Matrix, Linear Algebra, Differential Equation
MAT 2207

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April 2024

Matrix

Definition of Matrix:

A system of any $m \times n$ numbers arranged in a rectangular arrangement of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix.

Ex:

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & 7 \end{bmatrix} \qquad \text{is a } 2 \times 3 \text{ matrix.}$$

in general form:

$$A = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} = (\sigma_{ij})_{mxm}$$

Singular and Non-singular Matrix:

Let A be any square matrix. If $\det(A) = 0$, then A is called a singular matrix, and if $\det(A) \neq 0$, then A is called a non-singular matrix.

Ex: $\qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix}$

Then $|A| = \det(A) = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 4 - 4 = 0$
So, A is a singular matrix

Again, $|B| = \det(B) = \det \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix} = 12 - 10 = 2 \neq 0$
Hence, B is a non-singular matrix.

Inverse Matrix:

Let A and B be two $n \times n$ square matrices such that $AB = BA = I_n = I$, then B is said to be the inverse of A, and we write $B = A^{-1}$. Also, $A = B^{-1}$.

Ex: Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$.

$$\therefore\ AB = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -12+12 \\ 1-1 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\text{and } BA = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 3-3 \\ -4+4 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore AB = BA = I_2 = I$
Therefore, we can write $A = B^{-1}$ and $B = A^{-1}$.

[N.B.: The inverse of a matrix exists only when the matrix is non-singular, i.e., $|A| \neq \emptyset$.]
*** Multiplication of two matrices is possible only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Echelon Matrix:

Let $A = (a_{ij})_{m \times n}$ be any matrix. Then A is said to be an echelon matrix or is said to be in echelon form if:

1. all the non-zero rows (if any) precede the zero rows,
2. the number of zero entries preceding the first non-zero entry in each row increases by row.

Ex:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} \text{ is an echelon matrix, but } \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix} \text{ is not an echelon matrix.}$$

Rank of a Matrix:

Rank of a matrix is the largest non-zero row in the matrix of row echelon form.

Ex:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix A = 3, Rank of matrix B = 2

Find the rank of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 5 & 4 & -4 \\ 0 & 5 & 6 & -5 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 2r1 \\ r3' = r3 - 3r1 \\ r4' = r4 - 4r1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r4' \\ r4'' = r4' - r3' \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r4''' = r4'' + r3'' \end{bmatrix}$$

This is the echelon form of matrix A ; and the Rank of A is 3.

$$B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -5 & -10 & -5 & 5 \\ 0 & -4 & -8 & -4 & 4 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - r1 \\ r3' = r3 - 2r2 \\ r4' = r4 - 3r2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - 5r4' \\ r4'' = r4' + 4r2' \end{bmatrix}$$

This is the echelon form of matrix B ; and the Rank of B is 2.

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 3r1 \\ r3' = r3 - 2r1 \\ r4' = r2 - 3r4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r2' \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} r4''' = r4' - r3'' \end{bmatrix}$$

This is the echelon form of matrix C ; and the Rank of C is 4.

$$D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 2r1 \\ r3' = r3 - r1 \\ r4' = r4 - 2r2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r2' \\ r4'' = r4' - 4r3' \end{bmatrix}$$

This is the echelon form of matrix D ; and the Rank of D is 4.

Inverse Matrix Calculation:

Find the inverse of the matrix by using the formula [A:I]

example; find the inverse matrix of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Solution:

$$A = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{l} r2' = r2 - 3r1 \\ r3' = r3 - 2r1 \\ r4' = r4 - r2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & -1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r3'' = r3' - r2' \\ r4'' = 3r4' - r3' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r4''' = r4'' + 2r3'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1' = 3r1 + r2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 2 & 2 & -1 & 2 & 0 \\ 0 & 3 & 0 & -1 & -7 & 5 & -4 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1'' = r1' + 2r3'' \\ r2''' = r2'' - 4r3'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 3 & 0 & -6 \\ 0 & 3 & 0 & 0 & -6 & 3 & -3 & 3 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1''' = r1'' - 2r4''' \\ r2'''' = r2''' + r4'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1'''' = r1''' \times 1/3 \\ r2''''' = r2'''' \times 1/3 \\ r3''' = r3'' \times -1 \end{array}\right]$$

Home Work:

find the inverse matrix of $B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$

$$B = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{l} r2' = r2 + 2r1 \\ r3' = r3 + 4r1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & -2 & 1 & 0 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] \quad \left[\begin{array}{l} r3'' = -5r3' + 6r2' \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 0 & -3 & 1 & -2 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] \quad \left[\begin{array}{l} r1' = 5r1 + 2r2' \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 0 & 0 & 25 & -20 & 15 \\ 0 & 5 & 0 & 50 & -35 & 30 \\ 0 & 0 & -1 & 8 & -6 & -5 \end{array}\right] \quad \left[\begin{array}{l} r1'' = r1' - 3r3'' \\ r2'' = r2' - 6r3'' \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & 3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \quad \left[\begin{array}{l} r1''' = r1'' \times (-1/5) \\ r2''' = r2'' \times (1/5) \\ r3''' = r3'' \times (-1) \end{array}\right]$$

$$C = \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{c} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \\ r_4' = r_4 - r_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_4'' = 7r_4' - r_3' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & -5 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_1' = r_1 + 3r_2' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 7 & 0 & 0 & -1 & -34 & 21 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_1'' = 7r_1' - r_3' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 7 & 0 & 0 & 0 & -28 & 21 & 0 & -7 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & -49 & 0 & -7 & 56 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_1''' = r_1'' - r_4'' \\ r_3'' = r_3' + 8r_4'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -7 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & 6 & 0 & 1 & -7 \end{array}\right] \quad \left[\begin{array}{c} r_1'''' = r_1''' \times 1/7 \\ r_2'' = r_2' \times (-1) \\ r_3''' = r_3'' \times 1/7 \\ r_4''' = r_4'' \times (-1) \end{array}\right]$$

Eigenvalues and Eigenvectors:

A nonzero matrix X is an eigenvector of a square matrix A if there exists a λ such that $AX = \lambda X$. Then X is called the eigenvector of A with eigenvalue λ and λ is called the eigenvalue of A.

The equation $|A - \lambda I| = 0$ is called the characteristic equation of A.

Cayley-Hamilton theorem states that every square matrix A satisfies it's characteristic equation.

Example:

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ in the field of real numbers ($\mathbb{R}$). Also verify the Cayley-Hamilton theorem.

Sol$_n$:

Given matrix, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Characteristic matrix of A inverse:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \end{aligned}$$

Characteristic equation of A inverse:

$|A - \lambda I| = 0$
or,

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(3-\lambda) - 8 = 0$
or, $3 - 4\lambda + \lambda^2 - 8 = 0$
or, $\lambda^2 - 4\lambda - 5 = 0$
or, $\lambda^2 - 5\lambda + \lambda - 5 = 0$
or $\lambda = 1$ or $\lambda = -5$

thus, $\lambda = 1, -5$;
these are the eigenvalues of A.

For $x = 5$,

$$\begin{aligned}
(A - \lambda I)v &= 0 \\
\text{or,} \quad \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\
\text{or,} \quad \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0
\end{aligned}$$

Now, we can write,

$$\begin{aligned}
-4x_1 + 4x_2 &= 0 \\
2x_1 - 2x_2 &= 0 \\
\text{or,} \quad \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\sim \quad \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} & \quad [r_2' = 2r_2 + r_1] \\
\sim \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} & \quad [r_1' = r_1 \times \frac{1}{4}]
\end{aligned}$$

From this first row,

$$\begin{aligned}
-x_1 + x_2 &= 0 \\
x_1 = x_2 &= s \quad \text{(say)}
\end{aligned}$$

Taking $s = 1$, the eigenvector can be written as,

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now for $\lambda = -1$,

$$\begin{aligned}
(A - \lambda I)v &= 0 \\
\text{or,} \quad \begin{bmatrix} 1-(-1) & 4 \\ 2 & 3-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\
\text{or,} \quad \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0
\end{aligned}$$

Now, we can write,

$$
\begin{aligned}
2x_1 + 4x_2 &= 0 \\
2x_1 + 4x_2 &= 0 \\
\text{or,} \quad \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\sim \quad \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} & \quad [r_2' = r_2 - r_1] \\
\sim \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} & \quad [r_1' = r_1 \times \frac{1}{2}]
\end{aligned}
$$

From this first row,

$$
\begin{aligned}
x_1 + 2x_2 &= 0 \\
x_1 &= -2x_2
\end{aligned}
$$

Taking $x_2 = 1$, we get $x_1 = -2$. Thus, the eigenvector can be written as,

$$v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Cayley-Hamilton proof:

We have to show that $A^2 - 4A - 5I = 0$.

This will square the matrix A, which is:

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now, $A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ [Proved]

δ Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

We know $|A - \lambda I| = 0$

$$or, \begin{vmatrix} 1 & -3 & 3 & & \lambda & 0 & 0 \\ 3 & -5 & 3 & - & 0 & \lambda & 0 \\ 6 & -6 & 4 & & 0 & 0 & \lambda \end{vmatrix} = 0$$

$$or, \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

from this we get the eq^n:

$$= (1-\lambda)\left((-5-\lambda)\cdot(4-\lambda) - (-6)\cdot 3\right) - (-3)\left(3\cdot(4-\lambda) - 6\cdot 3\right) + 3\left(3\cdot(-6) - 6\cdot(-5-\lambda)\right) = 0$$
$$\Rightarrow (1-\lambda)\cdot(\lambda^2+\lambda-2) - (9\lambda+18) + (18\lambda+36) = 0$$
$$\Rightarrow \lambda^2+\lambda-2-\lambda^3-\lambda^2-9\lambda-18+18\lambda+36 = 0$$
$$\Rightarrow \lambda^3 - 12\lambda - 16 = 0$$

> To solve a cube polynomial, we assume a value for the variable where the left side of the equation equals the right side after calculation. Here, for $\lambda = -2$, we get $(-2)^3 - 12(-2) - 16 = 0$. We use that value to get the first component, here it's $\lambda - (-2)$ or $(\lambda + 2)$ in short.

$$\lambda^3 - 12\lambda - 16 = 0$$
$$or, \lambda^3 + 2\lambda^2 - 2\lambda^2 - 4\lambda - 8\lambda - 16 = 0$$
$$or, \lambda^2(\lambda+2) - 2\lambda(\lambda+2) - 8(\lambda+2) = 0$$
$$or, (\lambda+2)(\lambda^2 - 2\lambda - 8) = 0$$

here

$$(\lambda+2) = 0$$
$$\rightarrow \lambda = -2$$

$$or, (\lambda^2 - 2\lambda - 8) = 0$$
$$or, \lambda^2 + 2\lambda - 8 = 0$$
$$or, (\lambda - 4)(\lambda + 2) = 0$$
$$\rightarrow \lambda = 4$$
$$\rightarrow \lambda = -2$$

λ could be either -2 or 4 For λ = -2 , we get from (A-λI)v = 0:

$$\text{or,} \quad \begin{bmatrix} 1+2 & -3 & 3 \\ 3 & -5+2 & 3 \\ 6 & -6 & 4+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\text{or,} \quad \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} 3 & -3 & 3|0 \\ 0 & 0 & 0|0 \\ 0 & 0 & 0|0 \end{bmatrix} \begin{bmatrix} r_2' = r_2 - r_1 \\ x_3' = r_3 - 2r_1 \end{bmatrix} = 0$$

From the first row , we get :

$$3x_1 - 3x_2 + 3x_3 = 0$$

$$or, x_1 - x_2 + x_3 = 0$$

$$or, x_1 = x_2 - x_3$$

$$or, x_1 = a - b$$

$$\text{say,} \quad x_2 = a \quad x_3 = b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a-b \\ a \\ b \\ \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, we get $V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 4$, we get from $(A - \lambda I)v = 0$:

$$\text{or,} \quad \begin{bmatrix} 1-4 & -3 & 3 \\ 3 & -5-4 & 3 \\ 6 & -6 & 4-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\text{or,} \quad \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} -3 & -3 & 3|0 \\ 0 & -12 & 6|0 \\ 0 & -12 & 6|0 \end{bmatrix} \begin{bmatrix} r_2' = r_2 + r_1 \\ r_3' = r_3 + 2r_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -3 & 3|0 \\ 0 & -12 & 6|0 \\ 0 & 0 & 0|0 \end{bmatrix} \begin{bmatrix} r_3" = r_3' - r_2' \end{bmatrix}$$

From the second row, we get:

$$-12x_2 + 6x_3 = 0$$

$$2x_2 = x_3$$

Now, from the first row:

$$-3x_1 - 3x_2 + 3x_3 = 0$$

$$-x_1 - x_2 + 2x_2 = 0$$

$$x_1 = x_2$$

$$x_1 = c$$

$$\text{say,} \quad x_1 = c \quad x_2 = c \quad x_3 = 2c$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ 2c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Thus, we get $V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

δ Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

System of Equation

$$\begin{aligned}
a_{11}x_1 + \quad & a_{12}x_2 + \quad & a_{13}x_3 \quad & = b_1 \\
a_{21}x_1 + \quad & a_{22}x_2 + \quad & a_{23}x_3 \quad & = b_2 \\
\vdots \quad & \vdots \quad & \vdots \quad & \vdots \\
a_{n1}x_1 + \quad & a_{n2}x_2 + \quad & a_{n3}x_3 \quad & = b_n
\end{aligned}$$

or,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Here,

$$AX = B$$

Let,

$$C = [A|B]$$

Then,for finding solution of we look for:

(a) Consistent eq^n :
If Rank A = Rank C
(i) Unique sol^n : Rank A = Rank $C = n$;
(ii) Infinite sol^n : Rank A = Rank $C = n < r$;
(b) Inconsistent eq^n :
If Rank $A \neq$ Rank C

Example:
Test for consistency and solve:

$$\begin{cases} 5x + 3y + 7z &= 4, \\ 3x + 26y + 2z &= 9, \\ 7x + 2y + 10z &= 5. \end{cases}$$

Solution:

$$C = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{array}\right] \begin{bmatrix} r_2' = 5r_2 - 3r_1 \\ r_3' = 5r_3 - 7r_1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} r_3'' = 11r_3' + r_2' \end{bmatrix}$$

Here, the n = 3; Rank of A = 2; Rank of C = 2;
Thus it is a consistent equation with infinite amount of solutions.