

# 10.4.ex-13.2

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**Question:** Solve the Quadratic equation

$$x^2 + 4x + 5 = 0 \quad (0.1)$$

**Theoretical Solution:**

The roots of a Standard Quadratic equation of the form  $ax^2 + bx + c = 0$  are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (0.2)$$

Here,  $a = 1, b = 4$  and  $c = 5$ . On substituting the values in (0.2), we have

$$x = \frac{-4 \pm \sqrt{16 - 20}}{2} \quad (0.3)$$

$$x = -2 \pm i \quad (0.4)$$

**Computational Solution:**

Eigenvalues of Companion matrix:

The eigenvalues of a companion matrix are the roots of the characteristic polynomial that defines the matrix. For a polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 \quad (0.5)$$

Companion matrix A is defined as:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \quad (0.6)$$

Here,  $a_0 = 5, a_1 = 4$

$$A = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \quad (0.7)$$

We find the eigenvalues using the *QR* algorithm. The basic principle behind this algorithm is a similarity transform,

$$A' = X^{-1}AX \quad (0.8)$$

which does not alter the eigenvalues of the matrix  $A$ .

We use this to get the Schur Decomposition,

$$A = Q^{-1}UQ = Q^*UQ \quad (0.9)$$

where  $Q$  is a unitary matrix ( $Q^{-1} = Q^*$ ) and  $U$  is an upper triangular matrix whose diagonal entries are the eigenvalues of  $A$ .

To efficiently get the Schur Decomposition, we first use householder reflections to reduce it to an upper hessenberg form.

A householder reflector matrix is of the form,

$$P = I - 2\mathbf{u}\mathbf{u}^* \quad (0.10)$$

Householder reflectors transform any vector  $\mathbf{x}$  to a multiple of  $\mathbf{e}_1$ ,

$$P\mathbf{x} = \mathbf{x} - 2\mathbf{u}(\mathbf{u}^*\mathbf{x}) = \alpha\mathbf{e}_1 \quad (0.11)$$

$P$  is unitary, which implies that,

$$\|P\mathbf{x}\| = \|\mathbf{x}\| \quad (0.12)$$

$$\implies \alpha = \rho \|\mathbf{x}\| \quad (0.13)$$

$$(0.14)$$

As  $\mathbf{u}$  is unit norm,

$$\mathbf{u} = \frac{\mathbf{x} - \rho \|\mathbf{x}\| \mathbf{e}_1}{\|\mathbf{x} - \rho \|\mathbf{x}\| \mathbf{e}_1\|} = \frac{1}{\|\mathbf{x} - \rho \|\mathbf{x}\| \mathbf{e}_1\|} \begin{pmatrix} x_1 - \rho \|\mathbf{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (0.15)$$

Selection of  $\rho$  is flexible as long as  $|\rho| = 1$ . To ease out the process, we take  $\rho = \frac{x_1}{|x_1|}$ ,  $x_1 \neq 0$ . If  $x_1 = 0$ , we take  $\rho = 1$ .

Householder reflector matrix ( $P_i$ ) is given by,

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & I_{n-i} - 2\mathbf{u}_i\mathbf{u}_i^* \end{bmatrix} \quad (0.16)$$

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \quad (0.17)$$

Next step is to do Given's rotation to get the  $QR$  Decomposition.

The Givens rotation matrix  $G(i, j, c, s)$  is defined by

$$G(i, j, c, s) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\bar{s} & \cdots & \bar{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \quad (0.18)$$

where  $|c|^2 + |s|^2 = 1$ , and  $G$  is a unitary matrix.

Say we take a vector  $\mathbf{x}$ , and  $\mathbf{y} = G(i, j, c, s) \mathbf{x}$ , then

$$y_k = \begin{cases} cx_i + sx_j, & k = i \\ -\bar{s}x_i + \bar{c}x_j, & k = j \\ x_k, & k \neq i, j \end{cases} \quad (0.19)$$

For  $y_j$  to be zero, we set

$$c = \frac{\bar{x}_i}{\sqrt{|x_i|^2 + |x_j|^2}} = c_{ij} \quad (0.20)$$

$$s = \frac{\bar{x}_j}{\sqrt{|x_i|^2 + |x_j|^2}} = s_{ij} \quad (0.21)$$

Using this Givens rotation matrix, we zero out elements of subdiagonal in the hessenberg matrix  $H$ .

$$\begin{aligned} H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} &\xrightarrow{G(1,2,c_{12},s_{12})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \\ &\xrightarrow{G(2,3,c_{23},s_{23})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,4,c_{34},s_{34})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \\ &\xrightarrow{G(4,5,c_{45},s_{45})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} = R \quad (0.22) \end{aligned}$$

where  $R$  is upper triangular. For the given companion matrix,

Let  $G_k = G(k, k+1, c_{k,k+1}, s_{k,k+1})$ , then we deduce that

$$G_4 G_3 G_2 G_1 H = R \quad (0.23)$$

$$H = G_1^* G_2^* G_3^* G_4^* R \quad (0.24)$$

$$H = QR, \text{ where } Q = G_1^* G_2^* G_3^* G_4^* \quad (0.25)$$

Using this  $QR$  algorithm, we get the following update equation,

$$A_k = Q_k R_k \quad (0.26)$$

$$A_{k+1} = R_k Q_k \quad (0.27)$$

$$= (G_n \dots G_2 G_1) A_k (G_1^* G_2^* \dots G_n^*) \quad (0.28)$$

Running the eigenvalue code we get

$$x_1 = -2.0000000000000001 + -1.0000000000000003j \quad (0.29)$$

$$x_2 = -2.0000000000000001 + -1.0000000000000003j \quad (0.30)$$

Newton-Raphson iterative method:

$$f(x) = x^2 + 4x + 5 \quad (0.31)$$

$$f'(x) = 2x + 4 \quad (0.32)$$

Difference equation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (0.33)$$

$$x_{n+1} = x_n - \frac{x_n^2 + 4x + 5}{2x_n + 4} \quad (0.34)$$

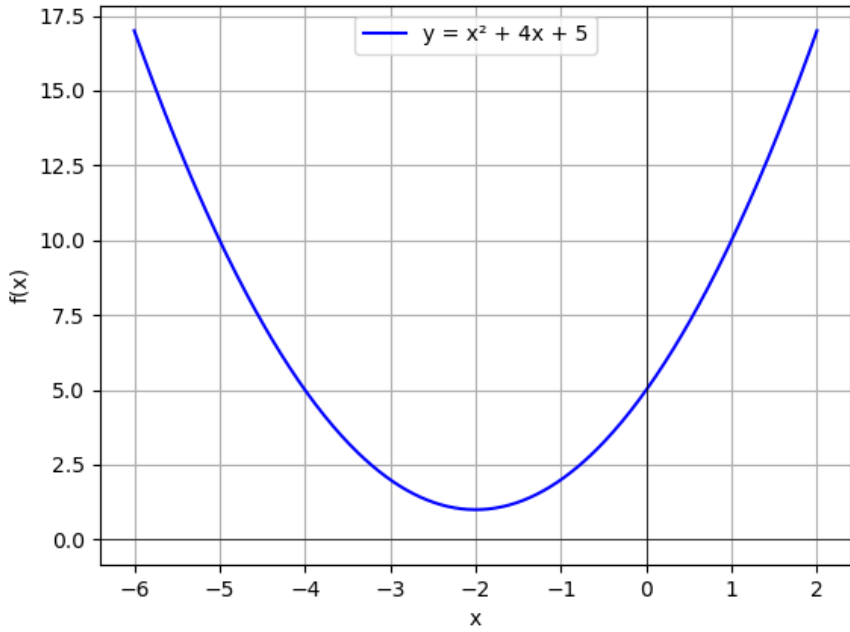
$$x_{n+1} = \frac{x_n}{2} - 1 - \frac{1}{2x_n + 4} \quad (0.35)$$

Picking two initial guesses,

$$x_0 = 0 + i \text{ converges to } -2.0 + 1.0000000000000002i \quad (0.36)$$

$$x_0 = 0 - i \text{ converges to } -2.0 + -1.0000000000000002i \quad (0.37)$$

**Plot:**



Acknowledgments:

Code from [<https://github.com/Dwarak-A/sprog/tree/c6f873e29ab9910f22e49f7421dfc8e5fd5c60fc/ncert/10/4/1/1/2/codes>].