ALGORITHM FOR APPROXIMATING NASH SOCIAL WELFARE WITH RADO VALUATIONS

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Abstract

This paper[3] proposes a constant factor approximation Nash social welfare algorithm under Rado valuation. The authors use a mixed integer programming relaxation algorithm. I restate the author's thoughts according to my own understanding. But there may be some deficiencies and mistakes, I hope readers understand.

Keywords: Nash social welfare, approximation algorithm, Rado valuations

Contents

1. Introduction

1.1 The reason why I choose this

To be honest, when I first got into game theory, I fantasized that it was a very macro subject, but over the course of this semester, I found that we are still learning game theory in a relatively small and abstract context. I want to learn some social game theory content through this article, so that I can understand the world from a new angle.

Of course, this article is not chosen by other students. As a student, I think it is normal to pursue high scores in a reasonable way. Therefore, because of interest and score I choose this topic as my term project assignment.

1.2 Relevant background knowledge

About origins of the problems, fair and efficient allocation of resources is a fundamental problem in many disciplines, including computer science, economics, and social choice theory; see, e.g., several excellent books written specifically on this problem.

The Nash social welfare emerged between the often conflicting requirements of fairness and efficiency.First, Nash emerged in 1950 as the only solution to bargaining games. It is also consistent with the competitive equilibrium concept of income equality in economics, and the proportional equity concept in networks . The above work considers symmetrical Nash social welfare problems. Since the 1970s, asymmetric goals have also been well studied and found many applications in different fields, such as bargaining theory , water resources allocation and climate agreements.

For NSW problem , it is where this paper makes significant progress. The (symmertric) NSW can be regarded as a balanced trade-off between two other popular social welfare concepts, which contain the utilitarian social welfare and the max-min fairness. The utilitarian social welfare means to maximize the sum of the valuations (we learn gernal, additive, subadditive valuations in class). The max-min fairness, people may know which in another way (Santa Claus Problem) means to maximizes the smallest valuations.

About computation complexity, the NSW problem is NP-hard though people consider about the number of agents is 2 with additive problem. Nowadays, in symmetric case, what people have done on this issue is that NP-hard to approximate within a factor better than 1.069 with additive valuations and better than 1.5819 with submodular valuations.

1.3 Major contribution of reference work

The work makes great progress towards both symmetric and asymmetric NSW problems. The work obtains a constant-factor approximation with Rado valuations. This is the first time I've heard this term. Rado valuation is a common generalization of OXSvaluations and weighted matroid functions. We can specify a Rado valuation by a bipartite $(G, V_i; E_i)$, edge costs c_i, and a matroid $M_i = (V_i, I_i)$.

The work secondly obtain a constant-factor approximation for the asymmetric NSW problem with Rado valuations which proves the maximum ratio between is bounded by a constant.

2. Preliminaries

In this chapter, I'm going to introduce definitions, assumptions and lemmas about this problem.

2.1 Basic Settings

We define G as a finite set of m indivisible items and $v_{ij} = v_i(\{j\})$ as the valuation of agent for item j. For any subset $S \subseteq G$, we use $\aleph_S$ as the characteristic vector of S. Also, we define a bipartite graph$(U, V; E)$, a matching σ from U to V, such that a mapping $U \to V \cup \{\emptyset\}$. Given $I \subseteq 2^V$is a nonempty collection of independent sets. Among other things, the collection needs to meet these conditions(We call it the independence axioms): **Monotonicity :** for $\forall Y \subseteq X$, if $X \in I$, then $Y \subseteq X$. **Exchange property:** for $\forall |X| < |Y|$,if $X, Y \in Y$,then there exists a $y \in Y \backslash X$.

The rank function r_M:$2^V \to Z_+$ associated with the matroid M is defined with $r_M(X)$ representing the size of the largest independent subset of $X \subseteq V$.

So that, we can obtain a fundamental property implied by Exchange property that every maximal independent set in X has size $r_M(X)$. Therefore, we call $r_M(X)$ the rank of the matroid and the maximal independent sets bases.

2.2 Valuation Function

About valuation function , we means a function $2^G \to \mathbb{R}_+$ with $v(\emptyset) = 0$ we define .

Several basic properties is below.

monotone: if $v(X) \leq v(Y), \forall X \subseteq Y \subseteq G$.

subadditive: if $v(X) + v(Y) \geq v(X \cup Y), \forall X, Y \subseteq G$.

submodular: if $v(X) + v(Y) \geq v(X \cup Y) + v(X \cap Y), \forall X, Y \subseteq G$.

Additive valuations and unit demand valuations satisfy all the above properties.

Gross Subtitute Valuations: For a price vector $p \in \mathbb{R}^G$and a subset $S \subseteq G$, we define $p(S) = \sum_{j \in S} p_j$. For a normal valuation function $v : 2^G \to \mathbb{R}_+$, the utility obtainable at prices p from a set $S \subseteq G$ is $v(S) - p(S)$. The set of optimal bundles at price p is called demand correspondence and is defined as the set of bundles maximizing the utility, i.e.,$D(v, p) = argmax_{S \subseteq G} v(S - p(S))$.

Definition 2.1. For a valuation $v : 2^G \to i$ is a gross substitutes valuation it for $\forall p, p' \in \mathbb{R}^G$, such that $p' \geq p$and $\forall S \in D(v, p)$, there $\exists S^{' \in D(v,p')}$. For an example, $S \cap \{j : p_j = p'_j\} \subseteq S'$.

Definition 2.2. For a valuation $v : 2^G \to \mathbb{R}_+$ is an $M^\natural - concave$ if for $\forall X, Y \subseteq G$ and $x \in X \backslash Y$.

$v(X) + v(Y) \leq max_{Z \subseteq Y \backslash X, |Z| \leq 1} v((X \backslash \{x\})) + v((Y \backslash Z) \cup \{x\})$

THERORM 2.3. The valuation fuction $v : 2^G \to \mathbb{R}_+$ is a gross substitutes valuation if and only if it is $M^{-}concave$. [2]

Definition 2.4. Assume we are given a bipartite graph $(G, V; E)$ with a cost function c:$E \to \mathbb{R}_+$ on the edges, and a matroid $M = (V, I)$. For a subset of items $S \subseteq G$,the Rado valuation function $v(S)$ is defined as the maximum cost of a matching M in $(G, V; E)$such that $\delta_G(M) \subseteq S$ and $\delta)V(M) \in I$,i.e.,

$$v(S) := max\{\sum_{e\in M} c(e) : M\ is\ a\ matching, \delta_G(M) \subseteq S, \delta_V(M) \in I\} \tag{2.1}$$

Lemma 2.5.[4]Every Rado valuation $v : 2^G \to \mathbb{R}_+$is an $M^\natural - concave\ function.$

2.3 Continuous Valuation Functions

For continuous valuation function , we means that $v : [0, 1]^{\mathcal{G}} \to \mathbb{R}_+$,we have $v(0) = 0$. We sightly abuse v to represent as both discrete and continuous valuations. So that, we extend notions from discrete valuations to a function $\mathbb{R}_+^{\mathcal{G}} \to \mathbb{R}_+$.

The continuous function is monotone if $f(x) \le f(y), \forall x \le y, x, y \in \mathbb{R}_+^{\mathcal{G}}$, and subadditive if $f(x + y) \le f(x) + f(y), \forall x, y \in [0, 1]^{\mathcal{G}}$

Whereas our overall result requires the continuous extension of Rado valuations, much weaker assumptions suffice for most parts of the argument, as formulated next.

Assumption 1. For every agent $i \in \mathcal{A}$ the continuous valuation function $v_i : [0, 1]^{\mathcal{G}} \to \mathbb{R}_+$is monotone,concave and subadditive.

Concave Extensions of Discrete Valuations. For any valuation function v:$2^{\mathcal{G}} \to \mathbb{R}_+$, we can define the concave closure $\overline{v} : [0, 1]^{\mathcal{G}} \to \mathbb{R}_+$as $\overline{v} := min_{p\in\mathbb{R}^{\mathcal{G}}}\{< p, x > +\alpha : p(S)+\alpha \ge v(S, \forall S \subseteq \mathcal{G}\}$. e.g.[2],as the minimum of linear functions,$\overline{v}$ is always concave which helps the proof of the work greatly.

THEOREM 2.6. Consider a Rado valuation $v : 2^{\mathcal{G}} \to \mathbb{R}_+$,given by a bipartite graph($\mathcal{G}$,V,E) with costs on the edges c:$E \to \mathbb{R}_+$, and a matroid $\mathcal{M}$=(V,I) with a rank function $r = r_{\mathcal{M}}$ we define in **Definition 2.4.**. Now, for $x \in [0, 1]^{\mathcal{G}}$, we define

$$v(x) := max \sum_{(j,k)\in E} c_{jk}z_{jk} \qquad s.t. : \sum_{k\in V} z_{jk} \le x_j,\ \ \forall j \in \mathcal{G}; \sum_{j\in\mathcal{G},k\in T} z_{jk} \le r(T), \forall T \subseteq V; z \ge 0. \tag{2.2}$$

Then, v = $\overline{v}$ is the concave extension of v, and satisfies Assumption 1.

By the light of THEOREM 2.6., in the rest of our project we will denote by $v : [0, 1]^{\mathcal{G}} \to \mathbb{R}_+$the continuous Rado valuation defined in (2).

2.4 Simple Upper Bounds

In the rest of our project, we will often use the following simple bounds.So it's worth introducing these concepts

Lemma 2.7. Let $n, c \in \mathbb{N}, S \subseteq [n]$,and $1 \le w_1, ..., w_n \le \gamma - 1$. For $i \in S, let\ k_i \in \mathbb{R}_+$such that $\sum_{i\in S} k_i \le c \cdot n$. Then

$(\prod_{i\in S} k_i^{w_i})^{1/\sum_{i=1}^{n} w_i} \leq c\cdot\gamma$

Lemma 2.8. Let $n, c \in \mathbb{N}, S \subseteq [n]$.For $i \in S, let\ k_i \in \mathbb{R}_+$ such that $\sum_{i\in S} k_i \leq c\cdot n$. Then

$(\prod_{i\in S} k_i^{1/n}) \leq c\cdot e^{1/e}$

In this section , I prove **Lemma 2.7.** shortly.

By the inequality of weighted arithmetic and geometric means we have:

$(\prod_{i\in S} k_i^{w_i})^{1/\sum_{i=1}^{n} w_i} = \prod_{i\in S} k_i^{\frac{w_i}{\sum_{i=1}^{n} w_i}} \cdot \prod_{i\in[n]\setminus S} 1^{\frac{w_i}{\sum_{i=1}^{n} w_i}} \leq \prod_{i\in S} \frac{w_i k_i}{\sum_{i=1}^{n} w_i} + \prod_{i\in[n]\setminus S} \frac{w_i}{\sum_{i=1}^{n} w_i} \leq (\gamma - 1)\frac{\sum_{i\in S} k_i}{\sum_{i=1} n w_i} + 1 \leq c\cdot\gamma$

3. The Theorem And The Proof

3.1 main ideas

We need to be clear about our goals.Our approach is based on a mixed-integer programming relaxation, using a careful combination of convex programming relaxations and combinatorial arguments.I'm going to start with my thoughts.

Let v_i be a continuous valuation above first and $w_i > 0$be the weight for each agent $i \in \mathcal{A}$. Given a fractional allocation $x = (x_1, ..., x_n) \in \mathbb{R}_+^{\mathcal{A}\times\mathcal{G}}$, we set

$NSW(x) = (\prod_{i\in\mathcal{A}} v_i(x_i))^{1/\sum_i w_i}$

We can find that the NSW problem with continuous valuation function. But before, the NSW problem is given with discrete valuation functions $v_i : 2^{\mathcal{G}} \to \mathbb{R}_+$. In order to apply convex programming techniques, we need to obtain a convex programming relaxation. What we mentioned in the previous chapter(Section 2.2.) is that Gross Substitute valuations are the subclass of discrete valuations. There a concave extension can be naturally defined.

Then, the following integer program can catch the asymmetric Nash social welfare program.

$maxNSW(x)\ s.t. \sum_{i\in\mathcal{A}} x_{ij} \leq 1, \forall j \in \mathcal{G}, x \in 0, 1^E.(NSW - IP)$

On the basis of the above formula, let OPT denote the optimum value. The natural relaxation of NSW-IP is

$$maxNSW(x)\ s.t. \sum_{i\in\mathcal{A}} x_{ij} \leq 1, \forall j \in \mathcal{G}, x \geq 0.(NSW - IP) \tag{3.1}$$

The objective is log-concave assuming the vi 's are concave functions. [1] showed that this relaxation has unbounded integrality gap already for additive valuations.

Then, the work propose a mixed integer programming relaxation instead of the natural relaxation of NSW-IP(3). For a set of items $\mathcal{H} \subseteq \mathcal{G}$, we introduce a mixed relaxation which requires the items in $\mathcal{H}$ to be allocated integrally and the rest can be allocated fractionally.

$$\begin{gathered} maxNSW(x) \\ \text{s.t.} \sum_{i\in\mathcal{A}} x_{ij} \leq 1\ \forall j \in \mathcal{G} \\ \text{x}_{ij} \in \{0, 1\}\ \forall j \in \mathcal{H}, \forall i \in \mathcal{A} \\ \text{x} \geq 0 \end{gathered}$$

The above formula implies a clear relaxation of NSW-IP: $OPT_{\mathcal{H}} \geq OPT$, when $OPT_{\mathcal{H}}$ is optimal value of mixed relaxation for set $\mathcal{H}$we define.

THEOREM 3.0.(Main) There exists a polynomial-time $256\gamma^3$-approximation algorithm for the Nash social welfare problem with Rado valuation functions. For additive valuation functions, there exists a polynomial-time 16γ -approximation algorithm.

Table 3.1: Summary of the best approximation algorithms for the NSW problem.

Valuations	Symmetric	Asymmetric
Additive	1.45	$O(\gamma)$
SPLC	1.45	$O(\gamma^3)$
Rado	O(1)	$O(\gamma^3)$
Subadditive	O(n)	$O(n)$

We note that even if the weights of the agents are bounded, an O(1)-approximation for the symmetric case does not yield an O(1)-approximation to the asymmetric case. It is a important theorem for our following project. booktabs

Tablea 1 summarizes the updated best approximation guarantees for the problem under various valuation functions.

Theorem 3.0. is shown by constructing an integer allocation $x \in \{0,1\}^{\mathcal{A}\times\mathcal{G}}$ and an item set $\mathcal{H}$ such that $NSW(x) \geq OPT_{\mathcal{H}}/(256\gamma^3)$.

We will prove it in the following five phases.

3.2 PHASE 1: Finding the item set $\mathcal{H}$

As can be seen from the title, the discussion focus of this section is finding the item set $\mathcal{H}$.

Firstly, we need to know more information about the set of $\mathcal{H}$. For the set $\mathcal{H}$, we aim to identify the set of the 'most important' items. Our result implies that the mixed integer relaxation that requires $\mathcal{H}$to be integrally allocated has a constant gap in contrast to the standard continuous relaxation. From [1], all the 'most' important items in spending restricted equilibrium will be included in $\mathcal{H}$.

We achieve the max NSW with each agent only receiving a single item. Here are the implementation details.

We define $w_ij = w_i log(v_ij), \forall i \in \mathcal{A}, j \in \mathcal{G}$ which implies a complete bipartite graph between $\mathcal{A}$ and $\mathcal{G}$. From the notion $v_ij = v_i(\{j\})$ above, we naturally let $\tau : \mathcal{A} \to \mathcal{G}$ denoting the optimal matching. The existence of τ with finite weight proves that the instance is feasible.We can refer to [1], where the algorithm starts with such a matching.

For example, we set $\tau(i)$ is the item matched to agent $i \in \mathcal{A}$. We define $\mathcal{H}$ as the set of items assigned by τ, i.e. $\mathcal{H} = \tau(\mathcal{A})$. In this project we see $\mathcal{H}$ as the set of most preferred items.

That's it. We found $\mathcal{H}$.

3.3 PHASE 2:Reduction to the Mixed Matching Relaxation

The purpose of this section is to approximate (Mixed relaxation) by another integer program.

We approximate (Mixed relaxation) by a second mixed integer program. We use variables $y \in \mathbb{R}_+^{\mathcal{A}\times(\mathcal{G}\backslash\mathcal{H})}$ denoting the fractional allocations of the items in $\mathcal{G}\backslash\mathcal{H}$.If x_i is obtained from y_i by setting $x_ij = 0, j \in \mathcal{H} and\ x_ij = y_ij, for j \in \mathcal{G}\backslash\mathcal{H}$, $v_i(y_i)$ denotes $v_i(x_i)$.

We need to introduce the following program as the mixed matching relaxation.

$$
\begin{aligned}
&max(\prod_{i\in\mathcal{A}}(v_i(y_i)+v_i\sigma(i))^{1/\sum_i w_i})\\
&s.t.\ \sum_{i\in\mathcal{A}} y_ij \leq 1, \forall j\in\mathcal{G}\backslash\mathcal{H}\\
&y_ij \geq 0\ \ \forall j\in\mathcal{G}\backslash\mathcal{H}, \forall i\in\mathcal{A}\\
&\sigma:\mathcal{A}\rightarrow\mathcal{H}\ is\ a\ matching.
\end{aligned}
$$

This program (mixed matching relaxation) differs from mixed relaxation in two parts.On one hand, the objective in (mixed matching relaxation) differs from NSW(x): for each agent , its value is given by the Rado valuation.

How does 'mixed' work? In Mixed matching relaxation, we evaluate the utility of each agent separately on $\mathcal{H}$ and $\mathcal{G}\backslash\mathcal{H}$ and the them to sum up. On the other hand, we requires items in $\mathcal{H}$ are allocated to the agents by a matching. In this situation, there will not be a relaxation of NSW-IP:the optimal integer solution may allocate multiple items in $\mathcal{H}$ to the same agent.

We show that the effect of both these changes is limited. We define (y,σ)be a feasible solution to mixed matching relaxation. We define $\overline{NSW}(y,\sigma)$ as the objective function value in mixed matching relaxation, and let $\overline{OPT_{\mathcal{H}}}$represent the optimal value.

The same as usual as we define $NSW(y,\sigma)$ as the Nash social welfare of the same allocation. Namely, $NSW(y,\sigma) = NSW(x), x_ij = y_ij, if\ j\in\mathcal{G}\backslash\mathcal{H}$and we have $x_ij = 1$, if $j=\sigma(i)$,and $x_ij = 0$ otherwise. Now we're going to introduce some lemmas.

Lemma 3.1. For a feasible solution (y,σ) to mixed matching relaxation ,we have

$$\overline{NSW}(y,\sigma) \geq NSW(y,\sigma) \geq \frac{1}{2}\overline{NSW}(y,\sigma)$$

THEOREM 3.2. Let $\mathcal{H}\subseteq\mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. For the optimum values $OPT_{\mathcal{H}}$ to mixed relaxation to $\overline{OPT_{\mathcal{H}}}$ to mixed matching relaxation , we have

$$OPT_{\mathcal{H}} \geq \frac{1}{\gamma}\overline{OPT_{\mathcal{H}}}$$

We use lemma 3.1. as well as lemma 2.7. to relate the optimum values and approximate solutions of mixed relaxation and mixed matching relaxation.

3.4 PHASE 3: Approximating the Mixed Matching Relaxation

For solving (Mixed+matching) still does not turn out to be easy, our next goal is to find a $2-$approximation solution to mixed matching relaxation; we do not know whether the problem is polynomial-time solvable. By THEOREM 3.2,this yields a (4γ)-approximation to mixed relaxation, a mixed integer relaxation of ANSW problem.

In Mixed matching, we need to allocate items $\mathcal{G}$ to the agents in $\mathcal{A}$ in order to maximize an objective function that is an approximation of NSW. Items in $\mathcal{G}\backslash\mathcal{H}$ can be allocated fractionally to the agents without any constraints. The items in H have to be allocated integrally via an assignment, thereby allocating exactly one item from $\mathcal{H}$ to each agent $\mathcal{A}$.

We can 2-approximate the exact computational complexity of mixed matching while it remains unsolved.

Denote $\mathcal{L} = \mathcal{G} \backslash \mathcal{H}$. Let $\mathcal{A}^{'}$ be the subset of agents that have positive value of the items in $\mathcal{G} \backslash \mathcal{H}, \mathcal{A} := \{\ i \in \mathcal{A} : v_i(\mathcal{G} \backslash \mathcal{H}) > \prime\}$ as some agents may only have positive value for items in $\mathcal{H}$.Restricting mixed matching relaxation to the items $\mathcal{L}$ and agents $\mathcal{A}^{'}$ and taking the objective yields an instance of EG:

$$max \sum_{i \in \mathcal{A}'} w_i log v_i(y_i)$$

$$s.t. : \sum_{i \in \mathcal{A}'} y_{ij} \leq 1, \forall j \in \mathcal{L}$$

$$y_{ij} \geq 0, \forall j \in \mathcal{L}, \forall i \in \mathcal{A}^{'}$$

The above is a convex program whenever the valuations v_i are concave, and we can solve it to an arbitrary precision in polynomial time if we have access to a supergradient oracle to the objective function.

On the other hand, suppose that in mixed matching the variables y are fixed. Under the fixed y, we can find an optimal assignment σ. Namely, an optimal assignment is exactly a maximum weight assignment in the bipartite graph (A, H; E) where the weight of an edge ij for $i \in A, j \in H$ is $\omega_{ij} := w_i log(v_i(y_i) + v_{ij})$.

Informally, mixed matching relaxation is a combination of two tractable problems. We show that an optimal solution y^* to the restriction of the problem to $\mathcal{L}$ and A, and an optimal assignment with respect to the fixed y^* gives a 2-approximation for mixed matching relaxation.

3.4.1 Properties of Eisenberg–Gale Program

Let us now consider the Eisenberg–Gale program (EG). For concave valuations v_i, the above is a convex program. An optimal solution y^* and the optimal Lagrange multipliers p_j for $j \in \mathcal{L}$ can be interpreted as the so-called Gale equilibrium in the market with divisible items $\mathcal{L}$, agents $\mathcal{A}^{'}$, and where agent i has valuation $v_i$and budget w_i. In particular, y represent the allocations and p_j for $j \in L$, specify the prices in the market equilibrium . In case of additive valuations this can be used to find a Fisher equilibrium, since Fisher and Gale equilibria coincide under homogeneous valuations .

Lemma 3.3. Let y^*be an optimal solution to EG with additive valuations.Then for any feasible solution $y^{'}$ and $any\ \mathcal{A}^{''} \subseteq \mathcal{A}^{'}$ it holds $\sum_i \in \mathcal{A}^{''} w_i \frac{v_i(y_i^{'})}{v_i(y_i^*)} \leq \sum_{i \in \mathcal{A}'} w_i$

Lemma 3.4. Let y^* be an optimal solution to EG. Then for any feasible solution $y^{'}$ and any $\mathcal{A}^{''} \subseteq \mathcal{A}^{'}$ it holds

3.4.2 The Approximation Guarantee for the Mixed Matching Relaxation

Lemma 3.5 Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. Let $\alpha > 0$and y^* be an optimal and y be a feasible solution of EG such that $v_i(y_i) \geq \frac{1}{\alpha} v_i(y_i^*)$ for all i$\in \mathcal{A}^{'}$. Let π be maximum weight assignment in the bipartite graph with colour classes $\mathcal{A}$and $\mathcal{H}$, and edge weights $w_{ij} = w_i log(v_i(y_i + vij)$for $i \in \mathcal{A}, j \in \mathcal{H}$.Then

$$\overline{NSW}(y, \pi) \geq \frac{1}{2\alpha}\overline{OPT}_{\mathcal{H}}$$

3.5 PHASE 4: A Sparse Approximate Solution for the Mixed Matching Relaxation

In this section we exploit the properties of Rado valuations. Assuming the agents have Rado valuation functions, we can find an approximate solution of mixed matching relaxation with a strong sparsity property. Even though the approximation ratio is weaker than given in Lemma 3.5 sparsity will be essential for the rounding in Phase 5.

THEOREM 3.6. Suppose the function v_i are Rado valuations. Let$\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. We can find a feasible solution (y, π) to mixed matching relaxation such that

$$(1)\overline{NSW}(y, \pi) \geq \tfrac{1}{4}\overline{OPT}_{\mathcal{H}}$$
$$(2)supp(y) \leq 2|\mathcal{A}| + |\mathcal{L}^+|$$
$$where\ \mathcal{L}^+ = \{j \in \mathcal{G}\backslash\mathcal{H} : \textstyle\sum_{i \in \mathcal{A}'} y_{ij} > 0\}, that\ is, \mathcal{L}^+\ is\ the\ set\ of\ allocated\ items\ in\ y.$$

Let us start with the special case of additive valuations. In this case, an exact solution y^* to the Eisenberg–Gale convex program (EG) can be found in strongly polynomial time [5] [6].

THEOREM 3.7. Assuming the valuations vi are additive, we can find an optimal solution y^* of (EG) in strongly polynomial time such that the support supp(y^*) is a forest.

For Rado valuations, we first prove that an optimal solution of EG can be found in polynomial time. We first show that this is a rational convex program, and use the variant of the ellipsoid method for rational polyhedron.

Lemma 3.8. Suppose that for each agent $i \in \mathcal{A}$, v_i is a Rado valuation given by a bipartite graph $(G, V_i; E_i)$, integer costs c_i:E_iß$Z$and a matroid Mi = (V_i, I_i) as in Definition 2.4. Let $T = max_{i \in \mathcal{A}}|V_i|$, and $C = max_i \in A||c_i||_\infty$. Let the weights $w_i > 0$ be rational numbers given as quotients of two integers at most U. Assume the matroids M_i are given by rank oracles. Then, (EG) has a rational solution with poly($|\mathcal{A}|, |\mathcal{G}|, T, logC, logU$) bit-complexity, and such a solution can be found in poly($|\mathcal{A}|, |\mathcal{G}|, T, logC, logU$) arithmetic operations and calls to the matroid rank oracles.

The next lemma shows that any feasible solution to EG can be sparsified by losing at most the half of the value for each agent. This is achieved in two steps, using the sparsity of basic feasible solutions to linear programs. Half of the valuation may be lost in the second step, where for the fractionally allocated items we aim to remove one of the fractional edges. The set to be deleted is identified by writing an auxiliary linear program.

Lemma 3.9. Suppose the functions v_i are Rado valuations, and let $\hat{y}$ be a feasible solution to (EG). Then, in polynomial time we can find a feasible solution y such that

$$(1)v_i(y) \geq \tfrac{1}{2}v_i(y)$$
(2) lsupp(y)l $\leq 2|\mathcal{A}'| + |\mathcal{L}^+| where\ \mathcal{L}^+ = \{j \in \mathcal{G}\backslash\mathcal{H} : \sum_{i \in \mathcal{A}'} y_{ij} > 0\}$

3.6 PHASE 5: ROUNDING THE MIXED SOLUTION

In this section , we make use of the initial choice of $\mathcal{H}$ as the set of items allocated in the best allocation with one item per agent. Using the property above , we carefully recombine the matching in the mixed assignment and the initial allocation of the items in H by swapping around alternating cycles. This enables the final rounding step to obtain an integer allocation.

Lemma 3.10. Let $\mathcal{H}$ be the set of preferred items, and let (y, π) be a solution to mixed matching relaxation as in THEOREM 3.6. Let (y, π^r) be a reduction of (y, π). Then in polynomial-time we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that

$$\overline{NSW}(y^r, \rho) \geq \frac{1}{32\gamma^2}\overline{NSW}(y, \pi)$$

When the valuations are linear, then we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that $\overline{NSW}(y^r, \rho) \geq \frac{1}{8}\overline{NSW}(y, \pi)$

We present the rounding for a sparse solution of mixed matching relaxation. We recall that by sparse we mean a feasible solution (y, π) of mixed matching relaxation satisfying $supp(y) \leq 2|\mathcal{A}^{'}| + |\mathcal{L}^{+}| where\ \mathcal{L}^{+} = \{j \in \mathcal{G}\backslash\mathcal{H} : \sum_{i \in \mathcal{A}'} y_{ij} > 0\}$.

Such a sparse solution is rounded by setting $2|\mathcal{A}|$ positive variables in y to 0.

For instance, a reduction of (y, π) and allocating the items according to the support of the reduction. Formally, by a reduction of (y, π) we mean a mixed integer solution (y^r, π) obtained as follows. We define $y(j \in \mathcal{L}^{+})$ as a fraction of $\forall$item j and then we set $\kappa(j)$ to get the item. We set $y^r_{\kappa(j)j} = y_{\kappa(j)j}$ and set $y^r_{ij} = 0$ when $i \neq \kappa(j)$.

Just to summarize, the agent $\kappa(j)$ keeps getting the same amount in reduction and no other agent receives any part of item j. By the round on $supp(y)$, this amounts to setting $\leq 2|\mathcal{A}|$ values $y_i j$to 0. Looking at the reduction from the agent i lost by reduction. Then, we have $\sum_{i \in \mathcal{A}'} d_i \leq 2|\mathcal{A}|$.

The reduction (y^r, π) might have an arbitrarily worse objective value than (y, π), but we show that we can find a different assignment ρ such that (y^r, ρ) is only worse by a constant factor than (y, π) no matter how the reduction is carried out. The assignment ρ is obtained as a combination of τ above and π .

Given a fixed reduction and the values d_i, ρ, let's go ahead and introduce the lemma.

Lemma 3.11. Let $\mathcal{H}$ be the set of most preferred items (y, π) a feasible solution to mixed matching relaxation and let $d_i \in \mathbb{N}, (d_i \geq 1)$for each $i \in \mathcal{A}$. In O($|\mathcal{A}|$) time, we can find an assignment ρ such that

$$\overline{NSW}(y, \rho) \geq \frac{1}{2}(\prod_{i \in \mathcal{A}}(d_i + 1)^{1/\sum_{i \in \mathcal{A}} w_i}\overline{NSW}(y, \pi)$$

and for each $i \in \mathcal{A}$ it holds either

$$(1) v_{i\rho(i)} \geq \frac{1}{d_i} v_i(y_i)$$

$$(2) \forall j \in calL, v_{ij} \leq \frac{1}{d_i + 1}(v_i(y_i) + v_{i\rho(i)})$$

Obviously, **Lemma 3.11.** states that starting with a feasible allocation y, we can find an assignment ρ that might have smaller $\overline{NSW}(y, \rho)$ than $\overline{NSW}(y, \pi)$ but has the following nice property for $\forall i \in \mathcal{A}$, we have two case :

In case (a), i values the item $\rho(i)$ at least as she values a $1/d_i$ fraction of y_i (and thus at least a 1/($d_i + 1$)fraction of $v_i(y_i + v_i\rho(i))$). Hence, agent i keeps a (1/($d_i + 1$)-fraction of her value just by keeping $\rho(i)$ even if we can take take away all items i gets from $\mathcal{L}$.

In case (b), every item $\mathcal{L}$ has a small value for i when compared to the combined value of y_i and $\rho(i)$. That is, i values y_i and $\rho(i)$ significantly more than any d_i items combined from $\mathcal{L}$. Looking at it from the other side, even if we were to take away any d_i in $\mathcal{L}$ items from i she will still keep a fraction of the value.

The essence of both cases is that the reduction will not hurt the agent too much.

3.6.1 Constructing the New Matching

In the previous content, we defined τ as an assignment to maximize $\prod_{i\in\mathcal{A}} v_{i\tau(i)}^{w_i}$ and $\mathcal{H}$ the set of items assigned by τ. We number the agents $\mathcal{A} = \{1, 2, ..., n\}$, and renumber the items $\mathcal{H} = \{\infty, \in, ..., \backslash\}$. In short, τ assigns items $i \in \mathcal{G}$ to agent $i \in \mathcal{A}$.

Claim 3.12. Let $i \in \mathcal{A}$ .Then either $v_{ii} > \frac{1}{d_i}v_i(y_i)$ or $\forall j \in \mathcal{L}$ it holds $v_{ij} \leq \frac{1}{d_i+1}(v_{ii} + v_i(y_i))$

Algorithm:let (y, π) be a feasible solution of mixed matching relaxation. We denote with Y_i the value i gets in y. In particular, whenever $\pi(i) = \tau(i)$ then we set $\rho(i) := \pi(i) = \tau(i)$ and otherwise exactly one of the following will be the case: $\rho(i) = \tau(i), \rho(i) = \pi(i) or \rho(i) = \emptyset$. Notation $\rho(i) = \emptyset$represents the case that i is not allocated any item from $\mathcal{H}$.

Consider the symmetric difference of the two assignments $\pi\triangle\tau$. Each component is an alternating cycle; we consider the components one-by-one. Take any component C of $\pi\triangle\tau$ with c agents and c items. Let the agents in the component be $a_1, a_2, ..., a_c$. The numbering is modulo c: $a_c + k = a_k$ for all $k \in Z$. By the convention on the numbering, the corresponding items are also numbered$a_1, a_2, ..., a_c$, and $(a_k, a_k) \in \tau$ for all $k \in [c]$. We order the agents around the cycle such that $(a_k, a_{k1}) \in \pi$ for all $k \in [c]$. Let $B := B(C) = t \in [c] : Y_{a_t} > d_{a_t} v_{a_t a_{t-1}}$. We consider two cases based on the size of B:

$|B| = 0$

$|B| \geq 1$ .In this section, we consider the following ratio measuring the change in the objective value by augmenting π over the previously mentioned path:

$$\phi(C, k, r) = \frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}})^{w_{a_k}} \prod_{t=k+1}^{r} (\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}})^{w_{a_t}}$$

Lemma 3.13. The assignment ρ can be constructed in linear time , it holds

$$\frac{\overline{NSW}(t, \pi)}{\overline{NSW}(t, \rho)} \leq 2 \cdot (\prod_{i\in\mathcal{A}} (d_i + 1)^{1/\sum_{i\in\mathcal{A}} w_i})$$

Lemma 3.14. Let $i \in \mathcal{A}$.Then we either have

$$(1) v_{i\rho(i)} \geq \frac{1}{d_i} v_i(y_i), or$$

$$(2) \forall j \in \mathcal{L}, it\ holds\ v_{ij} \leq \frac{1}{d_i + 1}(v_i(y_i) + v_{i\rho(i)})$$

Claim 3.15. For $\forall i \in \mathcal{A}$, if $Y_i \leq d_i v_{i\pi(i)}$ then

$$\frac{v_{i\pi(i)} + Y_i}{v_{ii} + Y_i} \leq \frac{(d_i + 1)v_{i\pi(i)}}{v_{ii}}$$

4. The Proof

In this chapter, I will try my best to prove the above theory and THEOREM.

PROOF of THEOREM 3.0. From Theorem 3.6 and Lemma 3.10, we can obtain a solution an $(128\gamma^2)$-approximate solution (y^r, ρ) to mixed matching relaxation such that for each item $\mathcal{L}^+$there is exactly one incident edge in supp(y^r). We can obtain a 0–1 valued solution x to (NSW-IP) by assigning each item in $\mathcal{H}$ according to ρ and each item $j \in \mathcal{L}^+$ to the unique agent i with $y^r_{ij} > 0$. Clearly, NSW(x) $\geq NSW(y^r, \rho)$. We obtain $NSW(x) \geq OPT_{\mathcal{H}}/(256\gamma^3) \geq OPT/(256\gamma^3)$ using Theorem 3.2. For additive valuations, we use the stronger bounds in the same results.

PROOF of Lemma 3.1.

We have $\overline{NSW}(y, \sigma) \geq NSW(y, \sigma)$. Under monotonicity :$2NSW(y, \sigma) \geq NSW(y, \emptyset) + NSW(0, \sigma) = \overline{NSW}(y, \sigma)$.

PROOF of THEOREM 3.2.

We first show that $\overline{OPT}_{\mathcal{H}} \geq \frac{1}{\gamma} OPT_{\mathcal{H}}$. Let x be an optimal solution to mixed relaxation . For i, let K_i be the set of x on $\mathcal{G}\backslash\mathcal{H}$ defined as $y_{ij} = x_{ij}$ for $j \in \mathcal{G}\backslash\mathcal{H}$ and $y_{ij} = 0$ otherwise.

We define $k_i = |K_i|$. Denote with S the sets of agents that receive at least one items from $\mathcal{H}$.

$\forall i \in S$, we define $\sigma(i) = max_{j \in K_i}\{v_{ij}\}$ and for ß $\in \mathcal{A}\backslash S$ we define $\sigma(i) = \emptyset$.

Therefore ,we have (y, σ) is a feasible solution of mixed matching relaxation. In short, for $i \in S$ we discard all items from set $K_i$above except the most valuable item to obtain (y, σ) . By, monotonicity and subadditivity, $\forall i \in S$, we have

$$v_i(x_i) \leq v_i(y) + \sum_{j \in K_i} v_{ij} \leq k_i \cdot (v_i(y) + v_{i\sigma(i)}$$

Therefore,

$$\frac{NSW(x)}{\overline{NSW}(y, \sigma)} = (\prod_{i \in S} \frac{v_i(x_i)^{w_i}}{v_i(y) + v_{i\sigma(i)}^{w_i}}) \leq (\prod_{i \in S} k_i^{w_i})^{\frac{1}{\sum_i w_i}}$$

From Lemma 2.8. Lemma 3.1., we gain $\frac{OPT_{\mathcal{H}}}{\overline{OPT}_{\mathcal{H}}} \leq \frac{NSW(x)}{\overline{NSW}(y,\sigma)}$. Therefore, we have the bound $\sum_{i \in S} k_i \leq |\mathcal{H}| = |\mathcal{A}| = n$.

PROOF of Lemma 3.3. We can extend to assume that $v(y^*) = w_i$. Hence, we need to prove $\sum_{i \in A'' v_i(y'_i)} \leq \sum_{i \in A'} v(y^*)$.

As y^* and p form a Fisher equilibrium, the previous inequality holds by the first welfare theorem.

PROOF of Lemma 3.5. We let π^* be a maximum weight matching in the bipartite graph with colour classes $\mathcal{A}, \mathcal{H}$ and edge weights $q^*_i = w_i log(v_i(y^*) + v_{ij})$. At the same, π^* is a matching maximizing $(\prod_{i \in \mathcal{A}}(v_i(y^*_i) + v_{i\pi^*(i)})^{w_i})^{1/\sum_{i \in \mathcal{A}} w_i}$

We have the bounds

$$\overline{NSW}(y,\pi) \geq \overline{NSW}(y,\pi^*) \geq \frac{1}{\alpha}\overline{NSW}(y^*,\pi^*) \tag{4.1}$$

The first inequality is by the definition of π as the maximum weight matching.

The second inequality is from the assumption $v_i(y_i) \geq \frac{1}{\alpha}v_i(y_i^*), \forall i \in \mathcal{A}$.

Then we need to prove $\overline{NSW}(y^*,\pi^*) \geq \frac{1}{2}\overline{OPT}_{\mathcal{H}}$, as well as (4,1) above , which implies the statement.

PROOF of THEOREM 3.6. By combining Lemmas 3.5, 3.8, 3.9, we obtain Theorem 3.5 for Rado valuations..

PROOF of Lemma 3.8. Let P be the set of feasible solutions and P^* the set of optimal solutions to EG. We note that $P \neq \emptyset$ since $z = 0$ is a feasible solution. Further, $P^* \neq \emptyset$ since P is bounded. Lemma 3.8. asserts that this is a nonempty polytope with vertex-complexity poly $(|\mathcal{A}|, |\mathcal{G}|, T, logC, logU)$, thus $(P^*, \sum_{i\in\mathcal{A}} |E_i|, \phi)$ is a well-desribed polytope for some $\phi \in (|\mathcal{A}|, |\mathcal{G}|, T, logC, logU)$ by [1].

We now describe the strong separation oracle to P^*. For a vector $z \in \mathbb{R}^{\times_{i\in A} E_i}$, we first check whether $z \in P$. Checking the first set of |A| constraints is straightforward. The submodular constraints can be verified by solving |A| submodular function minimization problems. We either conclude $z \in P$, or obtain a separating hyperplane for z and P^* that is also a separating hyperplane for z and P^*.

Nevertheless, we can run the ellipsoid method using the gradients as separating directions (without solving the LP). This ultimately leads to concluding $P^* = \emptyset$, since the algorithm returns a separating hyperplane for every$z \in \mathbb{R}^{\times_{i\in A} E_i}$. At this point, we consider the feasible solution $z \in P$with the largest objective value f(z) visited by the algorithm, and conclude that this solution must have been optimal. This is true since if no optimal solutions would have been visited, then every separating hyperplane we used would be a valid strong separator for P^* , and thus, we could not have reached the false conclusion $P^* = \emptyset$.

PROOF of Lemma 3.9. Let y be a solution of (EG) with $|supp(y^{'})| \leq |\mathcal{A}^{'}| + 2|\mathcal{L}^{+}(y^{'})| - |\mathcal{L}_{\infty}(y^{'})|$, given by ([1] Corollary 5.8.) .Let $D = \{j \in \mathcal{L}^{+}(\dagger^{'}) : \exists i, i^{'}, i \neq i^{'}\}$. Hence, $|D| = |\mathcal{L}^{+}(y^{'})| - |\mathcal{L}_1(y^{'})|$. For each $j \in D$, let D(j) be a set containing two different agents i, i getting the item j in $y^{'}$. Such two agents are picked arbitrarily, but fixed throughout the proof for each j. Let $A^{''} = \cup_{j\in D} D(j)$. We consider the following linear system with variables q. The value q_{ij} represents the fraction of $y^{'}_{ij}$ agent i keeps. By the above, if agent obtained u(i, j) value from $y^{'}_{ij}$ units of j then agent receives $q_{ij}u(i,j)$ value from $q_{ij}y^{'}_{ij}$ units of good j whenever $q_{ij} \in [0,1]$.

$$\begin{aligned} \sum_{j\in D} q_{ij}u(i,j) \geq \frac{1}{2}\sum_{j\in D} u(i,j), \forall i \in \mathcal{A}^{''} \\ q_{ij} + q_{i^{'}j} = 1, \forall j \in D, \{i, i^{'}\} = D(j) \\ q \geq 0 \end{aligned}$$

Let us define y : set $y_{ij} = 0$ if $q_{ij} = 0$ and $y_{ij} = y^{'}_{ij}$ for all other values. Then for any feasible q we have the second set of constraints together with non-negativity of q guarantees$q_{ij} \in [0,1]$ and hence we can treat the values $v_i(y_i) \geq q_{ij}v_i(y^{'}_i)$ as described before the statement of the lemma. By the first set of constraints and definition of y, we have

$$v_i(y_i) \geq \sum_{j \in D} q_{ij} u(i,j) + \sum_{j \in \mathcal{L} \setminus D} u(i,j) \geq \frac{1}{2} \sum_{j \in D} u(i,j) + \frac{1}{2} \sum_{j \in \mathcal{L} \setminus D} u(i,j) \geq \frac{1}{2} v_i(y^{'})$$

Therefore, any feasible solution of the linear system in q gives an allocation that satisfies the first condition of the lemma. Let us show that the system is indeed feasible. Namely, setting $q_{ij} = \frac{1}{2}$ for all $i \in \mathcal{A}^{''}$ and all $j \in D$ we see that the above system is feasible. Since, the system is feasible we can also find a basic feasible solution q. By counting the number of tight constraints we show that there are at least $\mathcal{L}^{+}(y^{'})| - |\mathcal{L}_1(y^{'})| - |\mathcal{A}^{''}|$ zeros in q. Thus, allocation y defined as$y_{ij} = q_{ij} y_{ij}$will have support smaller by at least $\mathcal{L}^{+}(y^{'})| - |\mathcal{L}_1(y^{'})| - |\mathcal{A}^{''}|$.

The maximum number of constraints is obviously $|\mathcal{A}^{''}| + |\mathcal{D}|$.Therefore, $|supp(q)| \leq |\mathcal{A}^{''}| + |\mathcal{D}|$. Crucially, by the second constraint we have $\mathcal{L}^{+}(y^{'}) = \mathcal{L}^{+}(y)$. Hence, we only need to compare $|supp(y)|$ and $|supp(y^{'})|$. The allocation $y^{'}$ has exactly $2|D|$ positive variables than $y^{'}$. By recalling we get $|supp(y)| \leq 2|\mathcal{A}^{'}| + |\mathcal{L}^{+}|$.

PROOF of Claim 3.12.

By the optimality of it then holds τ it then holds $v_{ii} \geq v_{ij}, \forall i \in \mathcal{L}$, $v_{ii} \geq \frac{1}{d_i v_i(y_i)}$.If $v_{ii} \geq \frac{1}{d_i} v_i(y_i)$ then case (a) holds. Otherwise, we have that $d_i v_{ii} \leq \frac{1}{d_i v_i(y_i)}$. Combining it with $v_{ij} < v_{ii}$,we have that

PROOF of Lemma 3.13. It suffices to prove the lemma for each of the connected components C of $\pi \triangle \tau$. For |B| = 0 the lemma holds trivially. So assume that |B| 1 for the rest of the proof. The procedure terminates in linear time, as we only require one pass through the agents and items in C.

To prove the bound on $\frac{\overline{NSW}(t,\pi)}{\overline{NSW}(t,\rho)}$, we show that for every interval $[k, r]$ the objective value "before averaging" decreases at most by factor $2^{w_{a_k}} \prod_{t=k}^{r} (d_{a_k} + 1)^{w_{a_k}}$

If interval [k,r] is not reversible, then the change in the objective function is captured by $(\frac{v_{a_k a_{k-1}}}{Y_{a_k}} + 1)^{w_{a_k}}$. We can get that $(\frac{v_{a_k a_{k-1}}}{Y_{a_k}} + 1)^{w_{a_k}} < 2^{w_{a_k}}$.

If, on the other hand, [k,r] is reversible, then the difference in the objectives is captured by

$$(\frac{v_{a_k a_{k-1}} + Y_{a_k}}{v_{a_k a_k} + Y_{a_k}})^{w_{a_k}} \prod_{t=k+1}^{r} (\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}})^{w_{a_t}} = (\frac{v_{a_k a_{k-1}} + Y_{a_k}}{Y_{a_k}} \cdot \frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}})^{w_{a_k}} \prod_{t=k+1}^{r} (\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}})^{w_{a_t}}$$

For the reversible [k,r],

$$\phi(C, k, r) = \frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}})^{w_{a_k}} \prod_{t=k+1}^{r} (\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}})^{w_{a_t}} < b \prod_{t=k}^{r} (d_{a_t} + 1)^{w_{a_k}}$$

Since $k \in B$ and $d_{a_t} \geq 1$ as well as $\frac{v_{a_k a_{k-1}} + Y_{a_k}}{Y_{a_k}} < 2$. So, the Lemma is proved.

PROOF of Lemma 3.14.

If $\rho(i)$ = i, that is, agent i receives the same item in ρ as in τ .

Hence, the lemma follows by Claim 3.12. For the rest of the proof we assume $\rho(i) \neq i$.

Hence, either $\rho(i) = \pi(i) or \rho(i) = \emptyset$.

We consider the component C of $\pi \triangle \tau$ containing an agent i. We use the notation before, denoting the agents in C by $a_1, a_2, ..., a_c$, and letting $i = a_k$.

In $\rho(a_k) = \pi(a_k) = a_{k-1}$, we can get $k \notin B$ to obtain $Y_{a_k} \leq d_{a_k} v_{a_k a_{k-1}}$.

In the other hand, we consider $\rho(a_k) = \emptyset$.we have that$k \in B$and also that the interval [k,r] with starting and k and ending in r that corresponds to some alternating path in C is not reversible. By Claim 3.15., we obtain

$$1 < (d_{a_k} + 1)^{-w_{a_k}} \cdot (\frac{Y_{a_k}}{v_{a_k j}} \cdot \frac{v_{a_k j}}{v_{a_k a_k}}) \cdot \prod_{t=2}^{r} (\frac{v_{a_t a_t}}{v_{a_t a_{t-1}}})^{w_{a_t}}$$

From above content , we know that $\forall j \in \mathcal{L}$ we have

$$1 \leq (\frac{v_{a_k a_k}}{v_{a_k j}})^{w_{a_t}} \prod_{t=2}^{r} (\frac{v_{a_t a_t}}{v_{a_t a_{t-1}}})^{w_{a_t}}$$

By combining the above two inequlities, we obtain $Y_{a_k} > (d_{a_k}) v_{a_k j}$. Hence, in case (b) holds.

5. Conclusion

Limited by my ability and knowledge, I can only understand the content of the article[3] as much as possible, and give corresponding proofs for the given lemmas and theorems after my own understanding (although I cannot prove it myself, but the knowledge in the article basically absorbed).

This article paves the way for the basic knowledge required for reference work. Then to complete the completion process of the task goal, a constant factor approximation algorithm for the Nash welfare problem with Rado values is given, assuming that the weight of the subject is limited by a constant. Finally, complete the corresponding proofs of the given lemmas and theorems.

This algorithm is based on the mixed integer programming relaxation method, which is divided into five stages. First, we find a suitable set $\mathcal{H}$. Approximate by another integer program. Find an approximate mixed integer solution. Find a sparse approximate mixed integer solution. Round mixed integer solutions to integer solutions.

The Rado valuation I get from this work is an interesting total alternative valuation; this may also be relevant to other issues in mechanism design. It can be used in algorithm design on issues such as weighted array functions and OXS valuation as further work.

6. Acknowledgements

In this paper[3], I feel that this is a certain degree of difficulty. But difficulty can also bring about the improvement of ability.

To be honest, after revisiting the class of this semester during the winter vacation, I feel that the teacher taught it really well, imparting knowledge in unknown fields like a senior. Although the depth of the course may not be particularly deep, it opened up a new perspective for me to think about future problems.

Thank you for meeting, and I wish Teacher Liu Zhengyang all the best.

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