9.3.11.4

EE24BTECH11058 - P.Shiny Diavajna

Question: Solve the differential equation $\frac{d^2y}{dx^2} - 1 = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 0$

Theoretical Solution:
Laplace Transform:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt \tag{0.1}$$

Properties of Laplace Transform:

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \tag{0.4}$$

Apply the properties to the given equation

$$y''(x) - 1 = 0 \tag{0.5}$$

$$\mathcal{L}(y'') - \mathcal{L}(1) = 0 \tag{0.6}$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) - \frac{1}{s} = 0 \tag{0.7}$$

Substituting the initial conditions in equation (0.8)

$$s^3\mathcal{L}(y) - 1 = 0 \tag{0.8}$$

$$\mathcal{L}(y) = \frac{1}{s^3} \tag{0.9}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) \tag{0.10}$$

$$y = \frac{x^2}{2}u(x) \tag{0.11}$$

Theoretical solution is

$$y(x) = \frac{x^2}{2}u(x) \tag{0.12}$$

Bilinear Transform:

$$s = \frac{2}{h}.\frac{1 - z^{-1}}{1 + z^{-1}} \tag{0.13}$$

Substitute (0.13) in (0.9)

$$Y(z) = \left(\frac{h}{2}\right)^3 \left(\frac{1+z^{-1}}{1-z^{-1}}\right)^3 \tag{0.14}$$

$$Y(z) = \left(\frac{h}{2}\right)^3 \left(\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+3z^{-2}-z^{-3}}\right) \tag{0.15}$$

$$Y(z)\left(1-3z^{-1}+3z^{-2}-z^{-3}\right) = \left(\frac{h}{2}\right)^3 \left(1+3z^{-1}+3z^{-2}+z^{-3}\right) \tag{0.16}$$

Multiply the above equation with z^3 and take the inverse z transform to obtain difference equation

$$y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = \left(\frac{h}{2}\right)^3 \left(\delta(n+3) + 3\delta(n+2) + 3\delta(n+1) + \delta(n)\right) \tag{0.17}$$

Since $n \geq 0$, $\delta(n+1) = \delta(n+2) = \delta(n+3) = 0$

$$y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = \left(\frac{h}{2}\right)^3 (\delta(n)) \tag{0.18}$$

Computational Solution:

: Method of finite differences:

The second derivative can be approximated as:

$$\frac{d^2y}{dx^2} = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \tag{0.19}$$

Substitute this into the differential equation:

$$\frac{y_{n+1} - 2y_n + y_{n+1}}{h^2} - 1 = 0 \tag{0.20}$$

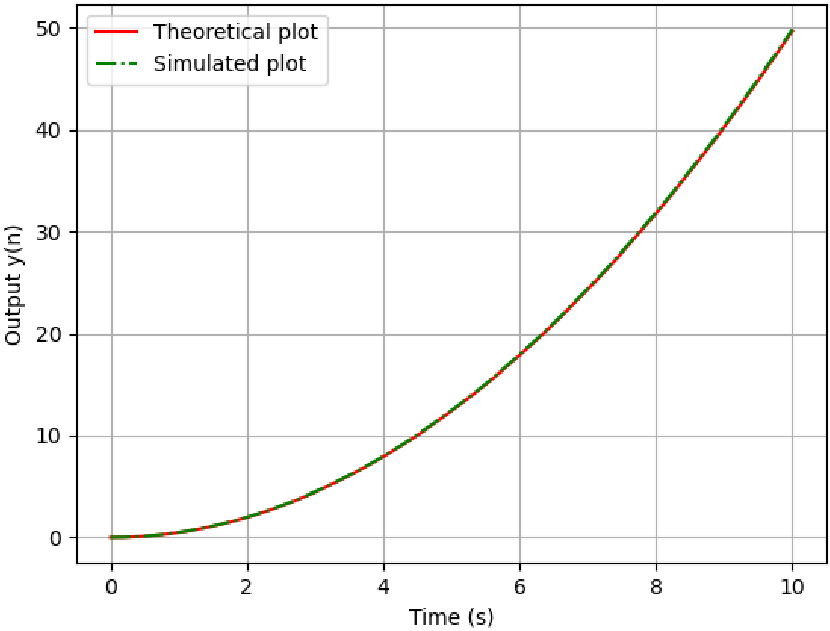
$$y_{n+1} = 2y_n - y_{n-1} + h^2 \tag{0.21}$$

Given $y(0) = 0$ and $y'(0) = 0$

$$y_1 = y_0 + y'(0).h \tag{0.22}$$

$$y_1 = 0 \tag{0.23}$$

This curve is generated by applying the finite difference method to the given problem and taking the values of $x_0 = 0, y_0 = 0, y_1 = 0$ and $h = 0.001$ and running the iterations for 1000 times


Theoretical plot
Simulated plot
50
40
30
20
10
0
0
2
4
6
8
10
Output y(n)
Time (s)