Matrix, Linear Algebra, Differential Equation
MAT 2207

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Matrix

Definition of Matrix:

A system of any $m \times n$ numbers arranged in a rectangular arrangement of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix.

Ex:

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & 7 \end{bmatrix} \qquad \text{is a } 2 \times 3 \text{ matrix.}$$

in general form:

$$A = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} = (\sigma_{ij})_{mxm}$$

Singular and Non-singular Matrix:

Let A be any square matrix. If $\det(A) = 0$, then A is called a singular matrix, and if $\det(A) \neq 0$, then A is called a non-singular matrix.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix}$

Then $|A| = \det(A) = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 4 - 4 = 0$
So, A is a singular matrix

Again, $|B| = \det(B) = \det \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix} = 12 - 10 = 2 \neq 0$
Hence, B is a non-singular matrix.

Inverse Matrix:

Let A and B be two $n \times n$ square matrices such that $AB = BA = I_n = I$, then B is said to be the inverse of A, and we write $B = A^{-1}$. Also, $A = B^{-1}$.

Ex: Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$.

$$\therefore \; AB = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -12+12 \\ 1-1 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\text{and } BA = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 3-3 \\ -4+4 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore AB = BA = I_2 = I$
Therefore, we can write $A = B^{-1}$ and $B = A^{-1}$.

[N.B.: The inverse of a matrix exists only when the matrix is non-singular, i.e., $|A| \neq \emptyset$.]

*** Multiplication of two matrices is possible only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Echelon Matrix:

Let $A = (a_{ij})_{m\times n}$ be any matrix. Then A is said to be an echelon matrix or is said to be in echelon form if:

1. all the non-zero rows (if any) precede the zero rows,
2. the number of zero entries preceding the first non-zero entry in each row increases by row.

Ex:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} \text{ is an echelon matrix, but } \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix} \text{ is not an echelon matrix.}$$

Rank of a Matrix:

Rank of a matrix is the largest non-zero row in the matrix of row echelon form.

Ex:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix A = 3, Rank of matrix B = 2

Find the rank of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 5 & 4 & -4 \\ 0 & 5 & 6 & -5 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 2r1 \\ r3' = r3 - 3r1 \\ r4' = r4 - 4r1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r4' \\ r4'' = r4' - r3' \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r4''' = r4'' + r3'' \end{bmatrix}$$

This is the echelon form of matrix A ; and the Rank of A is 3.

$$B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -5 & -10 & -5 & 5 \\ 0 & -4 & -8 & -4 & 4 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - r1 \\ r3' = r3 - 2r2 \\ r4' = r4 - 3r2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - 5r4' \\ r4'' = r4' + 4r2' \end{bmatrix}$$

This is the echelon form of matrix B ; and the Rank of B is 2.

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 3r1 \\ r3' = r3 - 2r1 \\ r4' = r2 - 3r4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r2' \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} r4''' = r4' - r3'' \end{bmatrix}$$

This is the echelon form of matrix C ; and the Rank of C is 4.

$$D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 2r1 \\ r3' = r3 - r1 \\ r4' = r4 - 2r2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r2' \\ r4'' = r4' - 4r3' \end{bmatrix}$$

This is the echelon form of matrix D ; and the Rank of D is 4.

Inverse Matrix Calculation:

Find the inverse of the matrix by using the formula [A:I]

example; find the inverse matrix of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Solution:

$$A = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

r2' = r2 - 3r1
r3' = r3 - 2r1
r4' = r4 - r2

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & -1 & 3 \end{array}\right]$$

r3'' = r3' - r2'
r4'' = 3r4' - r3'

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$$

r4''' = r4'' + 2r3''

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$$

r1' = 3r1 + r2

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 2 & 2 & -1 & 2 & 0 \\ 0 & 3 & 0 & -1 & -7 & 5 & -4 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$$

r1'' = r1' + 2r3''
r2''' = r2'' - 4r3''

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 3 & 0 & -6 \\ 0 & 3 & 0 & 0 & -6 & 3 & -3 & 3 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$$

r1''' = r1'' - 2r4'''
r2'''' = r2''' + r4''

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$$

r1'''' = r1''' x 1/3
r2''''' = r2'''' x 1/3
r3''' = r3'' x -1

$$C = \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

r2' = r2 - 2r1
r3' = r3 - r1
r4' = r4 - r1

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right]$$

r4'' = 7r4' - r3'

$$\sim \left[\begin{array}{cccc|cccc} 7 & 0 & 0 & -1 & -34 & 21 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right]$$

r1'' = 7r1' - r3'

$$\sim \left[\begin{array}{cccc|cccc} 7 & 0 & 0 & 0 & -28 & 21 & 0 & -7 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & -49 & 0 & -7 & 56 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right]$$

r1''' = r1'' - r4''
r3'' = r3' + 8r4''

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -7 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & & & & \end{array}\right]$$

r1'''' = r1''' x 1/7
r2'' = r2' x (-1)
r3''' = r3'' x 1/7
r4''' = r4'' x (-1)

Home Work:

find the inverse matrix of $B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$