

# 9.3.11.4

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**Question:** Solve the differential equation  $\frac{d^2y}{dx^2} - 1 = 0$  with initial conditions  $y(0) = 0$  and  $y'(0) = 0$

**Theoretical Solution:**

Laplace Transform:

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad (0.1)$$

Properties of Laplace Transform:

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - sy(0) - y'(0) \quad (0.2)$$

$$\mathcal{L}(1) = \frac{1}{s} \quad (0.3)$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \quad (0.4)$$

Apply the properties to the given equation

$$y''(x) - 1 = 0 \quad (0.5)$$

$$\mathcal{L}(y'') - \mathcal{L}(1) = 0 \quad (0.6)$$

$$s^2 \mathcal{L}(y) - sy(0) - y'(0) - \frac{1}{s} = 0 \quad (0.7)$$

Substituting the initial conditions in equation (0.8)

$$s^3 \mathcal{L}(y) - 1 = 0 \quad (0.8)$$

$$\mathcal{L}(y) = \frac{1}{s^3} \quad (0.9)$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) \quad (0.10)$$

$$y = \frac{x^2}{2} u(x) \quad (0.11)$$

Theoretical solution is

$$y(x) = \frac{x^2}{2} u(x) \quad (0.12)$$

**Bilinear Transform:**

$$s = \frac{2}{h} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \quad (0.13)$$

Substitute (0.13) in (0.9)

$$Y(z) = \left(\frac{h}{2}\right)^3 \left(\frac{1+z^{-1}}{1-z^{-1}}\right)^3 \quad (0.14)$$

$$Y(z) = \left(\frac{h}{2}\right)^3 \left(\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+3z^{-2}-z^{-3}}\right) \quad (0.15)$$

$$Y(z)(1-3z^{-1}+3z^{-2}-z^{-3}) = \left(\frac{h}{2}\right)^3 (1+3z^{-1}+3z^{-2}+z^{-3}) \quad (0.16)$$

Multiply the above equation with  $z^3$  and take the inverse  $z$  transform to obtain difference equation

$$y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = \left(\frac{h}{2}\right)^3 (\delta(n+3) + 3\delta(n+2) + 3\delta(n+1) + \delta(n)) \quad (0.17)$$

$$y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = \left(\frac{h}{2}\right)^3 \delta(n) - h^2\delta(n) \quad (0.18)$$

### Computational Solution:

: Method of finite differences:

The second derivative can be approximated as:

$$\frac{d^2y}{dx^2} = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad (0.19)$$

Substitute this into the differential equation:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} - 1 = 0 \quad (0.20)$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \quad (0.21)$$

Given  $y(0) = 0$  and  $y'(0) = 0$

$$y_1 = y_0 + y'(0) \cdot h \quad (0.22)$$

$$y_1 = 0 \quad (0.23)$$

This curve is generated by applying the finite difference method to the given problem and taking the values of  $x_0 = 0, y_0 = 0, y_1 = 0$  and  $h = 0.001$  and running the iterations for 1000 times

