Volume Rendering Digest (for NeRF)

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Neural Radiance Fields [3] employ simple volume rendering as a way to overcome the challenges of differentiating through ray-triangle intersections by leveraging a probabilistic notion of visibility. This is achieved by assuming the scene is composed by a cloud of light-emitting particles whose density changes in space (in the terminology of physically-based rendering, this would be described as a volume with absorption and emission but no scattering [4, Sec 11.1]. In what follows, for the sake of exposition simplicity, and without loss of generality, we assume the emitted light *does not* change as a function of view-direction. This technical report is a condensed version of previous reports [1, 2], but rewritten in the context of NeRF, and adopting its commonly used notation[1].

Transmittance. Let the density field $\sigma(\mathbf{x})$, with $\mathbf{x}\in\mathbb{R}^3$ indicate the differential likelihood of a ray hitting a particle (i.e. the probability of hitting a particle while travelling an infinitesimal distance). We reparameterize the density along a given ray $\mathbf{r}=(\mathbf{o}, \mathbf{d})$ as a scalar function $\sigma(t)$, since any point $\mathbf{x}$ along the ray can be written as $\mathbf{r}(t)=\mathbf{o}+t\mathbf{d}$. Density is closely tied to the transmittance function $\mathcal{T}(t)$, which indicates the probability of a ray traveling over the interval $[0, t)$ without hitting any particles. Then the probability $\mathcal{T}(t+dt)$ of *not* hitting a particle when taking a differential step dt is equal to $\mathcal{T}(t)$, the likelihood of the ray reaching t, times $(1 - dt \cdot \sigma(t))$, the probability of not hitting anything during the step:

$$\mathcal{T}(t + dt) = \mathcal{T}(t) \cdot (1 - dt \cdot \sigma(t)) \tag{1}$$

$$\frac{\mathcal{T}(t + dt) - \mathcal{T}(t)}{dt} \equiv \mathcal{T}'(t) = -\mathcal{T}(t) \cdot \sigma(t) \tag{2}$$

This is a classical differential equation that can be solved as follows:

$$\mathcal{T}'(t) = -\mathcal{T}(t) \cdot \sigma(t) \tag{3}$$

$$\frac{\mathcal{T}'(t)}{\mathcal{T}(t)} = -\sigma(t) \tag{4}$$

$$\int_a^b \frac{\mathcal{T}'(t)}{\mathcal{T}(t)} \, dt = -\int_a^b \sigma(t) \, dt \tag{5}$$

$$\log \mathcal{T}(t)\big|_a^b = -\int_a^b \sigma(t) \, dt \tag{6}$$

$$\mathcal{T}(a \to b) \equiv \frac{\mathcal{T}(b)}{\mathcal{T}(a)} = \exp\left(-\int_a^b \sigma(t) \, dt\right) \tag{7}$$

where we define $\mathcal{T}(a \to b)$ as the probability that the ray travels from distance a to b along the ray without hitting a particle, which is related to the previous notation by $\mathcal{T}(t) = \mathcal{T}(0 \to t)$.

[1]If you are interested in borrowing the LaTeX notation, please refer to: `https://www.overleaf.com/read/fkhpkzxhnyws`

Probabilistic interpretation. Note that we can also interpret the function $1-\mathcal{T}(t)$ (often called *opacity*) as a cumulative distribution function (CDF) indicating the probability that the ray *does* hit a particle sometime before reaching distance t. Then $\mathcal{T}(t)\cdot\sigma(t)$ is the corresponding probability density function (PDF), giving the likelihood that the ray stops precisely at distance t. (1-T(t))' = T(t)σ(t)

Volume rendering. We can now calculate the expected value of the light emitted by the particles in the volume as the ray travels from $t{=}0$ to D, composited on top of a background color. Since the probability density for stopping at t is $\mathcal{T}(t)\cdot\sigma(t)$, the expected color is

$$\boldsymbol{C} = \int_0^D \mathcal{T}(t)\cdot\sigma(t)\cdot\mathbf{c}(t)\,dt \;+\; \mathcal{T}(D)\cdot\mathbf{c}_{\text{bg}} \tag{8}$$

where $\mathbf{c}_{\text{bg}}$ is a background color that is composited with the foreground scene according to the residual transmittance $\mathcal{T}(D)$. Without loss of generality, we omit the background term in what follows.

Homogeneous media. We can calculate the color of some homogeneous volumetric media with constant color $\mathbf{c}_a$ and density σ_a over a ray segment $[a,b]$ by integration:

$$\begin{aligned}
\boldsymbol{C}(a\to b) &= \int_a^b \mathcal{T}(a\to t)\cdot\sigma(t)\cdot\mathbf{c}(t)\,dt && (9)\\
&= \sigma_a\cdot\mathbf{c}_a\int_a^b \mathcal{T}(a\to t)\,dt && \text{constant density/radiance} \quad (10)\\
&= \sigma_a\cdot\mathbf{c}_a\int_a^b \exp\left(-\int_a^t \sigma(u)\,du\right)\,dt && \text{substituting (7)} \quad (11)\\
&= \sigma_a\cdot\mathbf{c}_a\int_a^b \exp\left(-\,\sigma_a u\big|_a^t\right)\,dt && \text{constant density (again)} \quad (12)\\
&= \sigma_a\cdot\mathbf{c}_a\int_a^b \exp\left(-\sigma_a(t-a)\right)\,dt && (13)\\
&= \sigma_a\cdot\mathbf{c}_a\cdot\left.\frac{\exp\left(-\sigma_a(t-a)\right)}{-\sigma_a}\right|_a^b && (14)\\
&= \mathbf{c}_a\cdot\left(1-\exp\left(-\sigma_a(b-a)\right)\right) && (15)
\end{aligned}$$

Transmittance is multiplicative. Note that transmittance factorizes as follows:

$$\begin{aligned}
\mathcal{T}(a\to c) = &= \exp\left(-\left[\int_a^b \sigma(t)\,dt + \int_b^c \sigma(t)\,dt\right]\right) && (16)\\
&= \exp\left(-\int_a^b \sigma(t)\,dt\right)\exp\left(-\int_b^c \sigma(t)\,dt\right) && (17)\\
&= \mathcal{T}(a\to b)\cdot\mathcal{T}(b\to c) && (18)
\end{aligned}$$

This also follows from the probabilistic interpretation of $\mathcal{T}$, since the probability that the ray does not hit any particles within $[a,c]$ is the product of the probabilities of the two independent events that it does not hit any particles within $[a,b]$ or within $[b,c]$.

Transmittance for piecewise constant data. Given a set of intervals $\{[t_n,t_{n+1}]\}_{n=1}^N$ with constant density σ_n within the n-th segment, and with $t_1{=}0$ and $\delta_n{=}t_{n+1}{-}t_n$, transmittance is equal to:

$$\mathcal{T}_n = \mathcal{T}(t_n) = \mathcal{T}(0\to t_n) = \exp\left(-\int_0^{t_n}\sigma(t)\,dt\right) = \exp\left(\sum_{k=1}^{n-1} -\sigma_k\delta_k\right) \tag{19}$$

Volume rendering of piecewise constant data. Combining the above, we can evaluate the volume rendering integral through a medium with piecewise constant color and density:

$$\begin{aligned}
\boldsymbol{C}(t_{N+1}) &= \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} \mathcal{T}(t) \cdot \sigma_n \cdot \mathbf{c}_n \; dt && \text{piecewise constant} \quad (20)\\
&= \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} \mathcal{T}(0 \to t_n) \cdot \mathcal{T}(t_n \to t) \cdot \sigma_n \cdot \mathbf{c}_n \; dt && \text{from (18)} \quad (21)\\
&= \sum_{n=1}^{N} \mathcal{T}(0 \to t_n) \int_{t_n}^{t_{n+1}} \mathcal{T}(t_n \to t) \cdot \sigma_n \cdot \mathbf{c}_n \; dt && \text{constant} \quad (22)\\
&= \sum_{n=1}^{N} \mathcal{T}(0 \to t_n) \cdot (1 - \exp(-\sigma_n(t_{n+1} - t_n))) \cdot \mathbf{c}_n && \text{from (15)} \quad (23)
\end{aligned}$$

This leads to the volume rendering equations from NeRF [3, Eq.3]:

$$\boldsymbol{C}(t_{N+1}) = \sum_{n=1}^{N} \mathcal{T}_n \cdot (1 - \exp(-\sigma_n \delta_n)) \cdot \mathbf{c}_n, \quad \text{where} \quad \mathcal{T}_n = \exp\left(\sum_{k=1}^{n-1} -\sigma_k \delta_k \right) \tag{24}$$

Finally, rather than writing these expressions in terms of volumetric density, we can re-express them in terms of alpha-compositing weights $\alpha_n \equiv 1 - \exp(-\sigma_n \delta_n)$, and by noting that $\prod_i \exp x_i = \exp(\sum_i x_i)$ in (19):

$$\boldsymbol{C}(t_{N+1}) = \sum_{n=1}^{N} \mathcal{T}_n \cdot \alpha_n \cdot \mathbf{c}_n, \quad \text{where} \quad \mathcal{T}_n = \prod_{n=1}^{N-1} (1 - \alpha_n) \tag{25}$$

Alternate derivation. By making use of the earlier connection between CDF and PDF that $(1-\mathcal{T})' = \mathcal{T}\sigma$, and by assuming constant color $\mathbf{c}_a$ along an interval $[a, b]$:

$$\begin{aligned}
\int_a^b \mathcal{T}(t) \cdot \sigma(t) \cdot \mathbf{c}(t) \; dt &= \mathbf{c}_a \int_a^b (1 - \mathcal{T})'(t) \; dt && (26)\\
&= \mathbf{c}_a \cdot (1 - \mathcal{T}(t))\big|_a^b && (27)\\
&= \mathbf{c}_a \cdot (\mathcal{T}(a) - \mathcal{T}(b)) && (28)\\
&= \mathbf{c}_a \cdot \mathcal{T}(a) \cdot (1 - \mathcal{T}(a \to b)) && (29)
\end{aligned}$$

Combined with constant per-interval density, this identity yields the same expression for color as (24).

References

[1] Nelson Max and Min Chen. Local and global illumination in the volume rendering integral. Technical report, Lawrence Livermore National Lab (LLNL), Livermore, CA (United States), 2005.

[2] Nelson Max and Min Chen. Local and global illumination in the volume rendering integral. Technical report, Schloss Dagstuhl, Leibniz Center for Informatics (Germany), 2010. `https://drops.dagstuhl.de/opus/volltexte/2010/2709/pdf/18.pdf`.

[3] Ben Mildenhall, Pratul P. Srinivasan, Matthew Tancik, Jonathan T. Barron, Ravi Ramamoorthi, and Ren Ng. NeRF: Representing scenes as neural radiance fields for view synthesis. In *ECCV*, 2020.

[4] Matt Pharr, Wenzel Jakob, and Greg Humphreys. *Physically based rendering: From theory to implementation*. Morgan Kaufmann, 2016. `https://pbr-book.org/3ed-2018/Volume_Scattering/Volume_Scattering_Processes`.

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