Matrix, Linear Algebra, Differential Equation
MAT 2207

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Matrix

Definition of Matrix:

A system of any $m \times n$ numbers arranged in a rectangular arrangement of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix.

Ex:

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & 7 \end{bmatrix} \qquad\qquad \text{is a } 2 \times 3 \text{ matrix.}$$

in general form:

$$A = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} = (\sigma_{ij})_{mxm}$$

Singular and Non-singular Matrix:

Let A be any square matrix. If $\det(A) = 0$, then A is called a singular matrix, and if $\det(A) \neq 0$, then A is called a non-singular matrix.

Ex: $\qquad\qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad\qquad\qquad\qquad B = \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix}$

Then $|A| = \det(A) = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 4 - 4 = 0$

So, A is a singular matrix

Again, $|B| = \det(B) = \det \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix} = 12 - 10 = 2 \neq 0$

Hence, B is a non-singular matrix.

Inverse Matrix:

Let A and B be two $n \times n$ square matrices such that $AB = BA = I_n = I$, then B is said to be the inverse of A, and we write $B = A^{-1}$. Also, $A = B^{-1}$.

Ex: Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$.

$$\therefore AB = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -12+12 \\ 1-1 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and $BA = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4-3 & 3-3 \\ -4+4 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore AB = BA = I_2 = I$

Therefore, we can write $A = B^{-1}$ and $B = A^{-1}$.

[N.B.: The inverse of a matrix exists only when the matrix is non-singular, i.e., $|A| \neq \emptyset$.]

*** Multiplication of two matrices is possible only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Echelon Matrix:

Let $A = (a_{ij})_{m \times n}$ be any matrix. Then A is said to be an echelon matrix or is said to be in echelon form if:

1. all the non-zero rows (if any) precede the zero rows,

2. the number of zero entries preceding the first non-zero entry in each row increases by row.

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Ex:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} \text{ is an echelon matrix, but } \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix} \text{ is not an echelon matrix.}$$

Rank of a Matrix:

Rank of a matrix is the largest non-zero row in the matrix of row echelon form.

Ex:
$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix $A = 3$,   Rank of matrix $B = 2$

Find the rank of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$
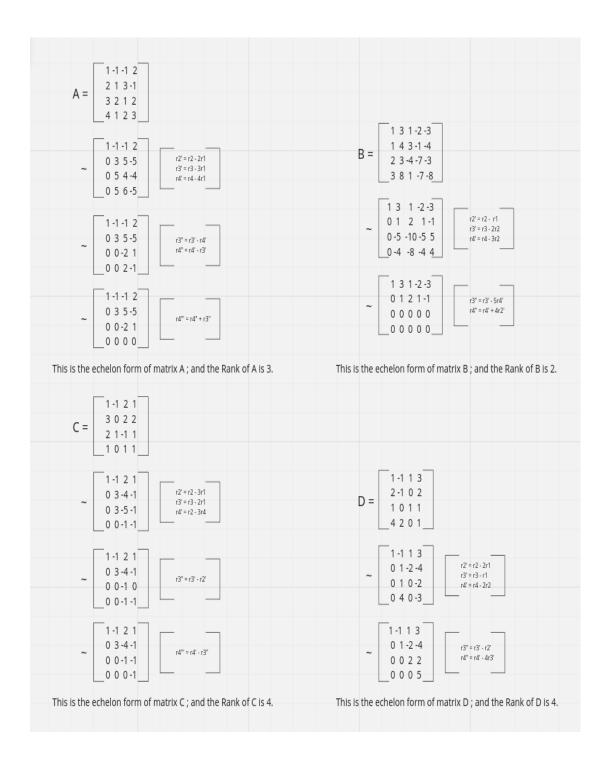
$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 5 & 4 & -4 \\ 0 & 5 & 6 & -5 \end{bmatrix} \qquad \begin{matrix} r2' = r2 - 2r1 \\ r3' = r3 - 3r1 \\ r4' = r4 - 4r1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \qquad \begin{matrix} r3'' = r3' - r4' \\ r4'' = r4' - r3' \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad r4''' = r4'' + r3''$$

This is the echelon form of matrix A ; and the Rank of A is 3.

$$B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -5 & -10 & -5 & 5 \\ 0 & -4 & -8 & -4 & 4 \end{bmatrix} \qquad \begin{matrix} r2' = r2 - r1 \\ r3' = r3 - 2r2 \\ r4' = r4 - 3r2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} r3'' = r3' - 5r4' \\ r4'' = r4' + 4r2' \end{matrix}$$

This is the echelon form of matrix B ; and the Rank of B is 2.

$$C = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \qquad \begin{matrix} r2' = r2 - 3r1 \\ r3' = r3 - 2r1 \\ r4' = r2 - 3r4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \qquad r3'' = r3' - r2'$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad r4''' = r4' - r3''$$

This is the echelon form of matrix C ; and the Rank of C is 4.

$$D = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & 0 & -3 \end{bmatrix} \qquad \begin{matrix} r2' = r2 - 2r1 \\ r3' = r3 - r1 \\ r4' = r4 - 2r2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad \begin{matrix} r3'' = r3' - r2' \\ r4'' = r4' - 4r3' \end{matrix}$$

This is the echelon form of matrix D ; and the Rank of D is 4.

Inverse Matrix Calculation:

Find the inverse of the matrix by using the formula [A:I]

example; find the inverse matrix of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Solution:

$A = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$ r1' = 3r1 + r2

$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$ r2' = r2 - 3r1
r3' = r3 - 2r1
r4' = r4 - r2

$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 2 & 2 & -1 & 2 & 0 \\ 0 & 3 & 0 & -1 & -7 & 5 & -4 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$ r1" = r1' + 2r3"
r2" = r2" - 4r3"

$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & -1 & 3 \end{array}\right]$ r3" = r3' - r2'
r4" = 3r4' - r3'

$\sim \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 3 & 0 & -6 \\ 0 & 3 & 0 & 0 & -6 & 3 & -3 & 3 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$ r1''' = r1" - 2r4'''
r2''' = r2" + r4"

$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$ r4''' = r4" + 2r3"

$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$ r1'''' = r1''' x 1/3
r2'''' = r2''' x 1/3
r3''' = r3" x -1

find the inverse matrix of $\quad B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$

$$B = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 0 & -3 & 1 & -2 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] \qquad r1' = 5r1 + 2r2'$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} r2' = r2 + 2r1 \\ r3' = r3 + 4r1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 0 & 0 & 25 & -20 & 15 \\ 0 & 5 & 0 & 50 & -35 & 30 \\ 0 & 0 & -1 & 8 & -6 & -5 \end{array}\right] \qquad \begin{array}{l} r1'' = r1' - 3r3'' \\ r2'' = r2' - 6r3'' \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & -2 & 1 & 0 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] \qquad r3'' = -5r3' + 6r2'$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & 3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \qquad \begin{array}{l} r1''' = r1'' \times (-1/5) \\ r2''' = r2'' \times (1/5) \\ r3''' = r3'' \times (-1) \end{array}$$

$$C = \begin{bmatrix} 1 & 3 & 1 & 1 & | & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & | & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & | & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 2 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & | & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & | & -1 & 0 & 0 & 1 \end{bmatrix}$$
r2' = r2 - 2r1
r3' = r3 - r1
r4' = r4 - r1

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & | & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & | & -6 & 0 & -1 & 7 \end{bmatrix}$$
r4'' = 7r4' - r3'

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & | & -5 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & | & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & | & -6 & 0 & -1 & 7 \end{bmatrix}$$
r1' = r1 + 3r2'

$$\sim \begin{bmatrix} 7 & 0 & 0 & -1 & | & -34 & 21 & -1 & 0 \\ 0 & -1 & 0 & 0 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & | & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & | & -6 & 0 & -1 & 7 \end{bmatrix}$$
r1'' = 7r1' - r3'

$$\sim \begin{bmatrix} 7 & 0 & 0 & 0 & | & -28 & 21 & 0 & -7 \\ 0 & -1 & 0 & 0 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & | & -49 & 0 & -7 & 56 \\ 0 & 0 & 0 & -1 & | & -6 & 0 & -1 & 7 \end{bmatrix}$$
r1''' = r1'' - r4''
r3'' = r3' + 8r4''

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & | & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & | & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & -7 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & | & 6 & 0 & 1 & -7 \end{bmatrix}$$
r1'''' = r1''' x 1/7
r2'' = r2' x (-1)
r3''' = r3'' x 1/7
r4''' = r4'' x (-1)

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Eigenvalues and Eigenvectors:

A nonzero matrix X is an eigenvector of a square matrix A if there exists a λ such that $AX = \lambda X$. Then X is called the eigenvector of A with eigenvalue λ and λ is called the eigenvalue of A.

The equation $|A - \lambda I| = 0$ is called the characteristic equation of A.

Cayley-Hamilton theorem states that every square matrix A satisfies it's characteristic equation.

Example:

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ in the field of real numbers ($\mathbb{R}$). Also verify the Cayley-Hamilton theorem.

$\underline{\textit{Sol}_n}$:

Given matrix, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Characteristic matrix of A inverse:

$$A - \lambda I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{bmatrix}$$

Characteristic equation of A inverse:

$|A - \lambda I| = 0$
or,

$$\begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} = 0$$

or, $(1 - \lambda)(3 - \lambda) - 8 = 0$
or, $3 - 4\lambda + \lambda^2 - 8 = 0$
or, $\lambda^2 - 4\lambda - 5 = 0$
or, $\lambda^2 - 5\lambda + \lambda - 5 = 0$
or $\lambda = 1$ or $\lambda = -5$

thus, $\lambda = 1, -5$;
these are the eigenvalues of A.

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For $x = 5$,

$$(A - \lambda I)v = 0$$

$$\text{or,} \quad \begin{bmatrix} 1 - 5 & 4 \\ 2 & 3 - 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\text{or,} \quad \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Now, we can write,

$$-4x_1 + 4x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

$$\text{or,} \quad \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sim \quad \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \quad [r_2' = 2r_2 + r_1]$$

$$\sim \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad [r_1' = r_1 \times \frac{1}{4}]$$

From this first row,

$$-x_1 + x_2 = 0$$
$$x_1 = x_2 = s \quad \text{(say)}$$

Taking $s = 1$, the eigenvector can be written as,

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now for $\lambda = -1$,

$$(A - \lambda I)v = 0$$

$$\text{or,} \quad \begin{bmatrix} 1 - (-1) & 4 \\ 2 & 3 - (-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\text{or,} \quad \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Now, we can write,

$$2x_1 + 4x_2 = 0$$
$$2x_1 + 4x_2 = 0$$
$$\text{or,} \quad \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \quad [r_2' = r_2 - r_1]$$
$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad [r_1' = r_1 \times \frac{1}{2}]$$

From this first row,

$$x_1 + 2x_2 = 0$$
$$x_1 = -2x_2$$

Taking $x_2 = 1$, we get $x_1 = -2$. Thus, the eigenvector can be written as,

$$v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Cayley-Hamilton proof:

We have to show that $A^2 - 4A - 5I = 0$.
  This will square the matrix A, which is:

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now, $A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ [Proved]