Homework1

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February 6, 2023

1 Problem 1

Prove the Nash Theorem mentioned in class in your own words.

Answer: In game theory, the Nash equilibrium, named after the mathematician John Nash, is the most common way to define the solution of a non-cooperative game involving two or more players. A simplified definition of Nash Theorem : In any finite game there is a Nash equilibrium if players are allowed to adopt mixed strategies.

The Proof is to find fixed point in a well defined domain.

We should know Brouwer's Fixed Point Theorem first. It says that we will define a function ϕ over the space of mixed strategy profiles. The space K is compact and and ϕ is continuous . Any continuous function ϕ that maps from K to K have at least one fixed point.Fixed point means that there exists x which is in ϕ, f(x) =x.

There is a particularly vivid explanation of Brouwer's Fixed Point Theorem : We put the campus map in any corner of the campus (assuming the campus is a convex compact set), then there must be at least one point on the map so that the point coincides with the actual location represented.

The idea of proof :

1. Construct a continuous function with special properties;

2. Apply Brauer's fixed point theorem, there must be fixed point s^*.Hence must be to prove that there is a fixed point;

3. Prove that the fixed point is a Nash equilibrium. (Because there must be a fixed point, there must be a Nash equilibrium)

Proof: For any strategy combination s, let $c_i(a_i; s) = max\{0, u_i(a_i, s_{-i}) - u_i(s)\}$

we define that function f as a mapping between two policy combinations. $f(x) = x'$

Each player$i$ in s' has each action a_i, we define it as $s'_i(a) = \frac{s_i(a_i)+c_i(s;a_i)}{1+\sum_{a'_i} c_i(s;a'_i)}$

The numerator can be viewed as a liitle adjustment of $s_i(a_i)$,if $c_i(s; a_i) > 0$, then we can increase the probability of a_i.The denominator is normalized to ensure that the probability sum of all the actions of each player is 1, and the sum of all the actions of each player is not normalized to $1 + \sum_{a'_i} c_i(s; a'_i)$ after all the updates are done.

In the mapping function, all the input and output are from set $\prod_{i\in N} \Delta(A_i)$, and the set is convex, bounded and closed why the reason is that the convex combination of two feasible policy combinations is still a feasible policy combination. We often call it convex compact set.

Now we apply Brauer's fixed point theorem on the function f(x). $f : \Delta \to \Delta$. It says that there exists fixed point s^*, let f(s^*) = s^*. That is, for each action of each player:

$s_i^*(a_i) = \frac{s_i^*(a_i)+c_i(s^*;a_i)}{1+\sum_{a_i'} c_i(s^*;a_i')}$

Because $c_i(s^*; a_i) \geq 0$, then there are two situations for the above formula to be established:

(1)$c_i(s^*; a_i) > 0$

(2)$c_i(s^*; a_i) = 0$

Considering (1), we can infer that $\forall i, \forall a_i \in A_i, c_i(s^*; a_i) > 0$, which means , in this situation, for each player the profit of each pure strategy is greater than the profit of mixed strategy s^*. This conclusion is obviously not established because any profit of mixed strategy can't be less than profit of all the pure strategy at the same time.

So assumption (1) fails.

Considering (2), this assumption means $u_i(s) \geq u_i(a_i, s_{-i})$ for every i and every $s \in S$, hence must be a Nash equilibrium.

Clearly, Δ is continous. Finally, it is easy to see that

$u_i(s_i^*, s^*{}_i) \geq u_i(a_i^*, s_{-i}^*)$

That is, the income of the mixed strategy s^* is greater than or equal to the income of all pure strategies, and it is easy to analyze that s^* must be a mixed strategy Nash equilibrium.

Because of Brauer's fixed point theorem, there must be fixed point s^*.So there must be a Nash equilibrium.

This concludes the proof of the existence of a Nash equilibrium.

2 Problem 2

Problem: Show that the two definition of Nash Equilibrium mentioned in class are equivalent. For convenience, we list these two definition.

Definition 1:

A pair of strategies (x, y) is NE iff

$x^T Ry \geq x^{'T} Ry, \forall x^{'} \in \Delta_m;$

$x^T Cy \geq x^T Cy^{'}, \forall y^{'} \in \Delta_n;$

Definition 2:

A pair of strategies (x, y) is NE iff

$x_i > 0 => e_i^T Ry \geq e_k^T Ry, \forall k \in [m]$

$y_i > 0 => x^T Ce_j \geq x^T Ce_l, \forall l \in [n]$

Answer: In the literature, we have two different definition of approximate NE. We assume that R,C $\in[0,1]^{n\times n}$.

We can transform the problem to another problem .

We assume that a pair of startegies (x,y) is NE, we can get that

$x^T Ry \geq x^{'T} Ry, \forall x^{'\in\Delta_m \iff x_i>0=>e_i^T Ry \geq e_k^T Ry, \forall k\in[m]}$

$x^T Cy \geq x^T Cy^{'}, \forall y^{'} \in \Delta_n \iff y_i > 0 => x^T Ce_j \geq x^T Ce_l, \forall l \in [n]$

Now we prove this assumption from two parts.

Necessity:

When we have $e_i^T Ry = e_j^T Ry$, referring to $e_i^T Ry \geq e_k^T Ry, \forall k \in [m], x_i > 0$, we can get that if $x_i, x_j > 0, (Ry)_i = (Ry)_j = x^T Ry = ||Ry||_\infty$

Considering another x' , the changes in the component of x' make a new value $x_1^{'}(Ry)_1 + ... + x_n^{'}(Ry)_n$, which can't compensate the loss of x' comparing with x.

We prove the necessity.

Sufficiency:

Referring to $x^T Ry \geq x^{'T} Ry, \forall x^{'} \in \Delta_m$, we consider the case of n additions.

We have $x_k > 0, k \in [m]$, because the sum of component of x and $x^{'}$ is 1 , we can change one component vector to change others. For convenience, we choose two different vectors in x and $x^{'}$.

Then we can reduce above formula to :

$x_{k1}(Ry)_{k1} + x_{k2}(Ry)_{k2} \geq x_{k1}^{'}(Ry)_{k1} + x_{k2}^{'}(Ry)_{k2}$

we can reduce above formula to :

$(x_{k1} - x_{k1}^{'})(Ry)_{k1} \geq (x_{k2}^{'} - x_{k2})(Ry)_{k2}$

Because $x_{k1} - x_{k1}^{'} = x_{k2}^{'} - x_{k2}$, we can get $(Ry)_{k1} \geq (Ry)_{k2}$

Now We discuss by category.

If $x_{k2} = 0$, then $x_{k2}^{'} - x_{k2} > 0$

If$x_{k2} > 0$,then for all situation, $x^{'}_{k2} - x_{k2}$ maybe positive or negative , so $(Ry)_{k1} = (Ry)_{k2}$

We prove the sufficiency.

Now we solve Problem 2.

Concise explanation:

Or we can think simpler. In this problem , we should show that the two definition of Nash Equilibrium mentioned in class are equivalent. One is the whole, and the other is discussed separately for each column and row. Considering R a array , y a column vector , we all know that matrix multiplication is the corresponding row multiplied by the corresponding column. When y is fixed, we know that x has the maximum payoff, then each component of x will have.Then each component also has the same form as the whole. The reverse is also the same theory, so the two definitions are equivalent.

3 Problem 3

Prolem: Prove that two kinds of approximate Nash Equilibrium can be reduced each other. (Hint: one can refer to this paper: Chen, Xi, Xiaotie Deng, and Shang-Hua Teng. "Settlingthe complexity of computing two-player Nash equilibria." Journal of the ACM (JACM) 56, no. 3(2009): 1-57.)

Two kinds of approximate Nash Equilibrium is "$\epsilon - Approximate$"NE and "$\epsilon - Well - Supported$"NE

"$\epsilon - Approximate$"NE: given any $\epsilon > 0$,

$x^T Ry \geq x^{'T} Ry - \epsilon, \forall x^{'} \in \Delta_m$;

$x^T Cy \geq x^T Cy^{'} - \epsilon, \forall y^{'} \in \Delta_n$;

"$\epsilon - Well - Supported$"NE: given any $\epsilon > 0$,

$x_i > 0 => e_i^T Ry \geq e_k^T Ry - \epsilon, \forall k \in [n]$

$y_i > 0 => x^T Ce_j \geq x^T Ce_k - \epsilon, \forall k \in [n]$

Answer:

We assume that (u,v) is an $\epsilon^2/8$ - approximate Nash equilibrium , then we have

$\forall u^{'} \in P^n, (u^{'})^T Av \leq u^T Av + \epsilon^2/8$;

$\forall v^{'} \in P^n, u^T Bv^{'} \leq u^T Bv + \epsilon^2/8$;

We define that a^i denotes the row of A and b^i denotes the column of B . And we define that i^* is the index of the max value. such as $a_{i^*}v = max_{1 \leq i \leq n} a_i v$. We define J_1 to denote the set of indices j: $1 \leq j \leq n$ such that $a_{i^*}v \geq a_j v + \epsilon/2$. Now by changing u_j to for all $j \in J_1$, and changing u_{i^*} to $u_{i^*} + \sum_{j \in J_1} u_j \leq \epsilon/4$. And we define $J_2 = \{j | 1 \leq j \leq n \; and \; \exists i, u^T b_i \geq u^T b_j + \epsilon/2\}$ and we have $\sum_{j \in J_2} v_j \leq \epsilon/4$. Let (x,y) be the vectors obtained by modifying u and v in the following manner: set all the $\{u_j | j \in J_1\}$and $\{v_j | j \in J_2\}$. to 0 . And we uniformly increases the probability of other strategies, making x and y a mixed strategy.

For all $i \in [n], |a_i y - a_i v| \leq \epsilon/4$,we assume that the value of each item a_i is 0 or 1. Considering every pair of i,j, $i \leq i, j \leq n$. We have $(a_i - a_j)y \; and \; (a_i - a_j)v$ not exceeding $\epsilon/2$.Then , any j that is beaten by some i in (x, y) has already been set to 0. At the end , (x,y) is an $\epsilon - well - supported \; Nash \; equilibrium$. We solve Problem 3 .

4 Problem 4

1. Give an example with general monotone valuations where an EF1 allocation is not Prop1.

Answer 1 : For this problem, I want to give two examples. When I discussed with my friend.He makes an interesting point that one of my examples that my example corresponds directly to EF, not precisely to EF1.

Example 1:

	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(1,2)$	$v_i(1,3)$	$v_i(2,3)$	$v_i(1,2,3)$
A	3	1	1	4	4	4	20
B	1	3	1	4	4	4	20
C	1	1	3	4	4	4	20

For general monotone valuations, $\forall X \subseteq Y, \forall agent\ i, v_i(X) \leq v_i(Y)$.I think the key to this problem is to make v(M) as big as possible so that the example will not meet Prop1. For the subject, we set a allocation $A_A = (1) A_B = (2) A_C = (3)$. Obviously it fits into EF, and it fits into EF1, too. For agent A, $v_A(M) = 20$, so $\forall g \in M \backslash A_A, v_A(A_1 \cup g) < \frac{v_A(M)}{3}$. Therefore, the example satisfies EF1 but not Prop1.

Example 2: To improve on the first example, we chose a more strict example that satisfies EF1 but not Prop1.

	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(1,2)$	$v_i(1,3)$	$v_i(2,3)$	$v_i(1,2,3)$
A	2	1	1	3	3	3	5
B	1	1	1	2	2	3	5

We set a allocation $A_A = (1), A_B = (2,3)$.It's easy to derive that the allocation doesn't meet Envy-free but EF1. For agent A, this example dosen't exist an extra allocation g which could obtain valuation more that $\frac{1}{2}v1(M)$. So, this allocation is not a Prop1.

Problem 2:(In this Problem, I discuss with my friend. We use the same code to print the figure)

Give an example with additive valuations where the round robin algorithm achieves better social-welfare ($\sum_i V_i(A_i)$) than the envy-cycle-elimination algorithm under certain choices.

Answer 2: For additive valuations, $v_i(S) = \sum_{j \in S} v_{ij}$

Our example is a scenario like this

	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(4)$	$v_i(5)$	$v_i(6)$
A_1	12	8	10	2	6	4
A_2	8	12	6	10	2	4
A_3	8	4	12	2	6	10
A_4	2	4	12	10	8	6

First, we apply Robin Algorithm to get relative social-welfare on it. Initially, we fix the order of agents arbitrarily. We set the order as $\{A, B, C, D\}$.

Round Robin Algorithm:

$Step1$: allocation :$\{A_{A_1} = 1, A_{A_2} = 2, A_{A_3} = 3, A_{A_4} = 4\}$

$Step2$: allocation :$\{A_{A_1} = (1,5), A_{A_2} = (2,6), A_{A_3} = 3, A_{A_4} = 4\}$

Because this example is relatively simple, I'm done in two rounds.

The social welfare by Round Robin Algorithm is 56.

Envy-Cycle-Elimination Algorithm:

Initially, we set a allocation $\{A_{A_1} = 1, A_{A_2} = 2, A_{A_3} = 6, A_{A_4} = 3\}$

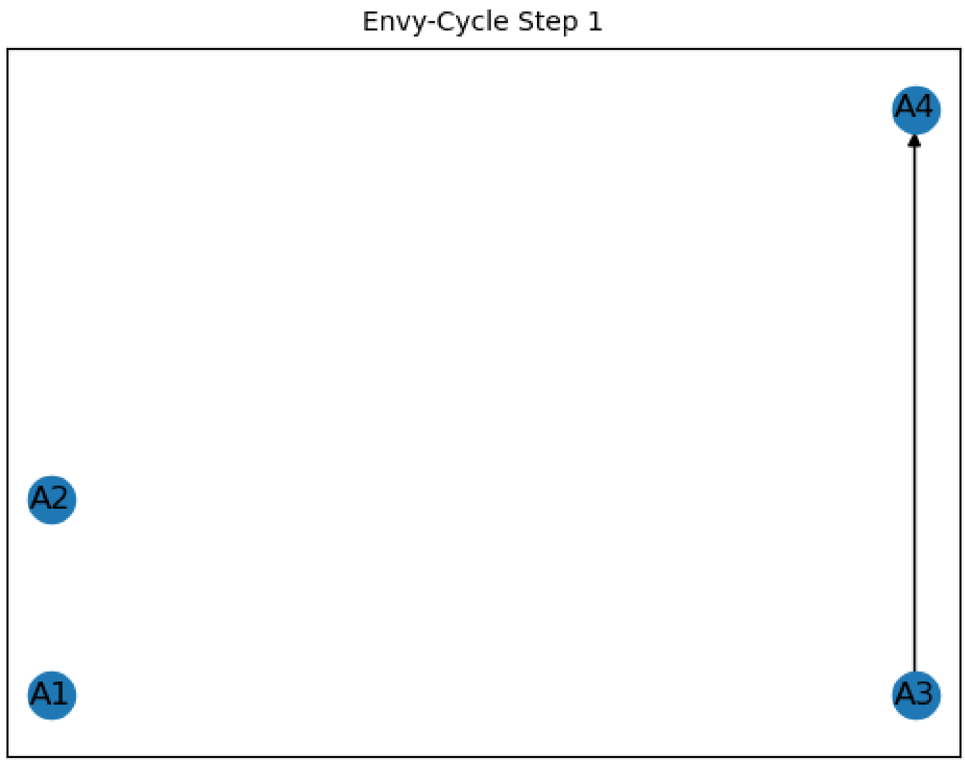

Figure 1: Step1

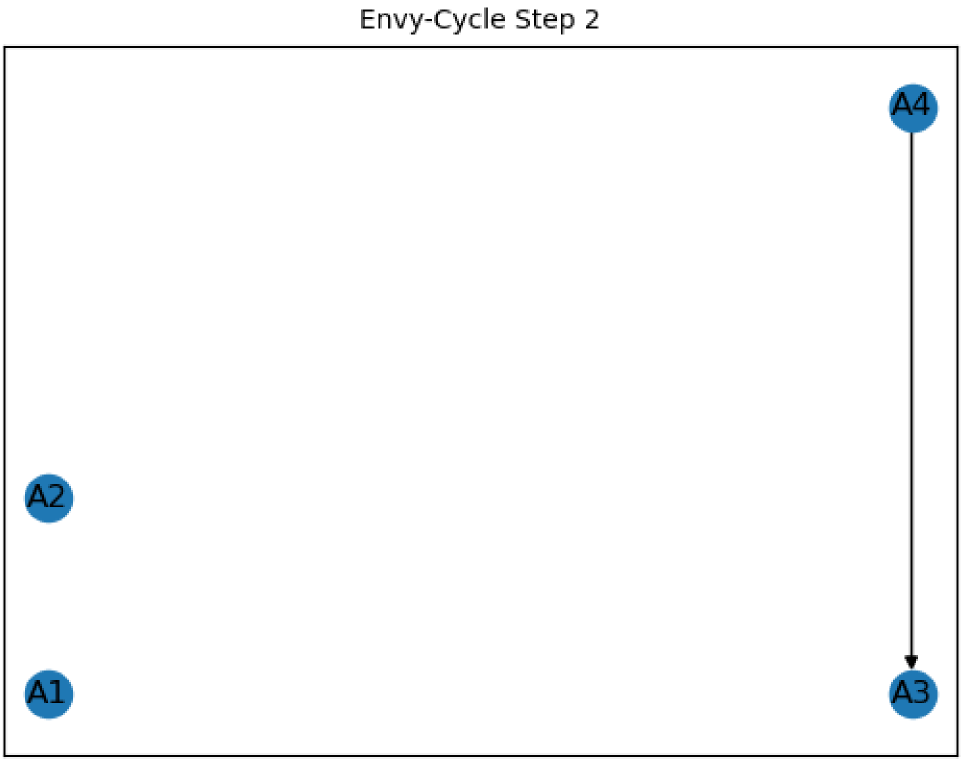

Figure 2: Step 2

Only A_3 envies A_4 .

Secondly,because A_3 is a source , we add item5 to agent A_3.

Because A_1 is a source , we add item 4 to agent A_4.(Indeed ,We need to minimize the outcome of method envy-cycle-elimination,So , I choose A_1 in Step 3).

Finally, our outcome is that allocation $\{A_{A_1} = (1, 4), A_{A_2} = 2, A_{A_3} = (6, 5), A_{A_4} = 3\}$, we output $= 54 < 56$. So, our example meets the requirements of the problem.

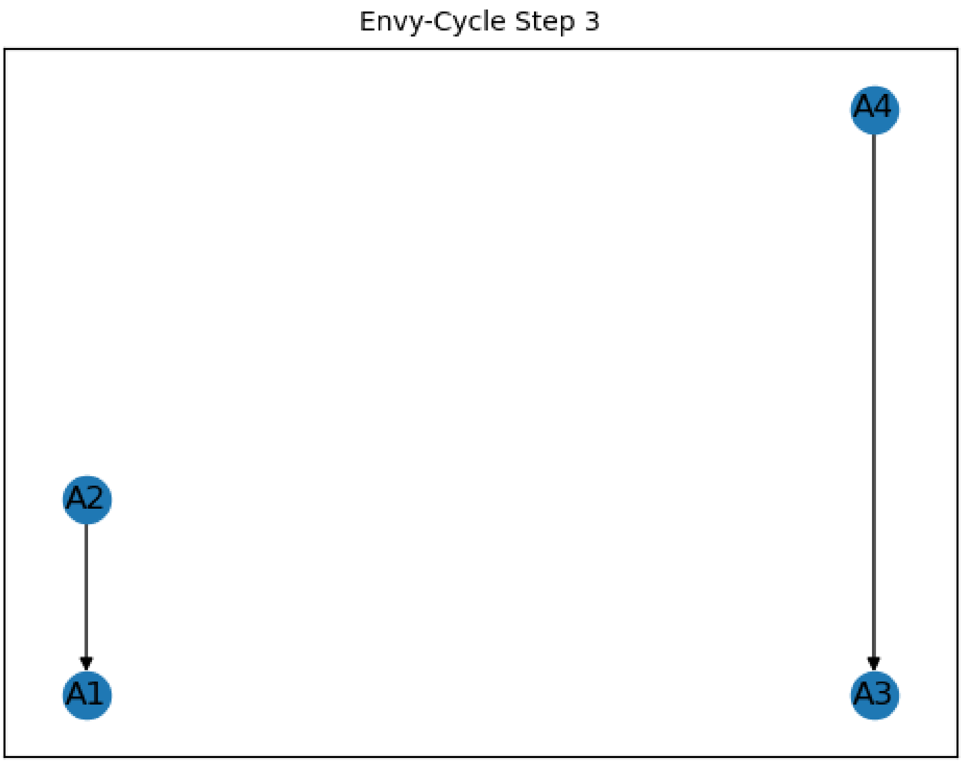

Figure 3: Step3

Problem 3: Give an example with additive valuations where an EF1+PO allocation is not EFX.

Answer3: Example:

	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(4)$
A_1	45	10	10	20
A_2	41	20	20	10

The allocation is $A_{A_1} = (1, 4), A_{A_2} = (2, 3)$.

For the definition of price,

	$p(1)$	$p(2)$	$p(3)$	$p(4)$
	45	20	20	20

It is known that EF1+PO$\rightarrow pEF1$.

For agent 1, $p(A_1) \geq p(A_2)$. For agent 2, $p(A_2) \geq p(A_1 \backslash 1)$. So the allocation is EF1+PO.

Because $v_2(A_2) \leq v_2(A_1)$ and $v_2(A_2) < v_2(A_1 \backslash 4) = 41$, the allocation isn't EFX.

Problem 4: For additive valuation functions, we showed $MMS_i \leq v_i(M)$for any agent i. Give an example with sub-additive valuation functions where this is not true, and in fact $MMS_i = v_i(M)$ for any agent i.

Answer 4:And in fact, because of this constraint, so looking at the problem, we can set up a very simple example that satisfies the problem.

Example:

	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(1,2)$	$v_i(1,3)$	$v_i(2,3)$	$v_i(1,2,3)$
A_1	2	2	2	2	2	2	2
A_2	2	2	2	2	2	2	2

We design this example by subadditive.

We set a allocation $A_{A_1} = (1, 3)$,$A_{A_2} = 2$. Easily, we have $MMS_i = v_i(M), MMS_i > \frac{v_i(M)}{n}$.

Problem 5: Prove that if an $\alpha - MMS$ allocation exists for an instance, then an $\alpha - MMS + PO$ allocation also exists.

Answer 5: If we want to get a PO, we can design a CE. If we want to a PO, the allocation must satisfy market clearring condition and MMB.I would add that in the universal context. As the definition from class, the price p is $maxv_i, \forall i \in N$ and we define good from $argmax(v_i(G))$ which satisfies market claering condition. We have $\mu_i \geq p_j$.

In the rest of proof, we can run bag filling with PO.

First, we asuume $\alpha - MMS$ allocation exists. For example, we define $A = (A_1, ..., A_n)$ for n agent.

For A $\alpha - MMS$, it holds $v_i(X) \geq \alpha \cdot v_i(A_i)$,for $\forall i$,and the bundle X, agents get now . The above formula holds that for agent i, i can't gain the bundle X over allocated bundle A_i at ratio α.

Step 2, adding the above bundle $v_i(X) \geq \alpha \cdot v_i(A_i)$ to a PO market. i.e. $v_i(Y) \geq \alpha \cdot v_i(A_i), \forall i \in N$. We gain new allocation $A^* = A_1, ..., A_n, Y$ is an $\alpha - MMS$ for agent can't get a bundle which they strictly prefer over A_i . Because we have Y, and Y is already as large as αX_i at price for agent i, the new allocation is $\alpha - MMS$.

In short, we give the example that there exists a $\alpha - MMS$ allocation, and we add an random bundle X in step1 to the PO market . However, the action doesn't destroy $\alpha - MMS$. Hence, if an $\alpha - MMS$ allocation exists for an instance, then an $\alpha - MMS + PO$ allocation also exists.The problem is proved.