We define $\lambda_h(u_h) = \dfrac{a_h(u_h, u_h)}{\|u_h\|^2}$

Then $(\lambda - \lambda_h(u_h))\|u_h\|^2$

$$= \lambda\|u_h\|^2 - a_h(u_h, u_h)$$

$$= \lambda\|u_h - u\|^2 + 2\lambda(u, u_h) - \lambda(u,u) - a_h(u_h, u_h)$$

For all three eigensolvers, we have

$$a_h(u,u) = (\nabla\times u, \nabla\times u)$$
$$= \lambda(u,u)$$

$\boxed{\text{since } u \in H_0(\text{curl}) \text{ and } \nabla\times(\nabla\times u) = \lambda u}$

$$a_h(u, u_h) = (\nabla\times u, \nabla\times u_h)$$
$$= \lambda(u, u_h) - \langle \nabla\times u, u_h\times n\rangle$$

So

$$(\lambda - \lambda_h(u_h))\|u_h\|^2 = \lambda\|u - u_h\|^2 + 2a_h(u, u_h) + 2\langle\nabla\times u, u_h\times n\rangle - a_h(u,u) - a_h(u_h, u_h)$$

$$= \lambda\|u - u_h\|^2 - a_h(u - u_h, u - u_h) + 2\langle\nabla\times u, u_h\times n\rangle$$

$$= \lambda\|u - u_h\|^2 - a_h(u - u_h, u - u_h) + 2\sum_e \langle\nabla\times u, [\![u_h]\!]_t\rangle_e$$

$$= \underbrace{\lambda\|u - u_h\|^2 - a_h(u - u_h, u - u_h) + 2\sum_e \langle\nabla\times u - \widehat{(\nabla\times u)}_h, [\![u_h - u]\!]_t\rangle_e}_{E_h}$$

$$+ \underbrace{2\sum_e \langle\widehat{(\nabla\times u)}_h, [\![u_h]\!]_t\rangle_e}_{AC_h}$$

Here $\widehat{(\nabla \times u)}_h$ is a single value approximation of $\nabla \times u$ on each edge e we need to specify.

For example, we can take

$$\widehat{(\nabla \times u)}_h = \overline{(\nabla \times u_h)}_e .$$

Then $AC_h = 2 \sum_e \langle \overline{(\nabla \times u_h)}_e, [\![u_h]\!]_t \rangle_e$. This is the $D_h^{⑥}$ in Prof. Li's notes.

As you can see, the first two terms in E_h are already bounded by the energy norm $\|u-u_h\|_h^2 := a(u-u_h, u-u_h)$. So we just need to choose $\widehat{(\nabla \times u)}_h$ such that the third term $2\sum_e \langle \nabla \times u - \widehat{(\nabla \times u)}_h, [\![u_h]\!]_t \rangle_e$ can be bounded by the energy norm.

// From the numerical results in Prof. Li's paper, it seems that λ_h approximates λ from below. Since our formula is

$$\lambda - \lambda_h = \frac{AC_h}{\|u_h\|^2} + \frac{E_h}{\|u_h\|^2}$$

I think if we can make AC_h be a small positive term, then the error can be reduced.

We may grab the penalty terms in $a(u-u_h, u-u_h)$ and add them to the AC_h, but these terms are negative, for example: $-\sum_e \frac{(\Phi_\mu(e))^2}{|e|} \int_e [\![u_h]\!]_t^2 \, ds$.

So I think adding these penalty terms may not help reduce the error, unless the ACs we define is too big.