2)]

The function T_n defined in Def. 4.1.66 satisfy the 3-term recursion

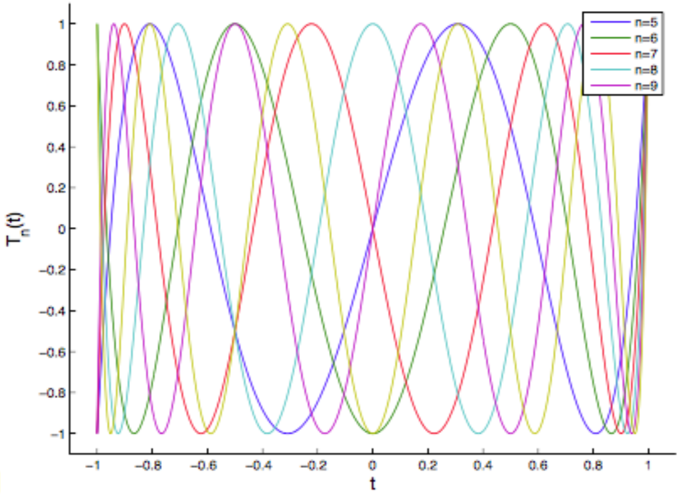
$$T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t) \quad , \quad T_0 \equiv 1 \, , \quad T_1(t) = t \, , \quad n \in \mathbb{N} \, . \tag{4.1.68}$$

$\Rightarrow \quad T_n \in \mathcal{P}_n$ w/ leading coefficient 2^{n-1}

Proof: $\cos((n+1)t) = 2\cos nt \cdot \cos t - \cos(n-1)t$ □



Chebychev polynomials $T_0, \dots, T_4$



Chebychev polynomials $T_5, \dots, T_9$

Zeros: $n \arccos t \in (2j+1)\frac{\pi}{2} \Rightarrow t_j = \cos\left(\frac{2j+1}{n}\,\frac{\pi}{2}\right)$, $j = 0, \dots, n-1$

$t_j \hat{=}$ Chebychev points $\hat{=}$ optimal nodes

⑩

Theorem 4.1.72. Minimax property of the Chebychev polynomials [9, Section 7.1.4.], [19, Thm. 32.2]

The polynomials T_n from Def. 4.1.66 minimize the supremum norm in the following sense:

$$\|T_n\|_{L^\infty([-1,1])} = \inf\{\|p\|_{L^\infty([-1,1])} : p \in \mathcal{P}_n, p(t) = 2^{n-1}t^n + \cdots\}, \quad \forall n \in \mathbb{N}.$$

Proof: (indirect proof), Assume $\exists\, q(t) = 2^{n-1}t^n + \ldots \in \mathcal{P}_n$

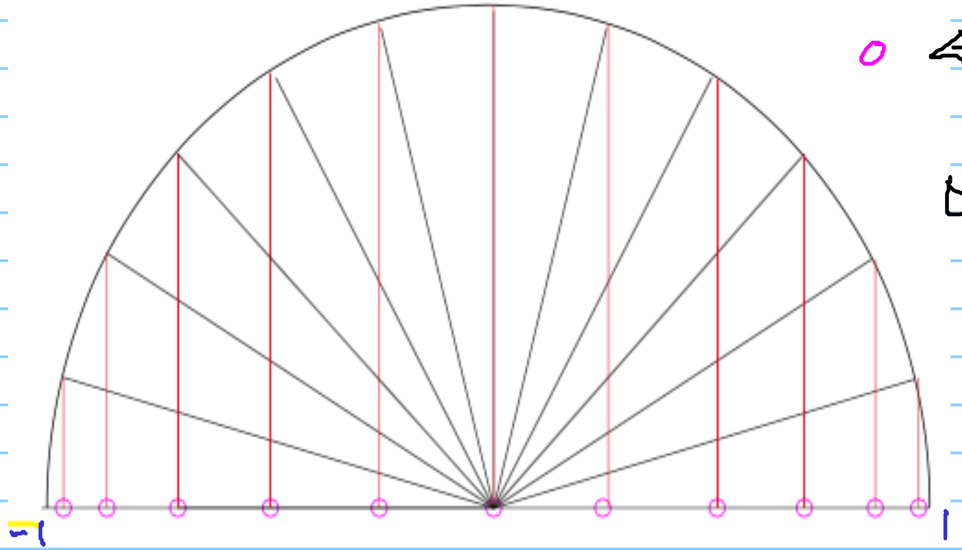
$\|q\|_\infty < \|T_n\|_\infty \Rightarrow (T_n - q)(\bar{t}) > 0$ in maxima $\bar{t}$ of T_n

$(T_n - q)(\bar{t}) < 0$ in minima $\bar{t}$ of T_n

n+1 sign changes

▷ $T_n - q \in \mathcal{P}_{n-1}$ has n zeros $\Rightarrow T_n - q = 0$ ↯ $\square$

▷ Optimal choice $w = 2^{-n} T_{n+1}$



o ← Chebychev nodes on $[-1,1]$

▷ Chebychev interpolation

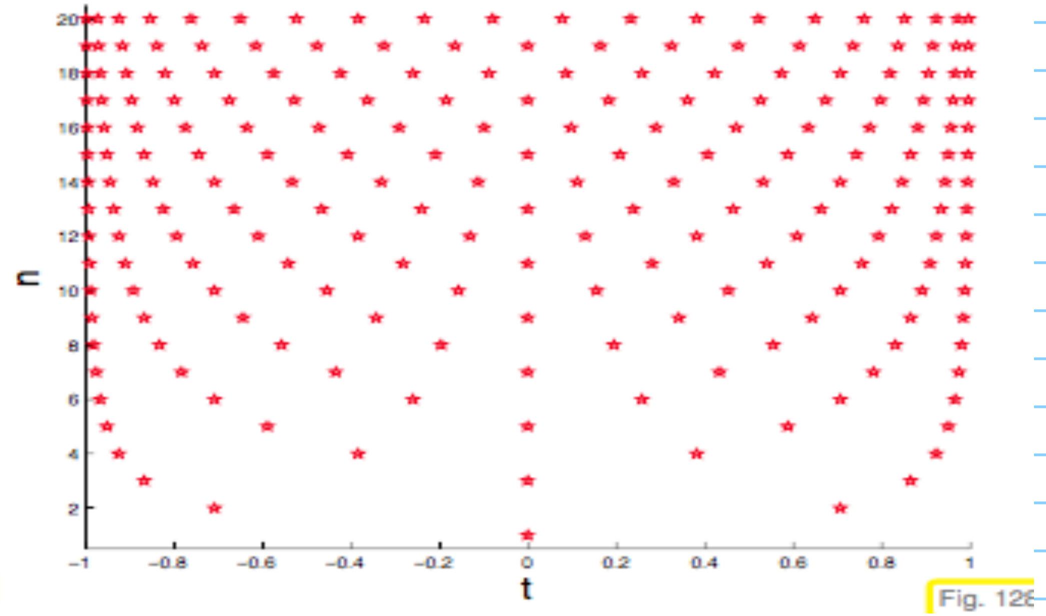


Fig. 128

C. N. cluster at endpoints of $[-1,1]$

The Chebychev nodes in the interval $I = [a,b]$ are

$$t_k := a + \tfrac{1}{2}(b-a)\left(\cos\left(\frac{2k+1}{2(n+1)}\pi\right) + 1\right), \quad k = 0,\ldots,n. \tag{4.1.77}$$

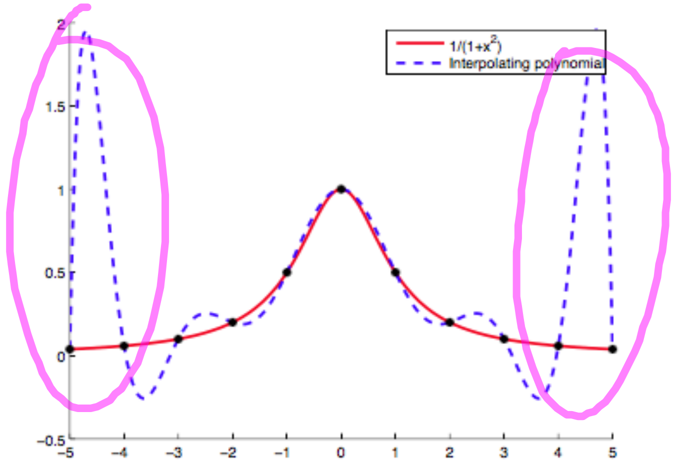
▷ $\|w\|_{L^\infty([a,b])} = 2^{-n}$

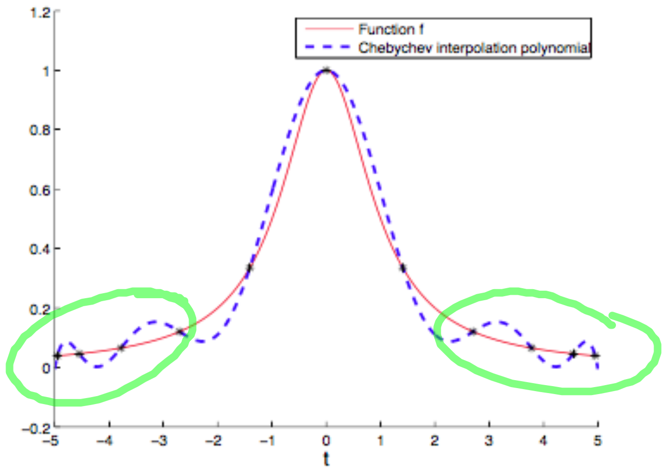
▷ For Chebychev interpolation on $[a,b]$

$f \in C^{n+1}([a,b])$:

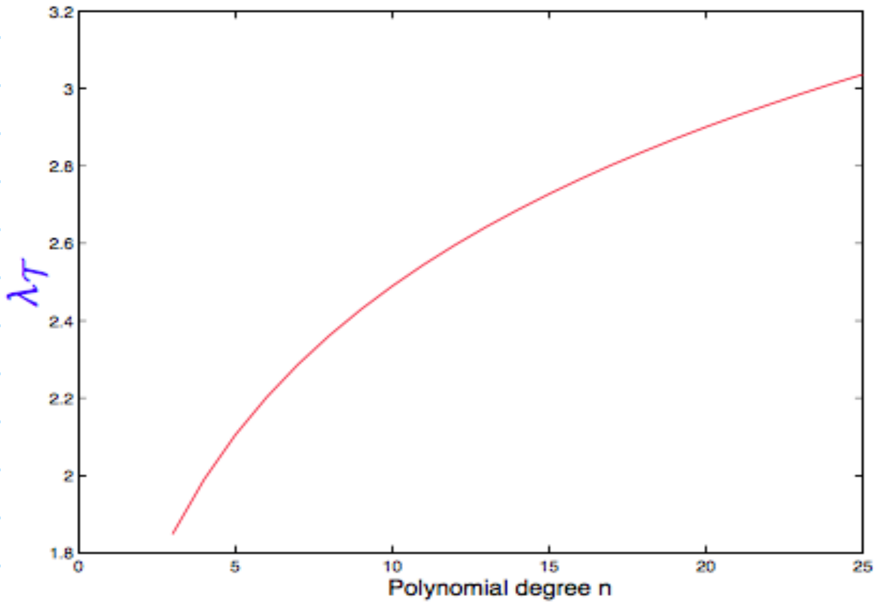
$$\|f - \mathsf{I}_\mathcal{T}(f)\|_{L^\infty([a,b])} \le \frac{2^{-n}}{(n+1)!}\left\|f^{(n+1)}\right\|_{L^\infty([a,b])} \left(\frac{b-a}{2}\right)^{n+1}$$

$f(t) = \frac{1}{1+t^2}$, $I = [-5,5]$





Sensitivity of Chebychev interpolation : Lebesgue constant λ_T



exponential blowup for equidistant nodes

Theoretical estimate

$$\lambda_{cheb} \le \frac{2}{\pi}\log(1+n)+1$$

λ_T connects interpolation & best approximation

$$\|f - L_T f\|_\infty = \|f - I_T[f(t_j)]_j\|_\infty =$$

[Note $L_T p = p \quad \forall p \in \mathcal{P}_n$]

$p \in \mathcal{P}_n$ arbitrary!

$$= \|(f-p) - I_T[(f-p)(t_j)]_j\|_\infty$$

$$\le \|f-p\|_\infty + \|I_T[(f-p)(t_j)]_j\|_\infty$$

$$\le \|f-p\|_\infty + \lambda_T \|[(f-p)(t_j)]_j\|_\infty$$

$$\le (1+\lambda_T)\|f-p\|_\infty$$

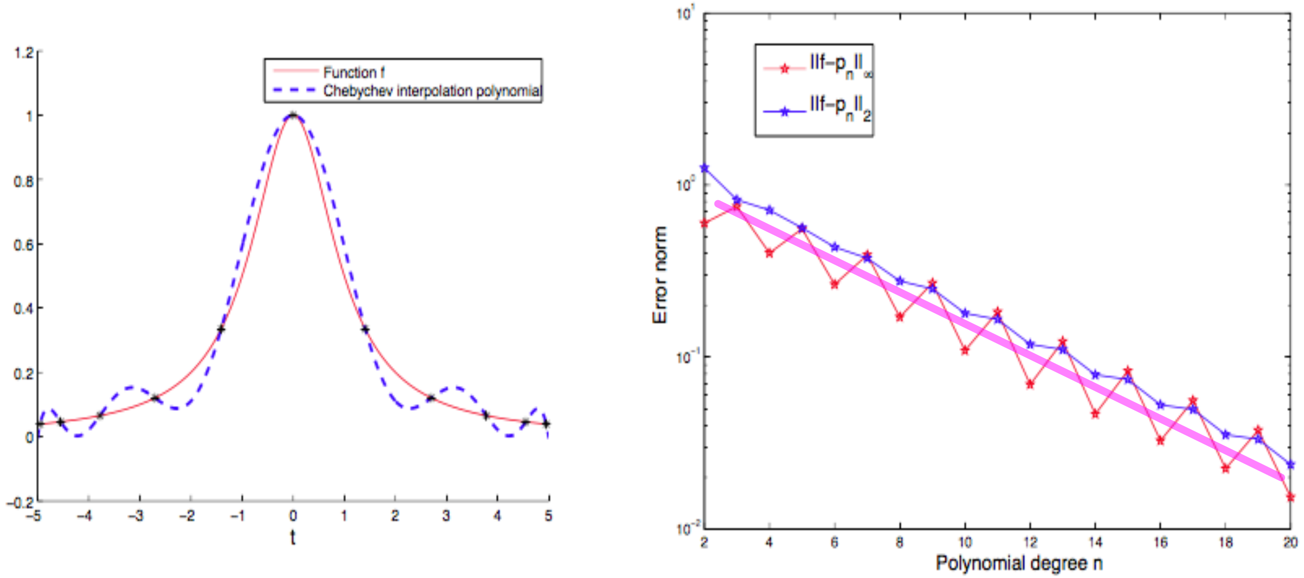
$$\Rightarrow \underbrace{\|f - L_T f\|}_{\text{interpolation}} \le (1+\lambda_T) \underbrace{\inf_{p\in\mathcal{P}_n} \|f-p\|_\infty}_{\|\cdot\|_\infty \text{ of best approx. error}}$$

$$\|f - \mathsf{L}_{\mathcal{T}} f\|_{L^\infty(I)} \le (2/\pi\log(1+n)+2)(1+\pi^2/2)^r \frac{(n-r)!}{n!}\left\|f^{(r)}\right\|_{L^\infty([-1,1])}.$$

$\rightarrow O(n^{-r})$ algebraic cvg. $\Updownarrow$ compare!

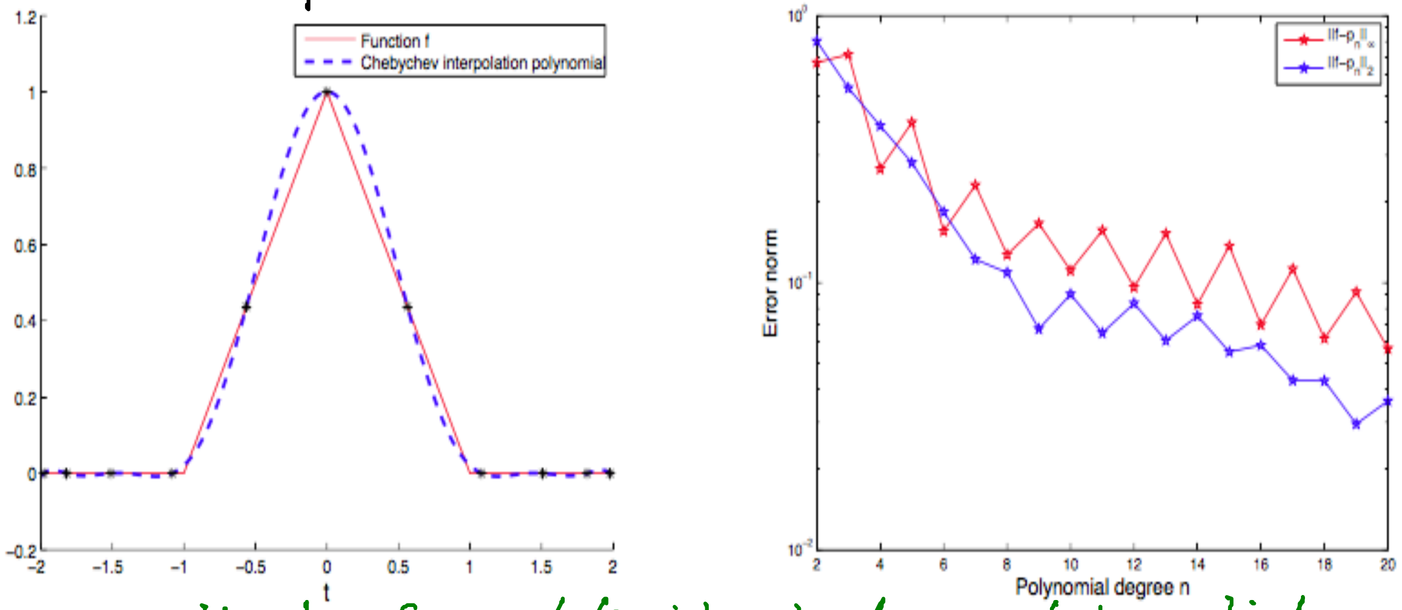
$$\|f - \mathsf{I}_{\mathcal{T}}(f)\|_{L^\infty([-1,1])} \le \frac{2^{-n}}{(n+1)!}\left\|f^{(n+1)}\right\|_{L^\infty([-1,1])}$$

① $f(t)=(1+t^2)^{-1}$, $I=[-5,5]$ (see Ex. 4.1.34): analytic in a neighborhood of I.
Interpolation with $n=10$ Chebychev nodes (plot on the left).



Exponential cvg $O(q^n)$

$f \hat{=}$ tent function $\notin C^1$



→ Estimates for analytic interpolands cannot be applied

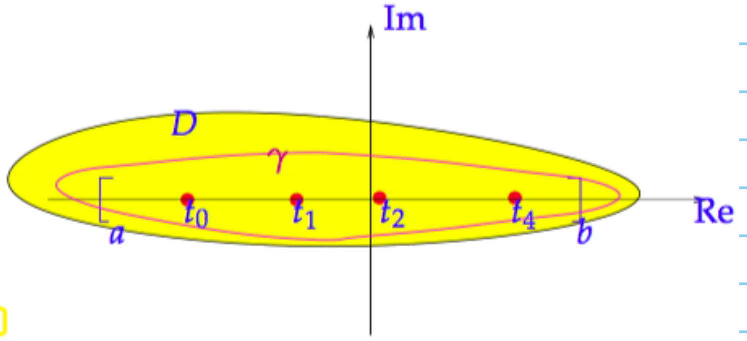
Chebychev interpolation: analytic interpolands

Focus: n-asymptotics $\qquad (I=[-1,1])$

$$\boxed{|f(t)-\mathsf{L}_{\mathcal{T}}f(t)| \le \left|\frac{w(t)}{2\pi\imath}\int_\gamma \frac{f(z)}{(z-t)w(z)}\,\mathrm{d}z\right| \le \frac{|\gamma|}{2\pi}\,\frac{\max_{a\le\tau\le b}|w(\tau)|}{\min_{z\in\gamma}|w(z)|}\cdot\frac{\max_{z\in\gamma}|f(z)|}{\operatorname{dist}([a,b],\gamma)}}$$

$$w = 2^{-n}T_{n+1} \Rightarrow \|w\|_{L^\infty[-1,1]} \le 2^{-n}$$

Choice of integration contour $\gamma\subset\mathbb{C}$ around $[-1,1]$



Choice of γ should cancel arccos

Definition 4.1.66. Chebychev polynomials → [19, Ch. 32]

The n^{th} Chebychev polynomial is $T_n(t):=\cos(n\arccos t)$, $\quad -1\le t\le 1, n\in\mathbb{N}$.

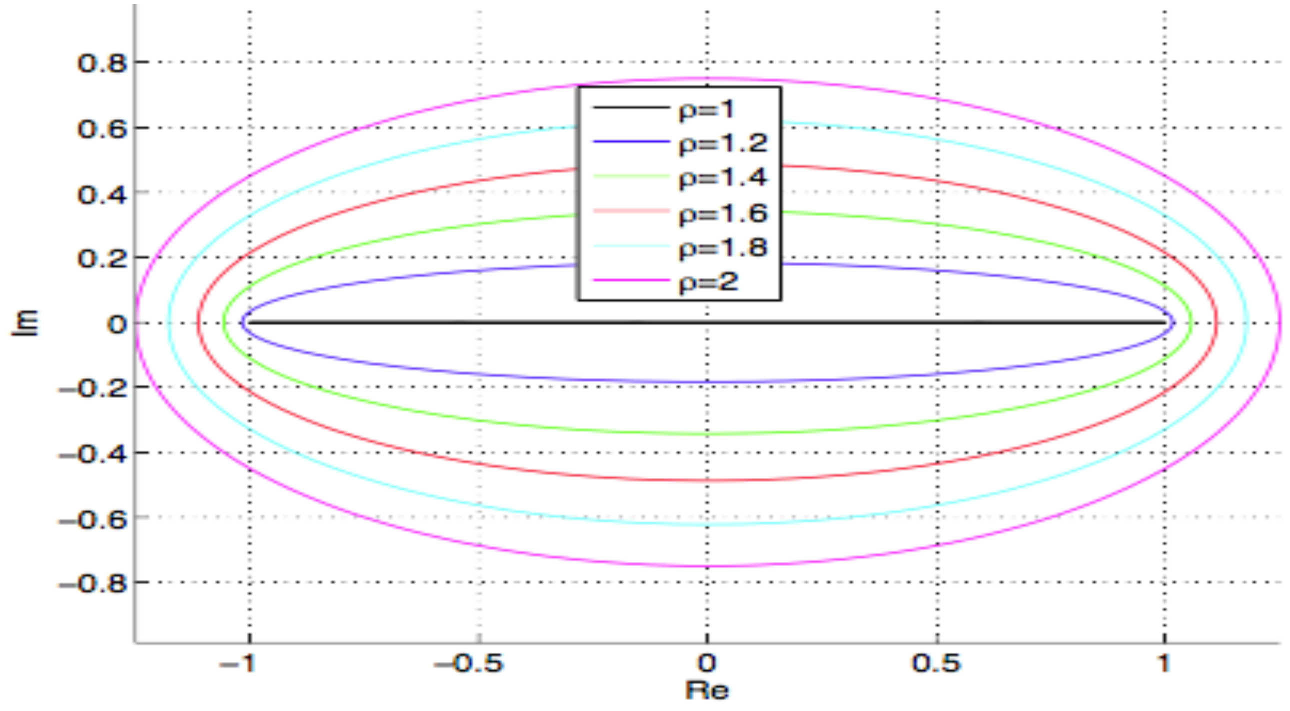
(13)

$$\gamma := \{z = \cos(\theta - \imath \log\rho),\ 0 \le \theta \le 2\pi\}$$
$$= \left\{z = \frac{1}{2}(\exp(\imath(\theta - \imath\log\rho)) + \exp(-\imath(\theta - \imath\log\rho)))\right\}$$
$$= \left\{z = \frac{1}{2}\left(\rho e^{\imath\theta} + \rho^{-1}e^{-iu\theta}\right)\right\}$$
$$= \Big\{z = \underbrace{\tfrac{1}{2}(\rho+\rho^{-1})\cos\theta}_{Re} + \imath\underbrace{\tfrac{1}{2}(\rho-\rho^{-1})\sin\theta}_{Im}\Big\}$$

$\rho > 1$

$\theta \hat{=}$ curve parameter

$\gamma(\theta) = A\cos\theta + B\sin\theta$, $\theta \in [0, 2\pi]$: ellipsis



$\Rightarrow$ Larger ρ $\Rightarrow$ larger ellipses

Lower bound of $|w| \Leftrightarrow |T_{n+1}|$ on γ

$s := n+1$

$$|T_s(\cos(\theta - \imath\log\rho))|^2 = |\cos(s(\theta - \imath\log\rho))|^2$$
$$= \cos(s(\theta - \imath\log\rho)) \cdot \overline{\cos(s(\theta - \imath\log\rho))}$$
$$= \frac{1}{4}(\rho^s e^{\imath s\theta} + \rho^{-s}e^{-\imath s\theta})(\rho^s e^{-\imath s\theta} + \rho^{-s}e^{\imath s\theta})$$
$$= \frac{1}{4}(\rho^{2s} + \rho^{-2s} + e^{2\imath s\theta} + e^{-2\imath s\theta})$$
$$= \frac{1}{4}(\rho^s - \underbrace{\rho^{-s}}_{<1})^2 + \underbrace{\frac{1}{4}(e^{\imath s\theta} + e^{-\imath s\theta})^2}_{=\cos^2(s\theta)\ge 0} \ge \frac{1}{4}(\rho^s - 1)^2,$$

$\downarrow$ Plug into

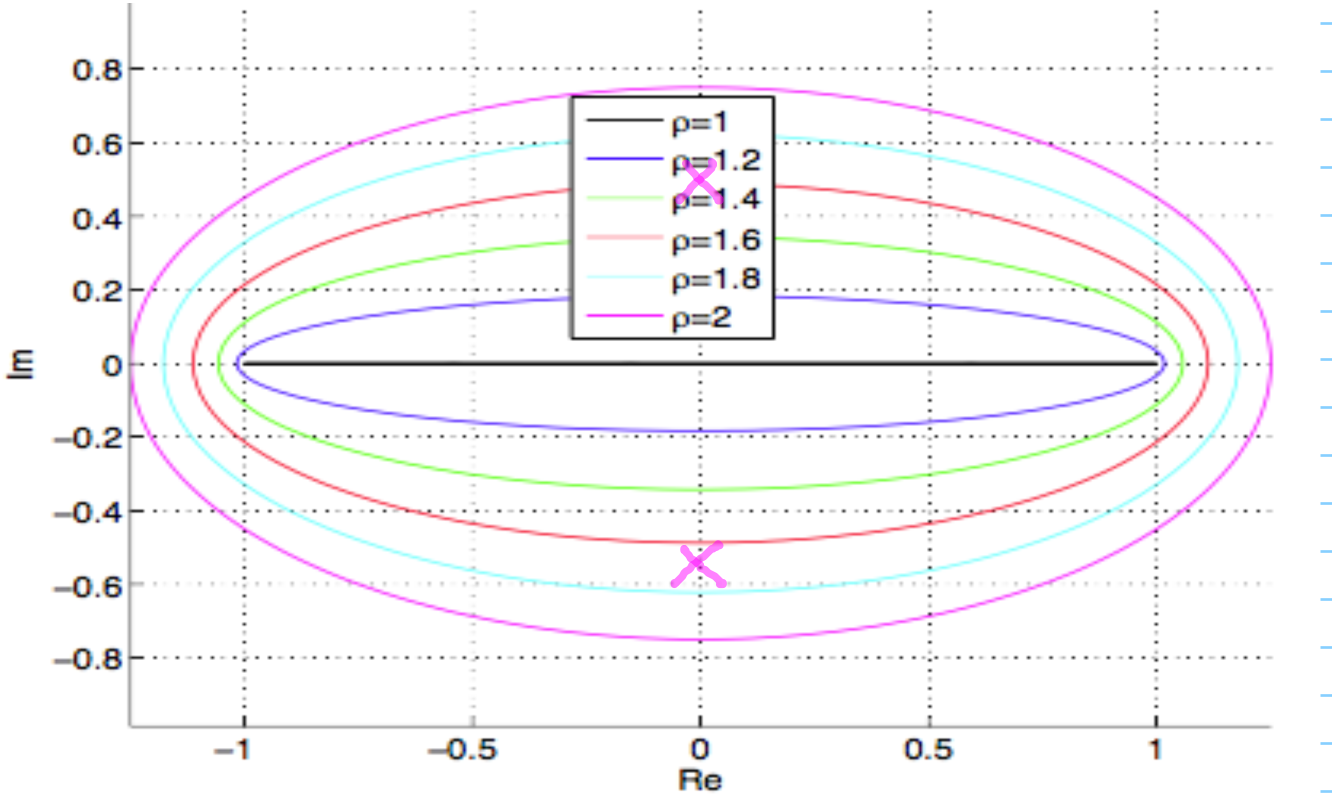
$$\|f - L_{\mathcal{T}}f\|_{L^\infty([-1,1])} \le \frac{4|\gamma|}{\pi}\underbrace{\frac{1}{(\rho^{n+1}-1)(\rho+\rho^{-1}-2)}}_{O(\rho^{-n})} \cdot \max_{z\in\gamma}|f(z)|$$

$\rho > 1$:

The larger ρ, the faster cvg.

Admissible ρ : $\gamma_\rho \subset D$ [domain of analyticity]

Ex: $f(t) = \frac{1}{1+(2t)^2} \Rightarrow D = \mathbb{C} \setminus \{\pm i/2\}$



Interior of γ_ρ must be contained in $D \Rightarrow \rho \lesssim 1.65$

4.2. Mean square best approximation

Goal: compute best approximants

4.2.1. Abstract theory

$X \hat{=} \mathbb{R}$-vector space $\longleftrightarrow$ function space

$V \subset X$ finite-dim. subspace with basis $\{b_1, \dots, b_N\}$ $\leftrightarrow$ space of polynomials

$(\cdot,\cdot)_X \hat{=}$ inner product on $X \Rightarrow$ mean square norm $\|v\|_X = (v,v)_X^{1/2}$

Sought: $q := \inf_{p \in V} \|f-p\|_X^2$ for $f \in X$

$$\|f-p\|_X^2 = \|f\|^2 - 2(f,p)_X + \|p\|_X^2$$

Basis rep.: $p = \sum_{j=1}^{N} \gamma_j b_j$, $\gamma_j \in \mathbb{R}$

$$\|f-p\|_X^2 = \|f\|^2 - 2\sum_j \gamma_j (f,b_j)_X + \sum_j \sum_k \gamma_j \gamma_k (b_j, b_k)_X$$

$$= \|f\|^2 - 2\underline{d}^T \underline{c} + \underline{c}^T M \underline{c} =: \phi(\underline{c})$$

$\underline{d} := [(f,b_j)_X]_{j=1}^N \in \mathbb{R}^N$ $\qquad \phi : \mathbb{R}^N \to \mathbb{R}$

$M := [(b_j,b_k)_X]_{j,k=1}^N \in \mathbb{R}^{N,N}$ $\qquad \phi \in C^\infty$

$\underline{c} := [\gamma_j]_{j=1}^N$

$\Rightarrow$ Find $\underline{c} \in \mathbb{R}^N$: $\Phi(\underline{c}) \longrightarrow \min$

$$\operatorname{grad} \Phi(\underline{c}) = -2\underline{d} + 2\mathbf{M}\underline{c} \overset{!}{=} 0 \qquad (*)$$

$$\operatorname{Hess} \Phi(\underline{c}) = \mathbf{M}$$

$$x := \sum \xi_i b_i \in X : \underline{x}^T \mathbf{M} \underline{x} = \Big\| \sum \xi_i b_i \Big\|_X^2 > 0 \Leftrightarrow x \neq 0$$

$\Rightarrow$ $\mathbf{M}$ s.p.d. !

$(*)$ LSE : $\underline{c} = \mathbf{M}^{-1}\underline{d}$: A formula!

$\hookrightarrow$ normal equations

Theorem 4.2.7. Mean square best approximation through normal equations

Given any $f \in X$ there is a unique $q \in V$ such that

$$\|f - q\|_X = \inf_{p \in V} \|f - p\|_X .$$

Its coefficients γ_j, $j = 1, \ldots, N$, with respect to the basis $\mathfrak{B}_V := \{b_1, \ldots, b_N\}$ of V ($q = \sum_{j=1}^N \gamma_j b_j$) are the unique solution of the normal equations

$$\mathbf{M}[\gamma_j]_{j=1}^N = \left[(f, b_j)_X\right]_{j=1}^N , \quad \mathbf{M} := \begin{bmatrix} (b_1,b_1)_X & \ldots & (b_1,b_N)_X \\ \vdots & & \vdots \\ (b_N,b_1)_X & \ldots & (b_N,b_N)_X \end{bmatrix} \in \mathbb{K}^{N,N} . \tag{4.2.8}$$

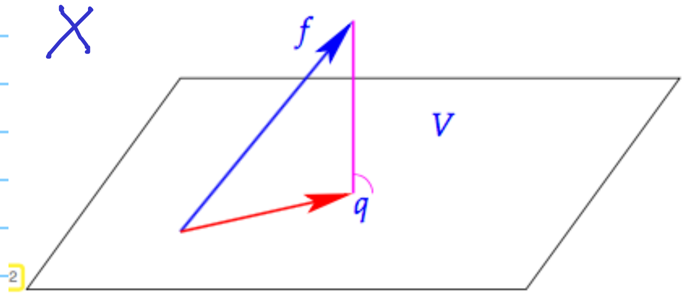
"Geometric" view:

Corollary 4.2.12. Best approximant by orthogonal projection

If q is the best approximant of f in V, then $f - q$ is orthogonal to every $p \in V$:

$$(f - q, p)_X = 0 \quad \forall p \in V \quad \Leftrightarrow \quad f - q \perp q .$$

$\uparrow$ best approximation error



$$\xi \to \|f - q + \xi p\|_X^2 = \|f-q\|_X^2 - 2\xi(f-q,p) + \xi^2\|p\|_X^2$$

a parabola with apex in $\xi = 0 \Rightarrow (f-q,p) = 0$ $\forall p \in V$

Aiming for simple N.E. (diagonal $\mathbf{M}$)

Definition 4.2.13. Orthonormal basis

A subset $\{b_1, \ldots, b_N\}$ of an N-dimensional vector space V with inner product ($\rightarrow$ Def. 4.2.1) $(\cdot,\cdot)_V$ is an orthonormal basis (ONB), if $(b_k, b_j)_V = \delta_{kj}$.

A basis of $\{b_1, \ldots, b_N\}$ of V is called orthogonal, if $(b_k, b_j)_V = 0$ for $k \neq j$.

$\{b_1, .., b_N\}$ ONB $\Rightarrow$ $\boxed{q = \sum_{j=1}^N (f, b_j)_X b_j}$ $(*)$

We already know how to compute ONBs !

Gram-Schmidt orthonormalization

$b_1 := \frac{p_1}{\|p_1\|_V}$ % 1st output vector
for $j = 2,\dots,k$ **do**
{ % Orthogonal projection
$b_j := p_j$
for $\ell = 1,2,\dots,j-1$ **do** (4.2.17)
{ $b_j \leftarrow b_j - (p_j, b_\ell)_V b_\ell$ }
if ($b_j = 0$) **then** STOP
else { $b_j \leftarrow \frac{b_j}{\|b_j\|_V}$ }
}

→ orthogonal projections

Theorem 4.2.18. Gram-Schmidt orthonormalization

When supplied with $k \in \mathbb{N}$ linearly independent vectors $p_1,\dots,p_k \in V$ in a vector space with inner product $(\cdot,\cdot)_V$, Algorithm (4.2.17) computes vectors $b_1,\dots,b_k$ with

$$(b_\ell, b_j)_V = \delta_{\ell j}, \quad \ell, j \in \{1,\dots,k\},$$
$$\mathrm{Span}\{b_1,\dots,b_\ell\} = \mathrm{Span}\{p_1,\dots,p_\ell\} \quad \|$$

for all $\ell \in \{1,\dots,k\}$.

Alg. for computing q : (i) G.S. $\rightarrow \{b_1,\dots,b_N\}$
(ii) (*)

4.2.2. Polynomial mean square best approximation

[$X = C^0([a,b])$, $V = \mathcal{P}_m$]

Inner product: $(f,g)_X = \int_a^b f(t)g(t)\,dt$: L^2-I.P.

Definition 4.2.24. Orthonormal polynomials → Def. 4.2.13

Let $(\cdot,\cdot)_X$ be an inner product on $\mathcal{P}_m$. A sequence $r_0, r_1, \dots, r_m$ provides orthonormal polynomials with respect to $(\cdot,\cdot)_X$, if

$$r_\ell \in \mathcal{P}_\ell, \quad (r_k, r_\ell)_X = \delta_{k\ell}, \quad \ell, k \in \{0,\dots,m\}. \tag{4.2.25}$$

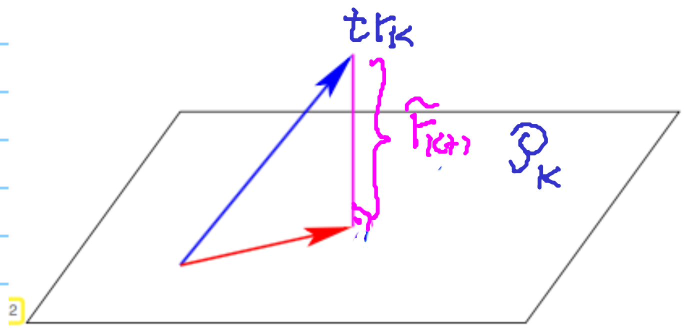
The polynomials are just orthogonal, if $(r_k, r_\ell)_X = 0$ for $k \neq \ell$.

→ increasing degree

$\{r_0,\dots,r_m\}$ ONPs : $t\,r_k(t) \in \mathcal{P}_{k+1} \setminus \mathcal{P}_k$

linearly indep ← leading coeff. $\neq 0$

$$\begin{cases} r_{k+1} \in \mathrm{Span}\{t r_k, r_k, r_{k-1}, \dots, r_0\} \\ r_{k+1} \perp \mathrm{Span}\{r_0, \dots, r_k\} \end{cases}$$



orthogonal projection

$$r_{k+1} = \frac{\tilde{r}_{k+1}}{\|\tilde{r}_{k+1}\|} \; ; \quad \tilde{r}_{k+1} \overset{(*)}{=} t r_k - \sum_{j=1}^{k} (t r_k, r_j)_X r_j \quad (+)$$

Assumption: $(tf, g)_X = (f, tg)_X \quad \forall f,g$

$$(t r_k, r_j)_X = (r_k, \underbrace{t r_j}_{\in \mathcal{P}_{j+1}})_X = 0 \quad \forall j \leq k-2 \quad \text{by orthogonality!}$$

→ Sum (+) collapses to a 3-term recursion.
↳ not normalized

Theorem 4.2.31. 3-term recursion for orthogonal polynomials

Given an inner product $(\cdot,\cdot)_X$ on $\mathcal{P}_m$, $m \in \mathbb{N}$, define $p_{-1} := 0$, $p_0 = 1$, and

$$p_{k+1}(t) := (t - \alpha_{k+1}) p_k(t) - \beta_k p_{k-1}(t) \ , \quad k = 0, 1, \ldots, m-1 \ ,$$

$$\text{with} \quad \alpha_{k+1} := \frac{(\{t \mapsto t p_k(t)\}, p_k)_X}{\|p_k\|_X^2} \ , \quad \beta_k := \frac{\|p_k\|_X^2}{\|p_{k-1}\|_X^2} \ . \tag{4.2.32}$$

Then $p_k \in \mathcal{P}_k$ with leading coefficient $= 1$, and $\{p_0, p_1, \ldots, p_m\}$ is an orthogonal basis of $\mathcal{P}_m$.

```
function [alpha, beta] = coeffortho(t,n)
% Vector t passes the points in the definition of the discrete L^2-inner
% product, n the maximal index desired
m = numel(t);   % Maximal degree of orthogonal polynomial
alpha(1) = sum(t)/m;
% Initialization of recursion; we store only the values of
% the polynomials at the points in T.
p1 = ones(size(t));
p2 = t-alpha(1);
% Main loop
for k=1:min(n-1,m-2)
  p0 = p1;  p1 = p2;
  % 3-term recusion (4.2.32),
  alpha(k+1) = dot(p1,(t.*p1))/norm(p1)^2;
  beta(k) = (norm(p1)/norm(p0))^2;
  p2 = (t-alpha(k+1)).*p1-beta(k)*p0;
end
```

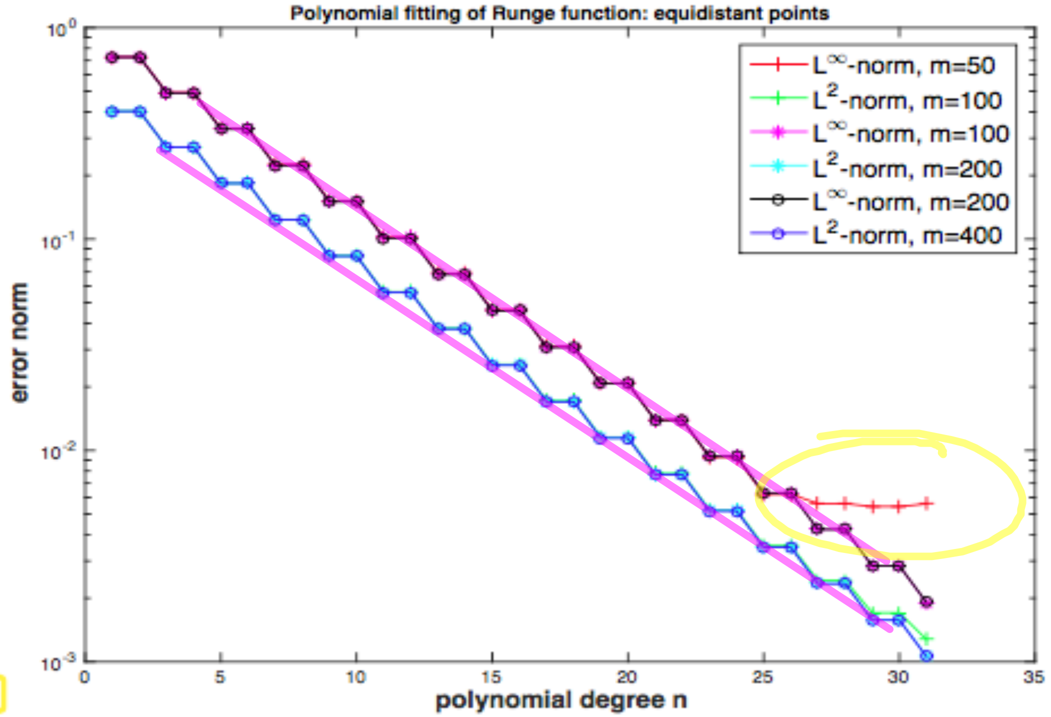
Practical (semi) inner products: discrete L^2-inner products

Given $\mathcal{T} = \{t_j\}_{j=0}^{m}$: $(f,g)_{\mathcal{T}} := \sum_{j=0}^{m} f(t_j) g(t_j)$

→ At least definite on $\mathcal{P}_k$, $k \leq m$.

Experiment: equidistant t_j in $[-1,1]$

$$f(t) = \frac{1}{1+(5t)^2}$$



→ exponential convergence

$$\|f-p\|_{\mathcal{T}}^2 = \sum_{j=0}^{m} ((f-p)(t_j))^2$$

4.3. Uniform Best-Approximation

$f \in C^0([a,b])$, Find $q = \underset{p \in \mathcal{P}_n}{\operatorname{argmin}} \|f-p\|_{L^\infty([a,b])}$

Theorem 4.3.2. Chebychev alternation theorem

Given $f \in C[a,b]$, $a < b$, *and a polynomial degree* $n \in \mathbb{N}$, *a polynomial* $q \in \mathcal{P}_n$ *satisfies*

$$q = \underset{p \in \mathcal{P}_n}{\operatorname{argmin}} \|f-p\|_{L^\infty(I)}$$

if and only if *there exist* $n+2$ *points* $a \le \xi_0 < \xi_1 < \cdots < \xi_{n+1} \le b$ *such that*

$$|e(\xi_j)| = \|e\|_{L^\infty([a,b])}, \quad j = 0,\dots,n+1,$$
$$e(\xi_j) = -e(\xi_{j+1}), \quad j = 0,\dots,n,$$

where $e := f - q$ *denotes the approximation error.*

$\xi_j \mathrel{\hat=}$ Chebychev alternants

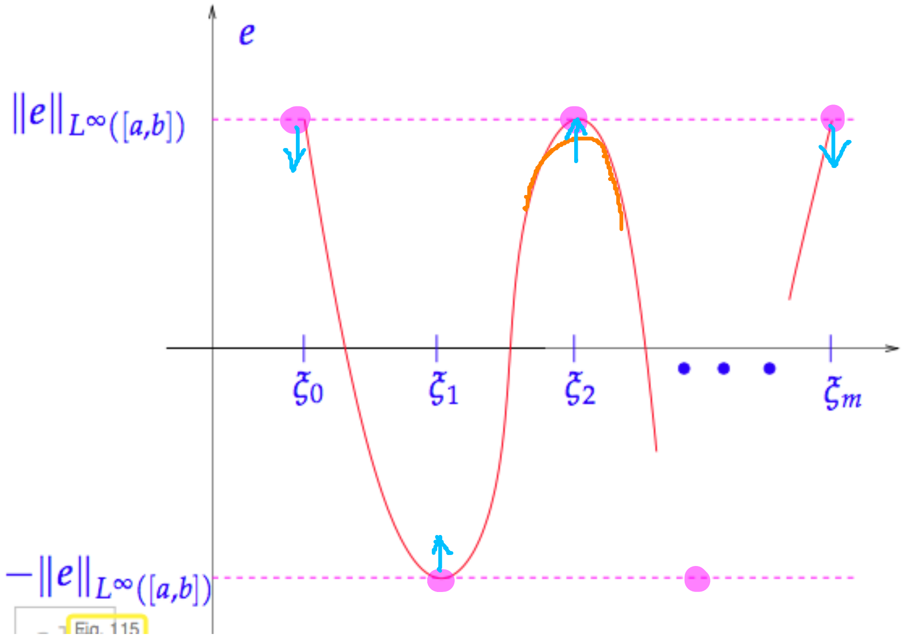


Fig. 115

Remez algorithm:

$$\text{CAT} \Rightarrow (+)\quad \underbrace{q(\xi_k) + (-1)^k \overset{=\|e\|_\infty}{\delta} = f(\xi_k), \; k = 0,\dots,n+1}$$

After choosing a basis for $\mathcal{P}_n$ $\to$ $(n+2)\times(n+2)$ LSE,
unknowns = expansion coeffs of q & δ

- Start from approximate alternants $\xi_j^{(0)}$, $j = 0,\dots,n+1$
- Solve $(+)$ $\to$ q
- Check (sampling) $\|f-q\|_\infty$ $\to$ STOP, if small enough
- New set of alternants $\mathrel{\hat=}$ extrema of $f-q$

4.5. Approximation by piecewise* polynomials

Recall : cubic splines , cubic Hermite , p.w. linear

Advantage : Cost $O(N)$, $N \hat{=}$ no. of data points

* w.r.t. to a mesh of $[a,b]$: $\mathcal{M} = \{a = x_0 < x_1 < \dots < x_m = b\}$

node

mesh cell / interval : $[x_{j-1}, x_j]$

4.5.1. Piecewise Lagrange Interpolation

↳ Separate Lagrange intp. on cells

General local Lagrange interpolation on a mesh

1. Choose local degree $n_j \in \mathbb{N}_0$ for each cell of the mesh, $j = 1, \dots, m$.
2. Choose set of *local* interpolation points (nodes)

$$\mathcal{T}^j := \{t_0^j, \dots, t_{n_j}^j\} \subset I_j := [x_{j-1}, x_j] \,, \quad j = 1, \dots, m \,,$$

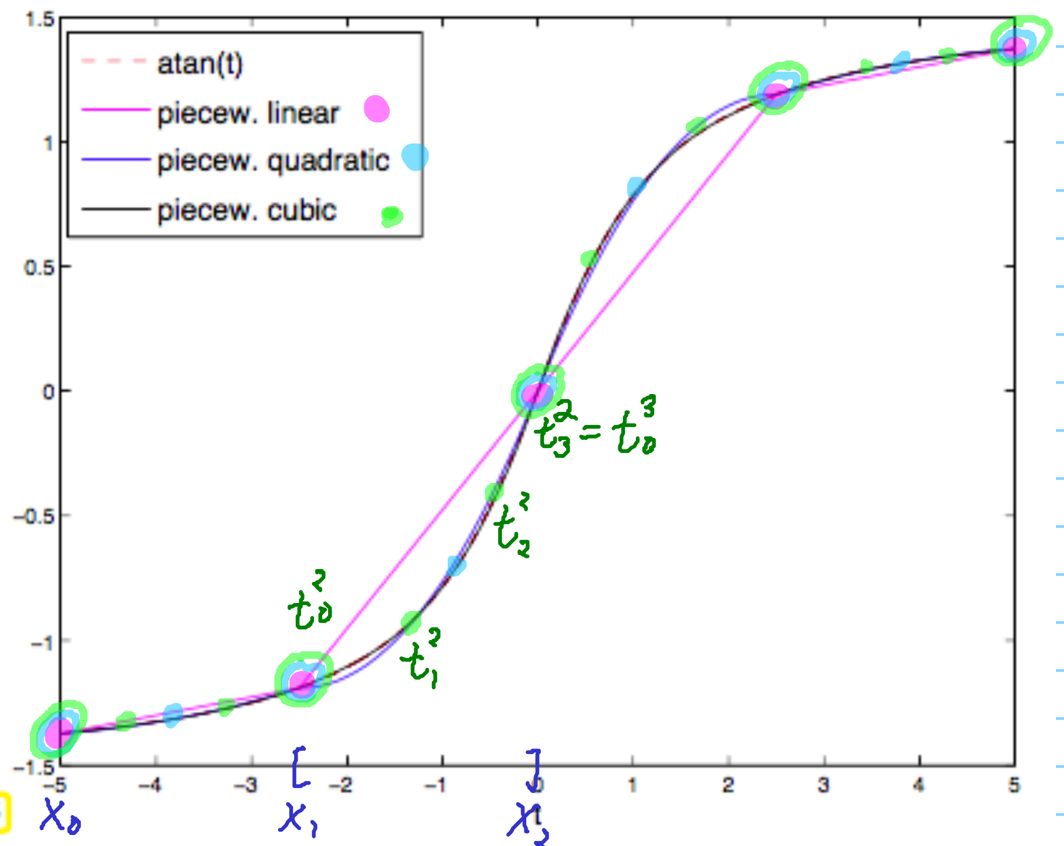
for each mesh cell/grid interval I_j.

3. Define piecewise polynomial interpolant $s : [x_0, x_m] \to \mathbb{K}$:

$$s_j := s_{|I_j} \in \mathcal{P}_{n_j} \quad \text{and} \quad s_j(t_i^j) = f(t_i^j) \quad i = 0, \dots, n_j \,, \quad j = 1, \dots, m \,. \tag{4.5.5}$$

Owing to Thm. 3.2.14, s_j is well defined.

↳ $I_{\mathcal{M}} \hat{=}$ interpolation operator



$s \in C^0([a,b])$, if $t_{n_j}^j = t_0^{j+1}$ $\forall j$

[Assum $f \in C^0([a,b])$]

In general $s \notin C^1$

Focus : Asymptotic cvg in maximum norm :

h-convergence : $n_j = n$ $\forall j$ [fixed local degree]

Consider sequence of meshes $\mathcal{M}_k$ with

meshwidth $h_k := \max_j |x_j^{(k)} - x_{j-1}^{(k)}| \xrightarrow{k \to \infty} 0$

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Find bound $\|f - I_{\mathcal{M}_k} f\|_{L^\infty([a,b])} \leq C\,T(h_k)$

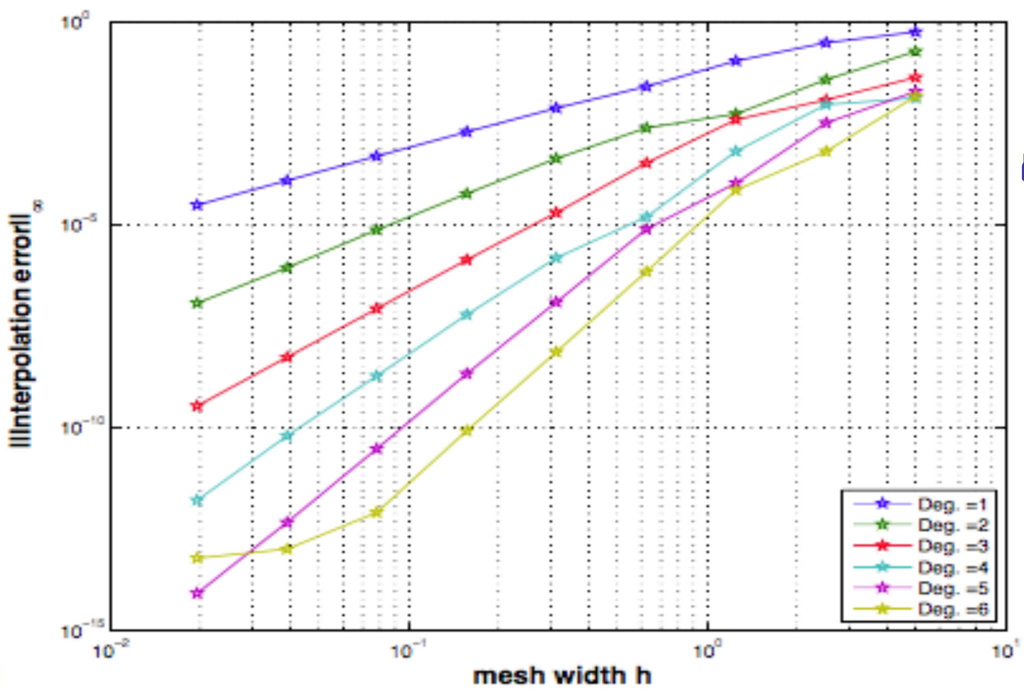
study asymptotics of T for $h_k \to 0$

Example: • $f(t) = \arctan(t)$ on $[-5,5]$

• $\mathcal{M}_k = \{-5 + j\,h_k\,10;\ j = 0,\dots,2^k\}$, $h_k = 2^{-k}$

["equidistant mesh"]

• equidistant local interpolation points (endpoints included)



◁ Alg. cvg.

$\|\text{error}\|_\infty \approx O(h^\alpha)$

$\alpha \mathrel{\hat=}$ rate

Empiric

$\alpha \approx n + 1$

$n \mathrel{\hat=}$ local pol. degree

Theory:

Known estimate

$I = [x_{j-1}, x_j]$

$$\|f - \mathsf{L}_\mathcal{T} f\|_{L^\infty(I)} \leq \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \max_{t\in I} \underbrace{|(t - t_0^j)\cdot\dots\cdot(t - t_n^j)|}_{\leq\ |x_j - x_{j-1}|^{n+1} \leq h_\mathcal{M}^{n+1}} \qquad (4.1.43)$$

$\max_j$

[$h_\mathcal{M} \mathrel{\hat=}$ meshwidth]

[Uniform local degree n]

$$(4.1.43) \quad \Rightarrow \quad \|f - s\|_{L^\infty([x_0,x_m])} \leq \frac{h^{n+1}}{(n+1)!} \left\| f^{(n+1)} \right\|_{L^\infty([x_0,x_m])}$$

$$h := h_\mathcal{M} := \max\{|x_j - x_{j-1}| : j = 1,\dots,m\}.$$

smoothness requirement

Remark: p-convergence: $\mathcal{M}$ fixed, raise local degree

4.5.2. Cubic Hermite interpolation

Definition 4.5.14. Piecewise cubic Hermite interpolant (with exact slopes) → Def. 3.4.1

[Piecewise cubic Hermite interpolant (with exact slopes) Given $f \in C^1([a,b])$ and a mesh $\mathcal{M} := \{a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b\}$ the piecewise cubic Hermite interpolant (with exact slopes) $s : [a,b] \to \mathbb{R}$ is defined as

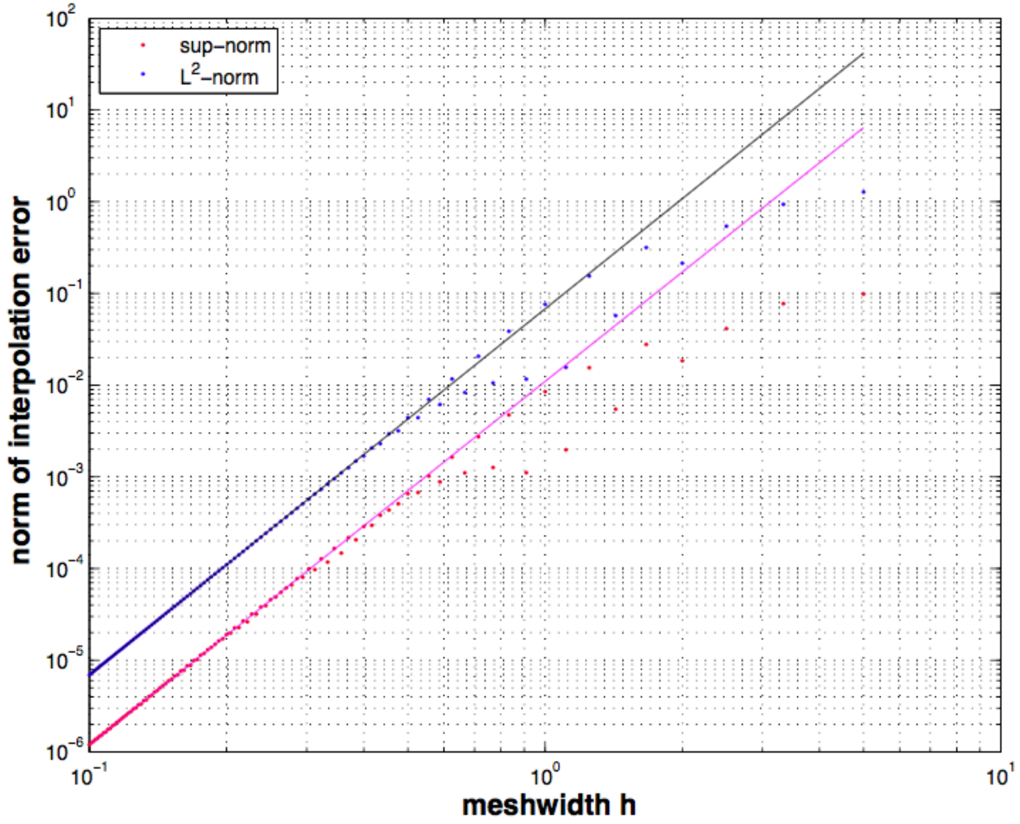
$$s_{|[x_{j-1},x_j]} \in \mathcal{P}_3\,,\quad j = 1,\dots,m \quad,\quad s(x_j) = f(x_j)\,,\ s'(x_j) = f'(x_j)\,,\quad j = 0,\dots,m\,.$$

(20) $\Rightarrow \quad s \in C^1$!

h-convergence :

Experiment : $f(t) = \frac{1}{1+t^2}$ on $[-5,5]$

- equidistant mesh



Alg cvg.

rate 4

$\|error\|_\infty = O(h^4)$

Theorem 4.5.17. Convergence of approximation by cubic Hermite interpolation

Let s *be the cubic Hermite interpolant of* $f \in C^4([a,b])$ *on a mesh* $\mathcal{M} := \{a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b\}$ *according to Def. 4.5.14. Then*

$$\|f - s\|_{L^\infty([a,b])} \leq \frac{1}{4!} h_{\mathcal{M}}^4 \left\|f^{(4)}\right\|_{L^\infty([a,b])},$$

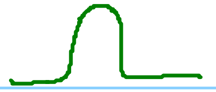
with the meshwidth $h_{\mathcal{M}} := \max_j |x_j - x_{j-1}|$.

4.5.3. Cubic spline interpolation

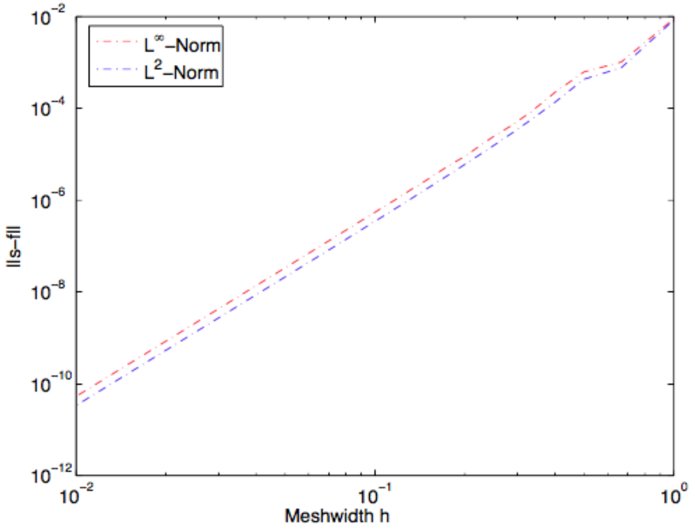
Exp. : h-convergence for complete cubic spline interpolant s on $[a,b]$:

$s(a) = f(a) \qquad s'(a) = f'(a)$
$s(b) = f(b) \qquad s'(b) = s'(b)$

$$f_1(t) = \frac{1}{1+e^{-2t}} \in C^\infty(I) \quad , \quad f_2(t) = \begin{cases} 0 & \text{, if } t < -\frac{2}{5}, \\ \frac{1}{2}(1 + \cos(\pi(t - \frac{3}{5}))) & \text{, if } -\frac{2}{5} < t < \frac{3}{5}, \\ 1 & \text{otherwise.} \end{cases} \in C^1(I).$$

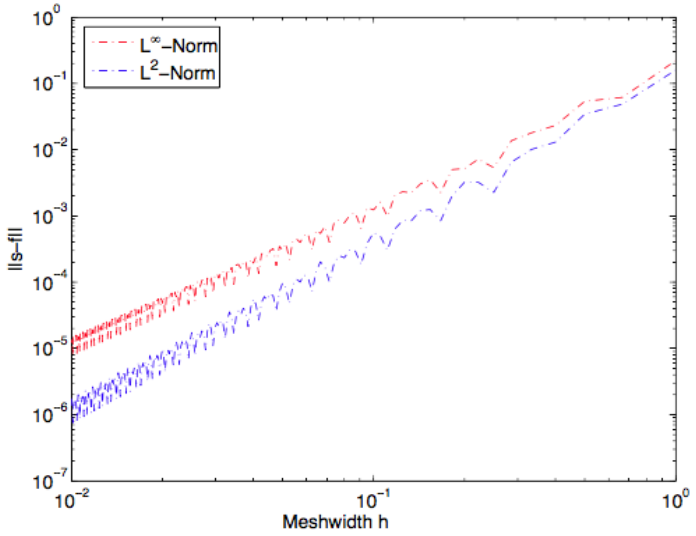


$\notin C^2$



$\|f_1 - s\|_{L^\infty([-1,1])} = O(h^4)$

Alg. cvg., rate 4



$\|f_2 - s\|_{L^\infty([-1,1])} = O(h^2)$

Alg. cvg., rate 2

Rule of thumb:

p.w. approximation w/ polynomials of degree p,

h-convergence: algebraic, maximal rate $p+1$

$\uparrow$

smoothness of f required!

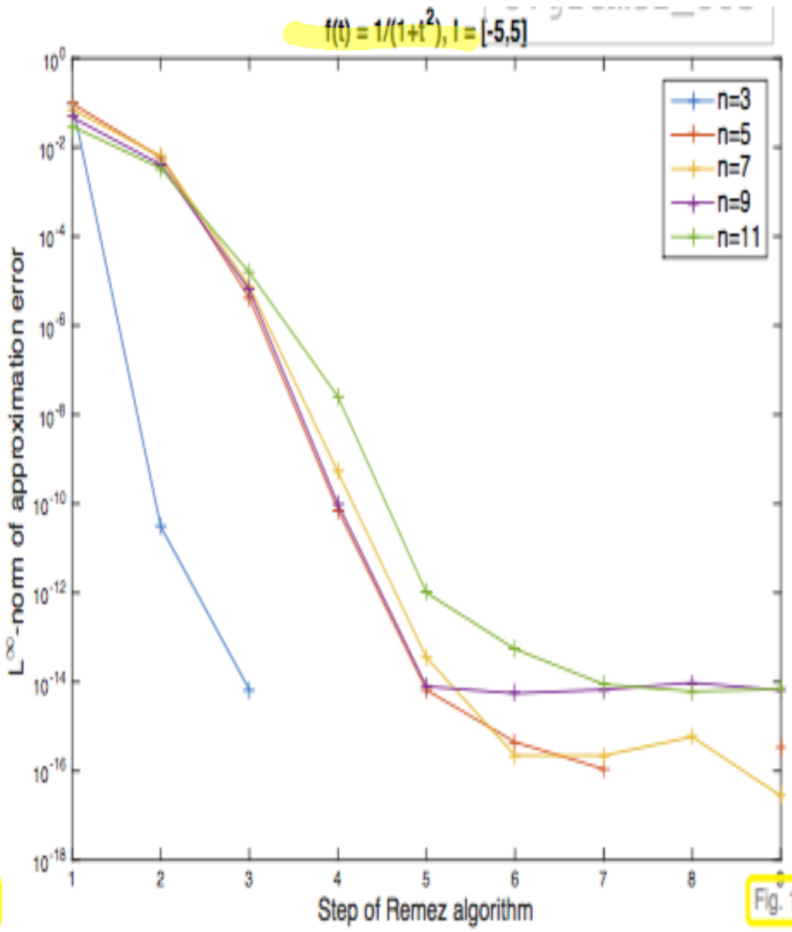
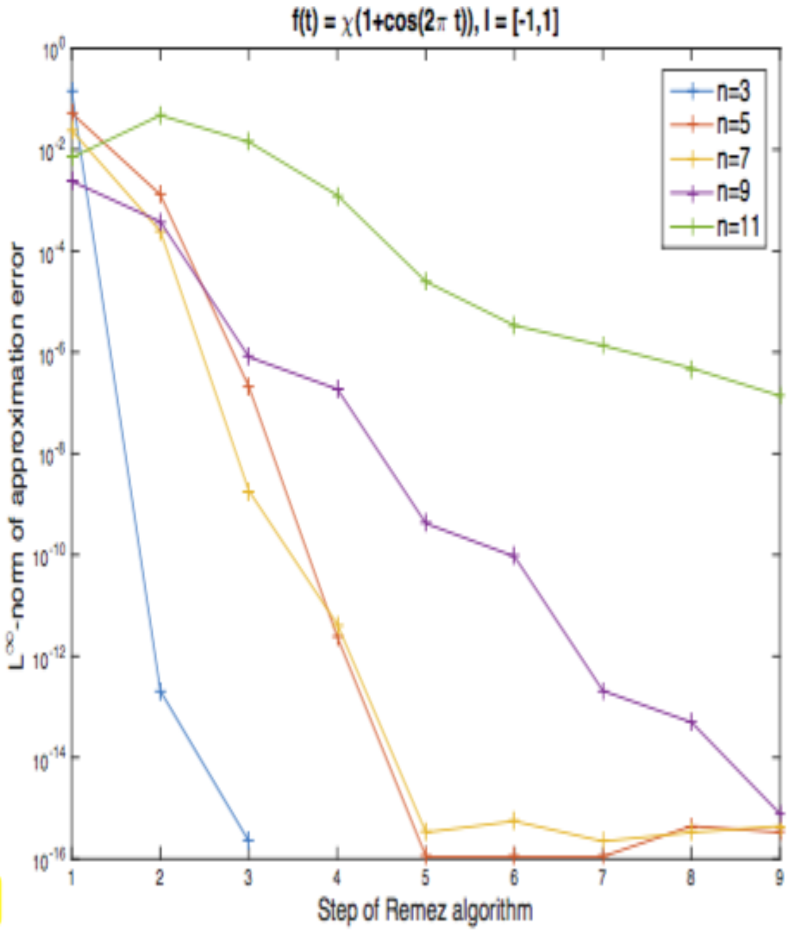

f(t) = 1/(1+t^2), I = [-5,5]
L∞-norm of approximation error
Step of Remez algorithm
n=3
n=5
n=7
n=9
n=11

Fig. 150


f(t) = χ(1+cos(2π t)), I = [-1,1]
L∞-norm of approximation error
Step of Remez algorithm
n=3
n=5
n=7
n=9
n=11