# HDG Eigenproblem

## 1. Local Solver

$$M \begin{pmatrix} q_h \\ u_h \end{pmatrix} = \sum_i N_i \hat{u}_h^i + \begin{pmatrix} 0 \\ f_h \end{pmatrix}$$

$$\begin{pmatrix} q_h \\ u_h \end{pmatrix} = \sum_{i \in \partial K} M^{-1} N_i \hat{u}_h^i + M^{-1} \begin{pmatrix} 0 \\ f_h \end{pmatrix}$$

$$\Rightarrow \quad \left. \begin{aligned} q_h &= \sum_i Q_i \hat{u}_h^i + Q_w f_h \\ u_h &= \sum_i U_i \hat{u}_h^i + U_w f_h \end{aligned} \right\}$$

Local solver for source problem

Now for eigenvalue problem $f_h = \lambda_h u_h$

$$\Rightarrow \quad (I - \lambda_h U_w) u_h = \sum_i U_i \hat{u}_h^i$$

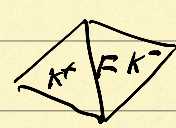
$$\left. \begin{aligned} u_h &= \sum_i (I - \lambda_h U_w)^{-1} U_i \hat{u}_h^i \\ \Rightarrow \quad q_h &= \sum_i Q_i \hat{u}_h^i + Q_w \lambda_h u_h \end{aligned} \right\}$$

Local solver for eigenproblem

## 2. Global Problem

$$\langle q_h \cdot n + \tau (u_h - \hat{u}_h), \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0$$

For face F




$$\sum_{F \in \partial K} \langle q_h^K \cdot n + \tau (u_h^K - \hat{u}_h^F), \mu \rangle_F = 0$$

$$\Rightarrow \quad (\quad)^{K^+} + (\quad)^{K^-} = 0$$

$$\Rightarrow \sum_{F\in\partial k} \left( \tilde{N}_F^T \begin{pmatrix} y_h \\ u_h \end{pmatrix} - \tau T \hat{u}_h^F \right) = 0$$

$$\sum_{F\in\partial k} \left( \tilde{N}_F^T \left[ \sum_i M_k^{-1} N_i^k \hat{u}_h^i + M_k^{-1} \begin{pmatrix} 0 \\ f_h \end{pmatrix} \right] - \tau T \hat{u}_h^F \right) = 0$$

Source problem

$$\sum_{F\in\partial k} \left( \tilde{N}_F^T \sum_i M_k^{-1} N_i^k \hat{u}_h^i - \tau T \hat{u}_h^F \right) = -\sum_{F\in\partial k} \tilde{N}_F^T M_k^{-1} \begin{pmatrix} 0 \\ f_h \end{pmatrix}$$

equation of $\hat{u}_h^F$

It is related to all the faces that sharing the elements with F.

$$\begin{pmatrix} * & * & * \end{pmatrix} \begin{pmatrix} \vdots \\ \hat{u}_h^F \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

$$\Rightarrow \quad A\hat{u} = b$$

Now for eigen problem

$$f_h = \lambda_h u_h = \lambda_h \sum_i (I - \lambda_h U_W)^{-1} U_i \hat{u}_h^i$$

For the global problem, the L.H.S is the same

But the r.H.S becomes

$$-\sum_{F\in\partial k} \tilde{N}_F^T M_k^{-1} \begin{pmatrix} 0 \\ \lambda_h u_h \end{pmatrix}$$

$$= -\sum \tilde{N}_F^T \begin{pmatrix} Q_W \\ U_W \end{pmatrix} \lambda_h u_h$$

$$= -\sum \tilde{N}_F^T \begin{pmatrix} Q_W \\ U_W \end{pmatrix} \lambda_h \sum (I - \lambda_h U_W)^{-1} U_i \hat{u}_h^i$$

$$= \lambda_h \sum \sum_i M(\lambda)\, \hat{u}_h^i$$

$$\begin{pmatrix} * & * & * \end{pmatrix} \begin{pmatrix} \hat{u}_h^i \end{pmatrix}$$

all the faces sharing the element with F

$$\Rightarrow \quad A\hat{u} = \lambda M(\lambda)\hat{u}$$

$\Rightarrow$ nonlinear eigenproblem

$\Rightarrow$ Newton iteration

$$F(\hat{u},\lambda) = \begin{pmatrix} A\hat{u} - \lambda M(\lambda)\hat{u} \\ \langle \hat{u},\hat{u}\rangle - 1 \end{pmatrix} = 0$$

$$\frac{\partial F(\hat{u},\lambda)}{\partial(\hat{u},\lambda)} \begin{pmatrix} \hat{u}' - \hat{u} \\ \lambda' - \lambda \end{pmatrix} + F(\hat{u},\lambda) = 0$$

$$\begin{pmatrix} A - \lambda M(\lambda) & [-M(\lambda) - \lambda M'(\lambda)]\hat{u} \\ 2\hat{u} & 0 \end{pmatrix} \begin{pmatrix} \hat{u}' - \hat{u} \\ \hat{\lambda}' - \hat{\lambda} \end{pmatrix} = -F(\hat{u},\lambda)$$

$N(\lambda) = M(\lambda) + \lambda M'(\lambda)$

$$(A - \lambda M(\lambda))(\hat{u}' - \hat{u}) - N(\lambda)(\lambda' - \lambda)\hat{u} = -(A\hat{u} - \lambda M(\lambda))\hat{u}$$

$$2\langle \hat{u}, \hat{u}' - \hat{u}\rangle = -(\langle \hat{u},\hat{u}\rangle - 1)$$

$$\Rightarrow \begin{cases} (A - \lambda M(\lambda))\hat{u}' = (\lambda' - \lambda) N(\lambda)\hat{u} \end{cases}$$

$| \quad \langle \ddot{u}, u \rangle = 1$ assume $\langle \hat{u}, \hat{u} \rangle = 1$

① solve $(A - \lambda M(\lambda))\eta = N(\lambda)\hat{u}$

$$\eta = \frac{\hat{u}'}{\lambda' - \lambda}$$

② $\frac{1}{\delta\lambda} := \langle \hat{u}, \eta \rangle = \frac{1}{\lambda' - \lambda}$

$\Rightarrow \lambda' = \lambda + \delta\lambda$

③ $\hat{u}' = \delta\lambda \cdot \eta$

Ignore the constant matrix

$$M(\lambda) = (I - \lambda_1 U_w)^{-1} U_i$$

$$(I - \lambda_1 U_w) M(\lambda) = U_i$$

$$-U_w M(\lambda) + (I - \lambda_1 U_w) M'(\lambda) = 0$$

$$\Rightarrow M'(\lambda) = (I - \lambda_1 U_w)^{-1} U_w M(\lambda)$$

$$= (I - \lambda_1 U_w)^{-1} U_w (I - \lambda_1 U_w)^{-1} U_i$$

Summary: step1: Local solver matrix $Q, U, Q_w, U_w, N_i$

step2: L.H.S of Global Eq.

$\}$ Same as source problem

step3: $M(\lambda)$ and $N(\lambda)$

step4: Iteration