MSCS 446 Numerical Analysis I
Written Assignment 4
Adhere to the Homework Guidelines
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1. (A) Let $g(x) = \dfrac{x^2+4}{5}$ be a fixed point iteration formula.
   (a) Find the fixed point(s) of $g(x)$.
   (b) Does the fixed point iteration converge for any $x_0 \in [0, 2]$? Prove it!

2. (A) Given $f(x) = x^3 + x^2 - 5x + 3$ answer the following questions.
   (a) What are the roots of $f(x)$ and what are their multiplicities?
   (b) Write the Newton fixed-point iteration function $g(x)$ for this particular function.
   (c) Determine the fixed points for $g(x)$ and show that this fixed-point iteration is linearly convergent for one root and quadratically convergent for the other root. Be careful, you will obtain a $\frac{0}{0}$ term. I recommend that you factor the numerator and denominator, then "cancel terms."

3. (A) A function, $f(x)$, has a zero of multiplicity m, call it α, if

$$f(x) = (x-\alpha)^m q(x)$$

   where $q(\alpha) \neq 0$. Suppose α is a zero of multiplicty m of f, where $f^{(m)}$ is continuous on an open interval containing α. Show that the following fixed-point (Modified Newton's) method has $g'(\alpha) = 0$ (has a rate of convergence of at least 2):

$$g(x) = x - m\frac{f(x)}{f'(x)}.$$

   See the hint at the end of this assignment.

4. (N) Find the first positive value ($\alpha = 2.066393863$) for which

$$e^{-0.2x^2} = \frac{\sin(x)}{x}$$

using the Secant method with a tolerance of 10^{-8}.

The Secant method is similar to Newton's method except that $f'(x)$ gets replaced by

$$f'(x) \approx \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}.$$

5. (N) The equation

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = 0.45$$

can be solved for x by using Newton's method with

$$\frac{2}{\sqrt{\pi}} \int_0^{x_k} e^{-t^2} dt \qquad \star \text{ note that a quadrature rule is needed}$$

as part of the function iteration (what is $f(x)$?) and $f'(x)$ is evaluated using the Fundamental Theorem of (Differential) Calculus sometimes called the First Fundamental Theorem of Calculus.

   (a) Find a solution to $f(x) = 0$ accurate to with 10^{-5} using Newton's method with $x_0 = 0.5$ and Simpson's Rule.

   (b) Repeat (a) using the Trapezoidal Rule.

   (c) It should take roughly 5 or less iterations to converge to the root – quadratic convergence! What is the minimum number of nodes to use in each of the quadrature rules to reach quadratic convergence? You may obtain these numbers using numerical experimentation.

Hint for Problem #3:
Warning: You may be tempted to think that $g'(x) = 1 - m\frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$ and conclude that $g'(\alpha) = 1 - m \neq 0$ but this reasoning is not correct; use the defintion of "zero of multiplicty m."

$$\begin{aligned}
g(x) &= x - m\frac{(x-\alpha)^m q(x)}{m(x-\alpha)^{m-1}q(x) + (x-\alpha)^m q'(x)} \\
&= x - m\frac{(x-\alpha)^m q(x)}{(x-\alpha)^{m-1}\left(mq(x) + (x-\alpha)q'(x)\right)} \\
g(x) &= x - m\frac{(x-\alpha)q(x)}{(mq(x) + (x-\alpha)q'(x))} \\
g'(x) &= 1 - m\frac{d}{dx}\left[\frac{(x-\alpha)q(x)}{(mq(x) + (x-\alpha)q'(x))}\right] \qquad \text{(use the quotient rule, then substitute in } \alpha)
\end{aligned}$$