MSCS 446 Numerical Analysis I
Written Assignment 2
Adhere to the Homework Guidelines
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The *Error Function*, defined

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{1}$$

gives the probability that any one of a series of trials will lie within x units of the mean, assuming that the trials have a normal distribution with mean 0 and standard deviation $\frac{\sqrt{2}}{2}$.

The *Fresnel Sine Function*, defined

$$\mathrm{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)\, dt, \tag{2}$$

is used in optics to describe far-field diffraction phenomena (essentially constructive and destructive wave phenomena).

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1. (A) Evaluate the limit $\lim_{x\to\infty} \operatorname{erf}(x)$ exactly by evaluating the integral

   $$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt.$$

   (Hint: Set $I = \operatorname{erf}(R)$ and write $I^2 = \frac{4}{\pi} \int_0^R e^{-x^2}\, dx \int_0^R e^{-y^2}\, dy = \int_0^R \int_0^R e^{-(x^2+y^2)}\, dxdy$ (by Fubini's theorem). Now transform the integral to polar coordinates using a quarter circle of radius, R, integrate, and let $R \to \infty$. Why not use a complete circle, where $0 \le \theta < 2\pi$?)

2. (A) Write the Taylor series expansion for $\ln(1+x)$ about $x_0 = 0$ without differentiating. Hint: What is the binomial expansion for $\frac{1}{1+x}$ and how is this function related to $\ln(1+x)$?

3. (N) Write a Taylor series expansion for $S(x)$ defined in (2), about $x = 0$, written in summation notation – you are permitted to integrate the Taylor series for $\sin(\frac{\pi}{2}t^2)$ (about $t = 0$) term-by-term. Now create a Jupyter function to perform the calculation $p_n(1) \approx \mathrm{S}(1)$ where $\mathrm{S}(x)$ is defined in (2). Report the number of terms required to estimate $\mathrm{S}(1)$ accurate to within 1×10^{-8}. You do not need to analytically justify your result, a numerical experiment will suffice.

4. (A) How many terms in a Taylor series expansion about $x_0 = 0$ are required to estimate $\arctan(1)$ to within 10^{-4}? Hint: again, differentiation is not necessary.

5. (N) (Burden & Faires, page 13, #24) Compare two algorithms for estimating the error function for efficiency. Note that numpy contains the function $\mathrm{erf}(x)$, so you can calculate `np.erf(1)` for comparing to your computations.

   (a) Integrate the Taylor series expanded about $x_0 = 0$ (Macluarin series) for e^{-x^2} to show that
   $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$$

   (b) Approximate $\mathrm{erf}(1)$ to within 10^{-7} using the series in part a.

   (c) The error function can also be expressed in the form
   $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$
   Approximate $\mathrm{erf}(1)$ to within 10^{-7} using this new series.

   (d) Which series is more efficient (requires less terms) for this level of accuracy?

   Hint for part c: what does the following code do?

```
k = 5, d = np.arange(1,2*k+1,2), p = np.prod(d)
```