Matrix, Linear Algebra, Differential Equation
MAT 2207

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Matrix

Definition of Matrix:

A system of any $m \times n$ numbers arranged in a rectangular arrangement of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix.

Ex:

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & 7 \end{bmatrix} \qquad \text{is a } 2 \times 3 \text{ matrix.}$$

in general form:

$$A = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} = (\sigma_{ij})_{mxm}$$

Singular and Non-singular Matrix:

Let A be any square matrix. If $\det(A) = 0$, then A is called a singular matrix, and if $\det(A) \neq 0$, then A is called a non-singular matrix.

Ex: $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix}$

Then $|A| = \det(A) = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 4 - 4 = 0$
So, A is a singular matrix

Again, $|B| = \det(B) = \det \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix} = 12 - 10 = 2 \neq 0$
Hence, B is a non-singular matrix.

Inverse Matrix:

Let A and B be two $n \times n$ square matrices such that $AB = BA = I_n = I$, then B is said to be the inverse of A, and we write $B = A^{-1}$. Also, $A = B^{-1}$.

Ex: Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$.

$$\therefore\ AB = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -12+12 \\ 1-1 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\text{and } BA = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 3-3 \\ -4+4 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore AB = BA = I_2 = I$

Therefore, we can write $A = B^{-1}$ and $B = A^{-1}$.

[N.B.: The inverse of a matrix exists only when the matrix is non-singular, i.e., $|A| \neq \emptyset$.]

*** Multiplication of two matrices is possible only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Echelon Matrix:

Let $A = (a_{ij})_{m \times n}$ be any matrix. Then A is said to be an echelon matrix or is said to be in echelon form if:

1. all the non-zero rows (if any) precede the zero rows,

2. the number of zero entries preceding the first non-zero entry in each row increases by row.

Ex:

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ is an echelon matrix, but $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix}$ is not an echelon matrix.

Rank of a Matrix:

Rank of a matrix is the largest non-zero row in the matrix of row echelon form.

Ex: $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

Rank of matrix A = 3, Rank of matrix B = 2

Find the rank of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 5 & 4 & -4 \\ 0 & 5 & 6 & -5 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 2r1 \\ r3' = r3 - 3r1 \\ r4' = r4 - 4r1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r4' \\ r4'' = r4' - r3' \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r4''' = r4'' + r3'' \end{bmatrix}$$

This is the echelon form of matrix A ; and the Rank of A is 3.

$$B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -5 & -10 & -5 & 5 \\ 0 & -4 & -8 & -4 & 4 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - r1 \\ r3' = r3 - 2r2 \\ r4' = r4 - 3r2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - 5r4' \\ r4'' = r4' + 4r2' \end{bmatrix}$$

This is the echelon form of matrix B ; and the Rank of B is 2.

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 3r1 \\ r3' = r3 - 2r1 \\ r4' = r2 - 3r4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r2' \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} r4''' = r4' - r3'' \end{bmatrix}$$

This is the echelon form of matrix C ; and the Rank of C is 4.

$$D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} r2' = r2 - 2r1 \\ r3' = r3 - r1 \\ r4' = r4 - 2r2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} r3'' = r3' - r2' \\ r4'' = r4' - 4r3' \end{bmatrix}$$

This is the echelon form of matrix D ; and the Rank of D is 4.

Inverse Matrix Calculation:

Find the inverse of the matrix by using the formula [A:I]

example; find the inverse matrix of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Solution:

$$A = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{l} r2' = r2 - 3r1 \\ r3' = r3 - 2r1 \\ r4' = r4 - r2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & -1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r3'' = r3' - r2' \\ r4'' = 3r4' - r3' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r4''' = r4'' + 2r3'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1' = 3r1 + r2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 2 & 2 & -1 & 2 & 0 \\ 0 & 3 & 0 & -1 & -7 & 5 & -4 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1'' = r1' + 2r3'' \\ r2''' = r2'' - 4r3'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 3 & 0 & -6 \\ 0 & 3 & 0 & 0 & -6 & 3 & -3 & 3 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1''' = r1'' - 2r4''' \\ r2'''' = r2''' + r4'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{l} r1'''' = r1''' \times 1/3 \\ r2''''' = r2'''' \times 1/3 \\ r3''' = r3'' \times -1 \end{array}\right]$$

Home Work:

find the inverse matrix of $B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$

$$B = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{c} r2' = r2 + 2r1 \\ r3' = r3 + 4r1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & -2 & 1 & 0 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] \quad \left[\begin{array}{c} r3'' = -5r3' + 6r2' \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 0 & -3 & 1 & -2 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] \quad \left[\begin{array}{c} r1' = 5r1 + 2r2' \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 0 & 0 & 25 & -20 & 15 \\ 0 & 5 & 0 & 50 & -35 & 30 \\ 0 & 0 & -1 & 8 & -6 & -5 \end{array}\right] \quad \left[\begin{array}{c} r1'' = r1' - 3r3'' \\ r2'' = r2' - 6r3'' \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & 3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \quad \left[\begin{array}{c} r1''' = r1'' \times (-1/5) \\ r2''' = r2'' \times (1/5) \\ r3''' = r3'' \times (-1) \end{array}\right]$$

$$C = \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{c} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \\ r_4' = r_4 - r_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_4'' = 7r_4' - r_3' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & -5 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_1' = r_1 + 3r_2' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 7 & 0 & 0 & -1 & -34 & 21 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_1'' = 7r_1' - r_3' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 7 & 0 & 0 & 0 & -28 & 21 & 0 & -7 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & -49 & 0 & -7 & 56 \\ 0 & 0 & 0 & -1 & -6 & 0 & -1 & 7 \end{array}\right] \quad \left[\begin{array}{c} r_1''' = r_1'' - r_4'' \\ r_3'' = r_3' + 8r_4'' \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -7 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & 6 & 0 & 1 & -7 \end{array}\right] \quad \left[\begin{array}{c} r_1'''' = r_1''' \times 1/7 \\ r_2'' = r_2' \times (-1) \\ r_3''' = r_3'' \times 1/7 \\ r_4''' = r_4'' \times (-1) \end{array}\right]$$

Eigenvalues and Eigenvectors:

A nonzero matrix X is an eigenvector of a square matrix A if there exists a λ such that $AX = \lambda X$. Then X is called the eigenvector of A with eigenvalue λ and λ is called the eigenvalue of A.

The equation $|A - \lambda I| = 0$ is called the characteristic equation of A.

Cayley-Hamilton theorem states that every square matrix A satisfies it's characteristic equation.

Example:

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ in the field of real numbers ($\mathbb{R}$). Also verify the Cayley-Hamilton theorem.

Sol$_n$:

Given matrix, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Characteristic matrix of A inverse:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \end{aligned}$$

Characteristic equation of A inverse:

$|A - \lambda I| = 0$
or,

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(3-\lambda) - 8 = 0$
or, $3 - 4\lambda + \lambda^2 - 8 = 0$
or, $\lambda^2 - 4\lambda - 5 = 0$
or, $\lambda^2 - 5\lambda + \lambda - 5 = 0$
or $\lambda = 1$ or $\lambda = -5$

thus, $\lambda = 1, -5$;
these are the eigenvalues of A.

For $x = 5$,

$$
\begin{aligned}
(A - \lambda I)v &= 0 \\
\text{or,} \quad \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\
\text{or,} \quad \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0
\end{aligned}
$$

Now, we can write,

$$
\begin{aligned}
-4x_1 + 4x_2 &= 0 \\
2x_1 - 2x_2 &= 0 \\
\text{or,} \quad \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\sim \quad \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} & \quad [r_2' = 2r_2 + r_1] \\
\sim \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} & \quad [r_1' = r_1 \times \frac{1}{4}]
\end{aligned}
$$

From this first row,

$$
\begin{aligned}
-x_1 + x_2 &= 0 \\
x_1 = x_2 &= s \quad \text{(say)}
\end{aligned}
$$

Taking $s = 1$, the eigenvector can be written as,

$$
v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Now for $\lambda = -1$,

$$
\begin{aligned}
(A - \lambda I)v &= 0 \\
\text{or,} \quad \begin{bmatrix} 1-(-1) & 4 \\ 2 & 3-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\
\text{or,} \quad \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0
\end{aligned}
$$

Now, we can write,

$$\begin{aligned}
2x_1 + 4x_2 &= 0 \\
2x_1 + 4x_2 &= 0 \\
\text{or,} \quad \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\sim \quad \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} & \quad [r_2' = r_2 - r_1] \\
\sim \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} & \quad [r_1' = r_1 \times \frac{1}{2}]
\end{aligned}$$

From this first row,

$$\begin{aligned}
x_1 + 2x_2 &= 0 \\
x_1 &= -2x_2
\end{aligned}$$

Taking $x_2 = 1$, we get $x_1 = -2$. Thus, the eigenvector can be written as,

$$v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Cayley-Hamilton proof:

We have to show that $A^2 - 4A - 5 = 0$.

$$|\text{Det}(A)| = \left| \text{Det} \left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right) \right| = |(1 \times 3) - (4 \times 2)| = |3 - 8| = 5$$

Thus,

Left side $= A^2 - 4A - 5 = 5^2 - 4 \times 5 - 5 = 25 - 20 - 5 = 0 =$ Right side [Proofed]