Linear combination

Let, $V(F)$ be a vector space where $v_1, v_2 \cdots v_n \in V$ and $\alpha_1, \alpha_2 \cdots \alpha_n \in F$

If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = u \in V$, then u is called the linear combination of the vectors $v_1, v_2 \cdots v_n$.

Linear Dependence

Let, $V(F)$ be a vector space where $v_1, v_2 \cdots v_n \in V$ and $\alpha_1, \alpha_2 \cdots \alpha_n \in F$

If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ [zero vector] at least one of elements of the set $\{\alpha_1, \alpha_2 \cdots \alpha_n\}$ is not zero, then the vectors $v_1, v_2 \cdots v_n$ are called linearly dependent.

Linear independent

Let, $V(F)$ be a vector space where $v_1, v_2 \cdots v_n \in V$ and $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n \in F$

If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ [zero vector] and all the elements of the set $\{\alpha_1, \alpha_2 \cdots \alpha_n\}$ are zero, then the vectors $v_1, v_2 \cdots v_n$ are called linearly independent.

Q) Write the linear combination of v with respect to v_1, v_2 and v_3, where
$v = (5,3,1)$, $v_1 = (1,1,-1)$, $v_2 = (1,-1,1)$, $v_3 = (-1,1,1)$

Soln

Let, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \longrightarrow (1)$ where $\alpha_1, \alpha_2, \alpha_3$ scalar.

or, $(5,3,1) = \alpha_1(1,1,-1) + \alpha_2(1,-1,1) + \alpha_3(-1,1,1)$

or, $(5,3,1) = (\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 - \alpha_2 + \alpha_3, -\alpha_1 + \alpha_2 + \alpha_3)$

Now we can write,

$$\alpha_1 + \alpha_2 - \alpha_3 = 5$$
$$\alpha_1 - \alpha_2 + \alpha_3 = 3$$
$$-\alpha_1 + \alpha_2 + \alpha_3 = 1$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 1 & -1 & 1 & 3 \\ -1 & 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} r_2' = r_1 - r_2 \\ r_3' = r_2 + r_3 \end{array}$$

Now we can write, from R_3

$$2\alpha_3 = 4$$
$$\Rightarrow \alpha_3 = 2$$

Again, from r_2

$2\alpha_2 - 2\alpha_3 = 2$

$\Rightarrow 2\alpha_2 - 2\cdot 2 = 2 \quad [\alpha_3 = 2]$

$\Rightarrow 2\alpha_2 = 6$

$\Rightarrow \alpha_2 = 3$

from r_1,

$\alpha_1 + \alpha_2 - \alpha_3 = 5$

$\Rightarrow \alpha_1 + 3 - 2 = 5$

$\Rightarrow \alpha_1 = 4$

Replacing $\alpha_1, \alpha_2, \alpha_3$ in eqn (1)

$V = 4V_1 + 3V_2 + 2V_3$

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Show that, the following vectors are linearly independent / dependent:

i. $V_1 = (1,1,1)$ $V_2 = (1,2,3)$, $V_3 = (2,-1,1)$

Sol^n

Let, $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = (0,0,0)$

or, $\alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(2,-1,1) = (0,0,0)$

or, $(\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3) = (0,0,0)$

Now we can write

$$\alpha_1+\alpha_2+2\alpha_3=0$$
$$\alpha_1+2\alpha_2-\alpha_3=0$$
$$\alpha_1+3\alpha_2+\alpha_3=0$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \quad r_3'' = r_3 - r_2$$

From row r_3

$5\alpha_3 = 0 \quad = \alpha_3 = 0$

From r_2

$\alpha_2 - 3\alpha_3 = 0$

$\Rightarrow \alpha_2 - 3\cdot 0 = 0 \quad \Rightarrow \alpha_2 = 0$

From r_1

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$

$\Rightarrow \alpha_1 + 0 + 0 = 0 \quad | \quad \alpha_1 = 0$

Here,

$\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

Since the value of every scalars are zero. The vectors are linearly independent.

Show that, the following vectors are linearly independent/dependent

i. $(1,2,3,4), (2,1,0,0), (2,4,6,8)$

Given, $v_1 = (3,1,1,-1)$, $v_2 (1,1,1,0)$, $v_3 = (0,-1,1,1)$

$v_4 = (0,2,0,-1)$

write vector v_1 as a linear combination of v_2, v_3 and v_4

Sol (i)

let, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0,0,0$

$\Rightarrow \alpha_1(1,2,3,4) + \alpha_2(2,1,0,0) + \alpha_3(2,4,6,8) = 0,0,0$

$\Rightarrow (\alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + \alpha_2 + 4\alpha_3, 3\alpha_1 + 6\alpha_3, 4\alpha_1 + 8\alpha_3)$
$= (0,0,0)$

Now we can write,

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$

$2\alpha_1 + \alpha_2 + 4\alpha_3 = 0$

$3\alpha_1 + 6\alpha_3 = 0$

$4\alpha_1 + 8\alpha_3 = 0$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 3 & 0 & 6 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_2' = 2r_1 - r_2 \\ r_3' = 3r_1 - r_3 \\ r_4' = 4r_1 - r_4 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_3'' = 2r_2' - r_3' \\ r_4' = 8r_3'' - 6r_4'' \end{array}$$

from R_2

$3\alpha_2 = 0$

$\Rightarrow \alpha_2 = 0$

from R_1

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$

$\Rightarrow \alpha_1 = -2\alpha_3$

$\Rightarrow \alpha_1 = -2a$ [say $\alpha_3 = a$]

let $a = 1$,

$\therefore \alpha_1 = -2, \alpha_2 = 0, \alpha_3 = 1$

Since the value of every scalar are ~~zero~~ not zero

$\therefore$ The vectors are linearly dependent,

(ii) ans

Let, $v_1 = \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$

$\Rightarrow \alpha_2(1,1,1,0) + \alpha_3(0,-1,1,1) + \alpha_4(0,2,0,-1) = (3,1,1,-1)$

$\Rightarrow \alpha_2, \alpha_2 - \alpha_3 + 2\alpha_4, \alpha_2 + \alpha_3, \alpha_3 - \alpha_4 = 3,1,1,-1$

Now we can write,

$$\alpha_2 = 3$$
$$\alpha_2 - \alpha_3 + 2\alpha_4 = 1$$
$$\alpha_2 + \alpha_3 = 1$$
$$\alpha_3 - \alpha_4 = -1$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} \\ \\ r_3'' = 2r_2' + r_3' \\ r_4'' = 2r_4' - r_3 \end{array}$$

Here, $0 = -2$ which is inconsistant

Therefore, the vector v_1 can not be expressed as a linear combination of v_2, v_3, v_4.