Maxwell.

$$\begin{cases} \mu w - \nabla \times \underline{u} = 0 & \text{in } \Omega \\ \nabla \times w - \varepsilon \omega^2 \underline{u} + \varepsilon \nabla p = \underline{\dot{j}} & \text{in } \Omega \\ \nabla \cdot \varepsilon \underline{u} = 0 & \text{in } \Omega \\ \underline{u} \times \vec{n} = g & \text{on } \partial\Omega \\ p = 0 & \text{on } \partial\Omega \end{cases}$$

$$\nabla \times \underline{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$$

$$\underline{u} \times n = u_1 n_2 - u_2 n_1$$

$$\nabla \times w = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x} \right)$$

Find $(w_h, \underline{u_h}, p_h, \hat{u}_h^t, \hat{p}_h) \in P_h \times \underline{V_h} \times P_h \times M_h(g) \times M_h(0)$ s.t.

① $(\mu w_h, r) - (\underline{u_h}, \nabla \times r) + \langle \hat{u}_h^t \times n, r \rangle_{\partial \mathcal{T}_h} = 0$

② $(w_h, \nabla \times \underline{v}) + \langle \hat{w}_h, \underline{v} \times n \rangle - (\varepsilon \omega^2 \underline{u_h}, \underline{v})$

$\quad - (p_h, \nabla \cdot \varepsilon \underline{v}) + \langle \varepsilon \hat{p}_h, \underline{v} \cdot n \rangle = (\underline{j}, \underline{v})$

③ $-(\varepsilon \underline{u_h}, \nabla q) + \langle \hat{u}_h^n \cdot n, \varepsilon q \rangle = 0$

④ $\langle \hat{w}_h, \delta_{\partial \mathcal{T}_h} \eta \rangle = 0$

⑤ $\langle \hat{u}_h^n \cdot n, \varepsilon \xi \rangle = 0$

where $\hat{w}_h = w_h + \tau_t (u_h \times n - \hat{u}_h^t \times n)$

$\hat{u}_h^n \cdot n = u_h^n \cdot n + \tau_n (p_h - \hat{p}_h)$

$\hat{u}_h^t \times n = \delta_{\partial \mathcal{T}_h} \hat{u}_h^t$

Notice $|\hat{u}_h^t \times n|$ is single value. $\hat{u}_h^t \times n$ on a face just changes sign depending on which element we are sitting at. So we can replace $\hat{u}_h^t \times n$ $= \delta \hat{u}_h^t$ where $\hat{u}_h^t$

single value

is single value and $\delta\tau_h$ changes sign on faces.

Local Equation: 

$$(\mu w_h, r) - (\underline{u}_h, (\tfrac{\partial r}{\partial y}, -\tfrac{\partial r}{\partial x})) = \langle -\hat{u}_h^t \times n, r \rangle$$

$$((\tfrac{\partial w_h}{\partial y}, -\tfrac{\partial w_h}{\partial x}), \underline{v}) + \langle \tau_h^t \underline{u}_h \times n, \underline{v} \times n \rangle - (\varepsilon \omega^2 \underline{u}_h, \underline{v}_h) - (p_h, \nabla\cdot\varepsilon\underline{v}) = \langle \tau_h^t \hat{u}_h^t \times n, \underline{v} \times n \rangle - \langle \varepsilon \hat{p}_h, \underline{v}\cdot n \rangle + (\underline{j}, v)$$

$$(\nabla\cdot\varepsilon \underline{u}_h, q) + \langle \tau_n p_h, \varepsilon q \rangle = \langle \tau_n \hat{p}_h n, \varepsilon q \rangle$$

$$\begin{pmatrix} A_{ww} & A_{wu} & 0 \\ A_{uw} & A_{uu} & A_{up} \\ 0 & A_{pu} & A_{pp} \end{pmatrix} \begin{pmatrix} w_h \\ u_h \\ p_h \end{pmatrix} = \begin{pmatrix} Z_{w\hat{u}} & 0 \\ Z_{u\hat{u}} & Z_{u\hat{p}} \\ 0 & Z_{p\hat{p}} \end{pmatrix} \begin{pmatrix} \hat{u}_h^t \\ \hat{p}_h \end{pmatrix} + \begin{pmatrix} 0 \\ (\underline{j}, v) \\ 0 \end{pmatrix}$$

$$A_{ww} = (\mu \omega_j, N_i)$$

$$A_{wu} = -(\underline{u}_j, (\tfrac{\partial w_i}{\partial y}, -\tfrac{\partial w_i}{\partial x}))$$

$$A_{uw} = -A_{wu}^T = (u_i, (\tfrac{\partial w_j}{\partial y}, -\tfrac{\partial w_j}{\partial x}))$$

$$A_{uu} = \langle \tau_h^t \underline{u}_j \times n, \underline{u}_i \times n \rangle - (\varepsilon \omega^2 \underline{u}_j, \underline{u}_i)$$

$$A_{up} = -(p_j, \nabla\cdot\varepsilon u_i)$$

$$A_{pu} = (\nabla\cdot\varepsilon u_j, p_i) = -A_{pu}^T$$

$N_w = \frac{(k+1)(k+2)}{2}$

$N_u = 2\cdot N_w$

$N_p = N_w$

$N_{\hat{u}} = k+1$

$N_{\hat{p}} = k+1$

$$A_{pp} = \langle \tau_n p_j, \varepsilon p_i \rangle$$

$$Z_{w\hat{u}} = \langle \delta_{\Gamma_h} \hat{u}_j \;\; w_i \rangle$$

$$Z_{u\hat{u}} = \langle \tau_t \delta_{\Gamma_h} \hat{u}_j , \underline{u}_i \times n \rangle$$

$$Z_{u\hat{p}} = - \langle \varepsilon \hat{p}_j , \underline{u}_i \cdot n \rangle$$

$$Z_{p\hat{p}} = \langle \tau_n \hat{p}_j , \varepsilon p_i \rangle$$

Global Equations:

$$\langle \hat{w}_h , \delta_{\Gamma_h} \eta \rangle = 0$$
$$\langle \hat{u}_h \cdot n , \varepsilon \xi \rangle = 0$$

$$\Rightarrow \langle w_h + \tau_t (\underline{u}_h \times n - \delta \hat{u}_h) , \delta \eta \rangle = 0$$
$$\langle u_h \cdot n + \tau_n (p_h - \hat{p}_h) , \varepsilon \xi \rangle = 0$$

$$\Rightarrow (N_{\hat{u}w} , N_{\hat{u}u}) \begin{pmatrix} w_h \\ \underline{u}_h \end{pmatrix} - N_{\hat{u}\hat{u}} \hat{u}_h = 0$$

$$(M_{\hat{p}u} , M_{\hat{p}p}) \begin{pmatrix} u_h \\ p_h \end{pmatrix} - M_{\hat{p}\hat{p}} \hat{p}_h = 0$$

$Z^T_{w\hat{u}} = N_{\hat{u}w} = \langle w_j , \delta \hat{u}_i \rangle$ $\qquad$ $N_{\hat{u}\hat{u}} = \langle \tau_t \delta \hat{u}_j \delta \hat{u}_i \rangle$

$Z^T_{u\hat{u}} = N_{\hat{u}u} = \langle \tau_t \underline{u}_j \times n, \delta \hat{u}_i \rangle$

$-Z^T_{u\hat{p}} = M_{\hat{p}u} = \langle \underline{u}_j \cdot n, \varepsilon \hat{p}_i \rangle$ $\qquad M_{\hat{p}\hat{p}} = \langle \tau_n \hat{p}_j, \varepsilon \hat{p}_i \rangle$

$Z^T_{p\hat{p}} = M_{\hat{p}p} = \langle \tau_n p_j, \varepsilon \hat{p}_i \rangle$

$$\begin{pmatrix} N_{\hat{u}w} & N_{\hat{u}u} & 0 \\ 0 & M_{\hat{p}u} & M_{\hat{p}p} \end{pmatrix} \begin{pmatrix} w_h \\ u_h \\ p_h \end{pmatrix} - \begin{pmatrix} N_{\hat{u}\hat{u}} & 0 \\ 0 & M_{\hat{p}\hat{p}} \end{pmatrix} \begin{pmatrix} \hat{u}_h \\ \hat{p}_h \end{pmatrix} = 0$$

$$\begin{pmatrix} w_h \\ u_h \\ p_h \end{pmatrix} = Loc \begin{pmatrix} \hat{u}_h \\ \hat{p}_h \end{pmatrix} + Loc\text{-}f$$