10.3.6.2.2

EE24BTECH11058 - P.Shiny Diavajna

Question:
2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.

Solution:
Let the number of days taken by 1 woman alone to finish the work be x
Let the number of days taken by 1 man alone to finish the work be y

Then,
The amount of work done by a woman in 1 day is $\frac{1}{x}$
The amount of work done by a man in 1 day is $\frac{1}{y}$

Let,

$$\frac{1}{x} = p \text{ and } \frac{1}{y} = q \tag{0.1}$$

Then the equations are:

$$2p + 5q = \frac{1}{4} \tag{0.2}$$

$$3p + 6q = \frac{1}{3} \tag{0.3}$$

Matrix Form:

$$\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{3} \end{pmatrix} \tag{0.4}$$

LU decomposition:
For the system of linear equations $\mathbf{Ax} = \mathbf{b}$, if $\mathbf{A}$ is non-singular, we can decompose it as product LU where L is lower triangular matrix and U is an upper triangular matrix.
The equation becomes

$$\mathbf{LUx} = \mathbf{b} \tag{0.5}$$

Taking

$$\mathbf{y} = \mathbf{Ux} \tag{0.6}$$

Substituting (0.6) in (0.5)

$$\mathbf{Ly} = \mathbf{b} \tag{0.7}$$

We solve for y in $\mathbf{Ly} = \mathbf{b}$ and then solve for $\mathbf{x}$ in $\mathbf{Ux} = \mathbf{y}$
Applying LU decomposition to matrix A,
For each column $j \geq k$, the entries of U in the kth row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m}.U_{m,j}, \forall j \geq k \tag{0.8}$$

For each row $i > k$, the entries of L in the kth column are updated as:

$$L_{j,k} = \frac{1}{U_{k,k}}\left(A_{j,k} - \sum_{m=1}^{k-1} .U_{m,k}\right), \forall i > k \tag{0.9}$$

We find $\mathbf{L}$ and $\mathbf{U}$ as follows:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \tag{0.10}$$

$$\mathbf{U} = \begin{pmatrix} 2 & 5 \\ 0 & -\frac{3}{2} \end{pmatrix} \tag{0.11}$$

Solving $\mathbf{Ly} = \mathbf{b}$ by forward substitution,

$$\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{3} \end{pmatrix} \tag{0.12}$$

$$y_1 = \frac{1}{4} \tag{0.13}$$

$$y_2 = -\frac{1}{24} \tag{0.14}$$

$$y = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{24} \end{pmatrix} \tag{0.15}$$

Solving $\mathbf{Ux} = \mathbf{y}$

$$\begin{pmatrix} 2 & 5 \\ 0 & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{24} \end{pmatrix} \tag{0.16}$$

$$p = \frac{1}{18} \tag{0.17}$$

$$q = \frac{1}{36} \tag{0.18}$$

Hence, solution is

$$x = \frac{1}{p} = 18 \tag{0.19}$$

$$y = \frac{1}{q} = 36 \tag{0.20}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \end{pmatrix} \tag{0.21}$$

Therefore,
The time taken by 1 woman alone to finish the work is 18 days
The time taken by 1 man alone to finish the work is 36 days

Plot: