2.a

$$\mathbb{E}[\bar{y}_w] = \mathbb{E}[w_1 Y_1 + w_2 Y_2] \quad , \quad w_1 = w \text{ and } w_2 = 1-w$$

$$= \mathbb{E}[w_1 Y_1] + \mathbb{E}[w_2 Y_2]$$

$$= w_1 \mathbb{E}[Y_1] + w_2 \mathbb{E}[Y_2]$$

$$= w\,\mathbb{E}[Y_1] + (1-w)\mathbb{E}[Y_2]$$

$$= \mathbb{E}[Y_2] + w(\mathbb{E}[Y_1] - \mathbb{E}[Y_2])$$

As $\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = \mu$:

$\mathbb{E}[\bar{y}_w] = \mu + w(\mu - \mu) = \mu$ $\longrightarrow$ Always unbiased regardless of w

2.b

$$Var(\bar{y}_w) = Var(w_1 Y_1 + w_2 Y_2)$$

$$= Var(w Y_1 + (1-w) Y_2)$$

$$= Var(w Y_1) + Var((1-w) Y_2)$$

$$= w^2 Var(Y_1) + (1-w)^2 Var(Y_2)$$

$$= w^2 \sigma_1^2 + (1-w)^2 \sigma_2^2$$

2.c

$$Var[\bar{y}_w] = w^2\sigma_1^2 + (1-2w+w^2)\sigma_2^2$$

$$= w^2\sigma_1^2 + \sigma_2^2 - 2w\sigma_2^2 + w^2\sigma_2^2$$

Take derivative to find minimum

$$\frac{d}{dw} Var[\bar{y}_w] = 2\sigma_1^2 w - 2\sigma_2^2 + 2\sigma_2^2 w$$

$$= 2w(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2$$

when $2w(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0$

$$w(\sigma_1^2+\sigma_2^2) = \sigma_2^2$$

$$w = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

Thus $w_1 = w = \dfrac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$, $w_2 = 1-w = 1 - \dfrac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \dfrac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$

Hence, w_i is inversely proportional to σ_i

$$\Longrightarrow w_i \propto \frac{1}{\sigma_1}$$