2D HDG with triangular mesh and adaptivity.

1. ☐ Mesh
   - vertices ✓
   - elements + edges ✓
   - dirichlet bdy ✓

2. ☐ Basis & Quadrature
   - ϕ, $\frac{\partial \phi}{\partial x_i}$, ✓
   - Qud on K, ∂K ✓
   - normal ✓

3. ☐ Local matrix — Local solver
4. ☐ Global matrix — global solver

$\Rightarrow$ HDG

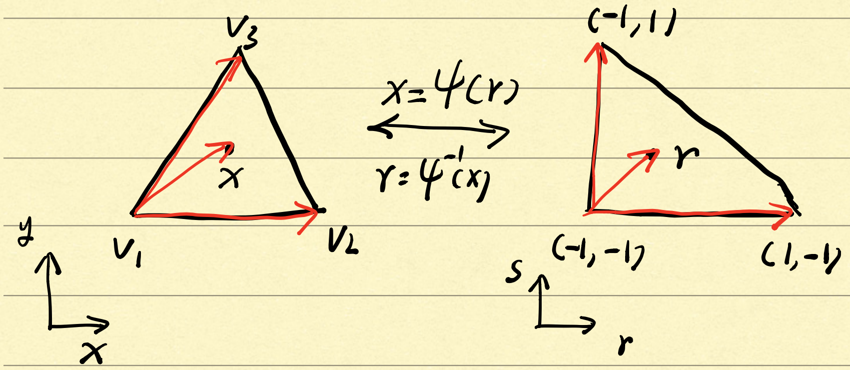
5. ☐ Functional
6. ☐ Adaptive Refinement

7. ☐ Error and visualization.

Basis & Quadrature.

P_k: dim=2 $\quad N_k = \frac{(k+2)(k+1)}{2} = C_{k+2}^{2}$

① General Triangle $\rightleftarrows$ Reference triangle



$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A\left[\begin{pmatrix} r \\ s \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right]$$

Barycentric coordinates: $\quad 0 \le \lambda_i \le 1, \ \sum \lambda_i = 1$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{x} = \lambda_1 \vec{V_1} + \lambda_2 \vec{V_2} + \lambda_3 \vec{V_3}$$

$$\begin{pmatrix} r \\ s \end{pmatrix} = \vec{r} = \lambda_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\Rightarrow r = -\lambda_1 + \lambda_2 - \lambda_3 \quad \Rightarrow \lambda_2 = \frac{r+1}{2} \quad \Rightarrow \quad \lambda_1 = -\frac{r+s}{2}$

$\quad\ s = -\lambda_1 - \lambda_2 + \lambda_3 \quad \Rightarrow \lambda_3 = \frac{s+1}{2}$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \vec{x} = -\frac{r+s}{2} \vec{V_1} + \frac{r+1}{2} \vec{V_2} + \frac{s+1}{2} \vec{V_3}}$$

$$= \frac{r+1}{2}(\vec{V_2} - \vec{V_1}) + \frac{s+1}{2}(\vec{V_3} - \vec{V_1}) + \vec{V_1}$$

$$\vec{x} - \vec{V_1} = \frac{1}{2}\left(\vec{V_2} - \vec{V_1}, \vec{V_3} - \vec{V_1}\right)\left(\begin{pmatrix} r \\ s \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)$$

$$\frac{\partial \vec{x}}{\partial \vec{r}} = \begin{pmatrix} x_r & x_s \\ y_r & y_s \end{pmatrix} = \left(\frac{\vec{V_2} - V_1}{2}, \frac{\vec{V_3} - V_1}{2} \right)$$

Jacobian $J = x_r y_s - x_s y_r$

② Basis on Ref Triangle.

$$u(\vec{r}) = \sum_{n=1}^{N_k} \hat{u}_n \phi_n(\vec{r}) = \sum_{n=1}^{N_k} u(r_i)\, l_i(r)$$

Legendre interpolation

modal

nodal

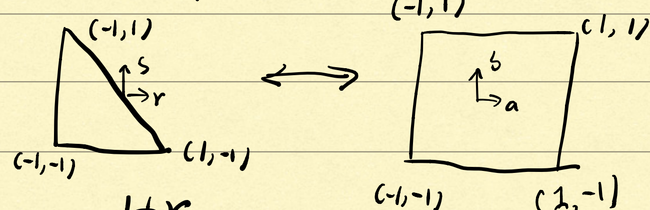
Vandermonde Matrix

* Relation $\quad V\hat{u} = \begin{pmatrix} \vdots \\ u(r_i) \\ \vdots \end{pmatrix}$ where $V = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \phi_1(r_i), & \phi_2(r_i) & \cdots & \phi_n(r_i) \\ \vdots & \vdots & & \vdots \end{pmatrix}$

$$V^T \begin{pmatrix} l_1(r) \\ \vdots \\ l_{N_k}(r) \end{pmatrix} = (\phi_1(r) \cdots \phi_n(r)) \qquad V_{ij} = \phi_j(r_i)$$

since $\phi_i(r) = \sum_j \phi_i(r_j)\, l_j(r)$ by interpolation

Dubiner basis



$$\begin{cases} a = 2\dfrac{1+r}{1-s} - 1 \\ b = s \end{cases}$$

Legendre poly.

Jacobi Polynomial

$\boxed{\begin{array}{l} \text{For } i = 0 : N \\ \quad \text{For } j = 0 : N-i \end{array}}$

$$\phi_m(\vec{r}) = \sqrt{2}\, P_i(a)\, P_j^{(2i+1,0)}(b)\,(1-b)^i$$

i	
0	0 1, ⋯ k
1	0, 1, ⋯ k-1
⋮	
k	0

$$m = j + (k+1)i + 1 - \frac{i}{2}(i-1) \qquad i,j \geq 0 \quad i+j \leq k$$

ϕ_m is a polynomial of degree $i+j$

and $(i,j) \neq (p,q)$, $\phi_{ij} \perp \phi_{pq}$

ref triangle

☆ $\phi_n(\vec{r})$ is an orthonormal basis on 

③ Quadrature. / Nodal points

Need ① $(\phi_{ij}, \phi_{pq})_K$

② $(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial r})_K$ $(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial s})_K$

③ $(f, \phi_{ij})_K$

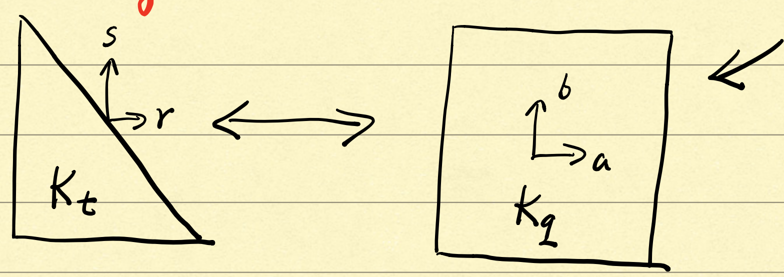
Volume Integral

④ $\langle \phi_{ij}|_F, l_p \rangle_F$

⑤ $\langle l_i, l_j \rangle_F$

Face integral.

Volume Integral



use Gauss Quadrature here.

$$\begin{cases} a = 2\frac{1+r}{1-s} - 1 \\ b = s \end{cases} \Rightarrow \begin{cases} r = \frac{(a+1)(1-b)}{2} - 1 \\ s = b \end{cases}$$

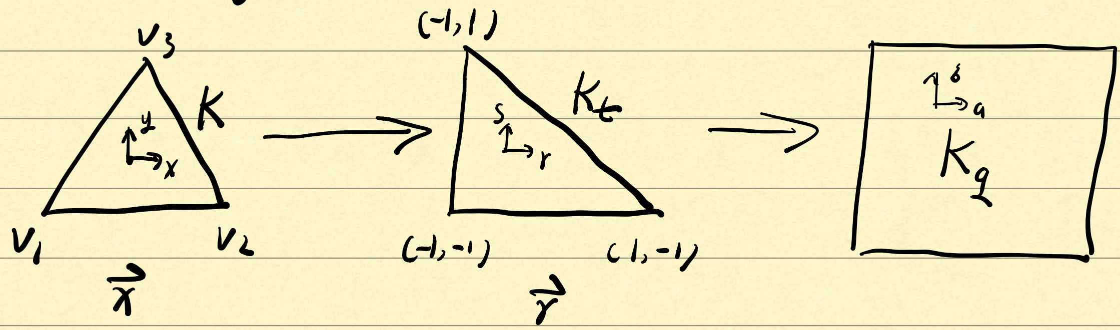
$$\frac{\partial(r,s)}{\partial(a,b)} = \begin{pmatrix} \frac{1-b}{2} & -\frac{a+1}{2} \\ 0 & 1 \end{pmatrix}$$

Jacobian $\left|\frac{\partial(r,s)}{\partial(a,b)}\right| = \frac{1-b}{2}$

$$\Rightarrow \boxed{\int_{K_t} f(r,s)\,dr\,ds = \int_{K_q} f\left(\frac{(a+1)(1-b)}{2}-1,\, b\right)\frac{1-b}{2}\,da\,db}$$

Use Gauss Quadrature on a squre (1D tensor product)

More generally:



$$\vec{x} = -\frac{r+s}{2}\vec{v}_1 + \frac{r+1}{2}\vec{v}_2 + \frac{s+1}{2}\vec{v}_3 \qquad \begin{cases} r = \frac{(a+1)(1-b)}{2}-1 \\ s = b \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(r,s)}$$

$$\int_K f(\vec{x})\,dx\,dy = \int_{K_t} f(\vec{x}(r,s))\det J\,dr\,ds$$

$$= \int_{K_q} f(\vec{x}(\vec{r}(a,b))\det J \cdot \frac{1-b}{2}\,da\,db$$

$$= \int_{K_q} F(a,b)\,da\,db$$

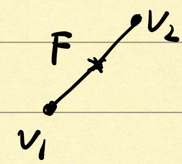
$$= \int^1 \int^1 F(a,b)\,da\,db = \int^1 \sum w_j F(a,x_j)\,da$$

$$\int_{-1}\int_{-1} \cdots \int_{-1} \cdots$$

$$= \sum_{i,j} w_j w_i F(x_i, x_j)$$

$$= (\cdots w_i \cdots)\Big(F(x_i, x_j) \Big)\begin{pmatrix} \vdots \\ w_j \\ \vdots \end{pmatrix} \qquad \checkmark$$

Face Integral.



$$\vec{r} = \frac{v_1+v_2}{2} + t\,\frac{v_1-v_2}{2}$$

$$-1 \le t \le 1$$

$$f(\vec{r})\big|_F = f\left(t\,\frac{v_1-v_2}{2} + \frac{v_1+v_2}{2}\right)$$

$$= \tilde{f}(t) \qquad -1 \le t \le 1$$

$$\int_F f(\vec{r})\,ds$$

$$= \int_{-1}^{1} f\left(t\,\frac{v_1-v_2}{2} + \frac{v_1+v_2}{2}\right) \left|\frac{v_1-v_2}{2}\right| dt$$

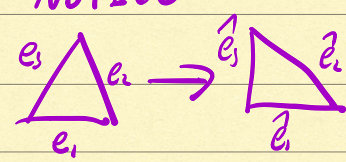
$$\frac{h_e}{2}$$

$$= \int_{-1}^{1} \tilde{f}(t) \left|\frac{v_2-v_1}{2}\right| dt$$

$$= \sum w_i \tilde{f}(x_i)\,\frac{h_e}{2}$$

$$= \sum w_i f\left(x_i\,\frac{v_1-v_2}{2} + \frac{v_1+v_2}{2}\right)\frac{h_e}{2}$$

NOTICE



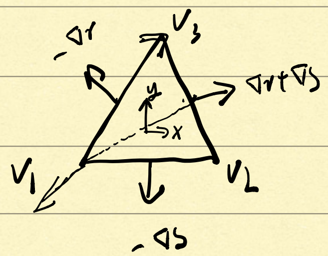
$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} r+1 \\ s+1 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

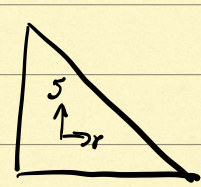
e_1: $-1 \le r \le 1$, $s = -1$

e_2: $r + s = 0$

e_3: $r = -1$ $\quad -1 \le s \le 1$.

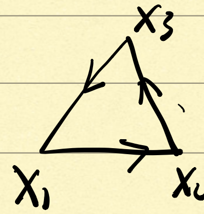
Normal:



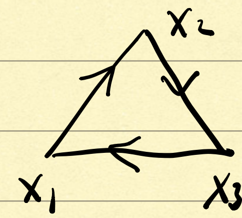


① check the 3 vertices is clockwise or counterclockwise

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



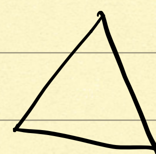
$\Delta > 0$ counterclock



$\Delta < 0$ clockwise

Rotation matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ ≪ counterclockwise Rotate 90°



$$n_1 = -\text{sign}(\Delta) R (X_2 - X_1)$$
$$n_2 = -\text{sign}(\Delta) R (X_3 - X_2)$$ outer normal
$$n_3 = -\text{sign}(\Delta) R (X_1 - X_3)$$

or $R' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ clockwise rotate 90°

$$n_i = \text{sign}(\Delta) R' (X_{i+1} - X_i)$$

Partial derivatives

$$\begin{cases} a = 2\dfrac{1+r}{1-s} - 1 \\ b = s \end{cases}$$

$$\phi(r,s) = \sqrt{2}\, P_i(a)\, P_j^{(2i+1,0)}(b)\,(1-b)^i$$

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial a}\frac{\partial a}{\partial r} + \frac{\partial \phi}{\partial b}\underbrace{\frac{\partial b}{\partial r}}_{=0}$$

$$= \frac{\partial \phi}{\partial a}\frac{\partial a}{\partial r}$$

$$= \frac{2}{1-s}\frac{\partial \phi}{\partial a}$$

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial a}\frac{\partial a}{\partial s} + \frac{\partial \phi}{\partial b}\frac{\partial b}{\partial s}$$

$$= \frac{\partial \phi}{\partial a}\cdot\frac{2(1+r)}{(1-s)^2} + \frac{\partial \phi}{\partial b}$$

$$\phi_{ij} = \sqrt{2}\, P_i(a)\, P_j^{(2i+1,0)}(b)\,(1-b)^i$$

$$\frac{\partial \phi_{ij}}{\partial a} = \sqrt{2}\, P_i'(a)\, P_j^{(2i+1,0)}(b)\,(1-b)^i$$

$$\frac{\partial \phi_{ij}}{\partial b} = \sqrt{2}\, P_i(a)\, P_j'^{(2i+1,0)}(b)\,(1-b)^i$$

$$+ \sqrt{2}\, P_i(a)\, P_j^{(2i+1,0)}(b)\cdot i(1-b)^{i-1}\cdot(-1)$$

DG SPACES on element K

$\deg = k \qquad N_k = \dfrac{(k+2)(k+1)}{2}$

$P_k(K)$: $\phi_m = \phi_{ij} \qquad 0 \le i, j \le k \quad i+j \le k.$

$[P_k(K)]^2$: $(\phi_m, 0)$, $(0, \phi_m)$

$P_k(F)$: $l_i \qquad 0 \le i \le k$

$\uparrow$ Legendre.

To make $\hat{u}_h \in P_k(F)$ has single value, we need to define the oritation of each face, e.g. $1 \longrightarrow 2$

//

HDG Method For Poission problem

$k, \quad N_u = \dfrac{(k+2)(k+1)}{2} \quad N_q = 2 \cdot N_u \quad N_{\hat{u}} = k+1$

$\Rightarrow DOF = N_u + N_q + N_{\hat{u}}$

$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} r \\ s \end{pmatrix} + b \qquad \phi_m \begin{pmatrix} x \\ y \end{pmatrix} = \phi_m \left(A \begin{pmatrix} r \\ s \end{pmatrix} + b \right)$

$u_h \in \mathrm{span}\{\phi_m\}_{m=1}^{N_u}$

$q_h \in \mathrm{span}\{(\phi_m, 0)\}_{m=1}^{N_u} \oplus \mathrm{span}\{(0, \phi_m)\}_{m=1}^{N_u}$

$\hat{u}_h \in \mathrm{span}\{l_i\}_{i=1}^{N_{\hat{u}}}$

Local Equations

$$\begin{cases} (q_h, r) - (u_h, \nabla \cdot r) + \langle \hat{u}_h, r \cdot n \rangle = 0 \\ (\nabla \cdot q_h, w) + \langle (\hat{q}_h - q_h) \cdot n, w \rangle = (f, w) \end{cases}$$

$\because \hat{q}_h n = q_h n + \tau(u_h - \hat{u}_h)$

$$\Rightarrow \begin{cases} (q_h, r) - (u_h, \nabla\cdot r) = -\langle \hat{u}_h, r\cdot n\rangle \\ (\nabla\cdot q_h, w) + \langle \tau u_h, w\rangle = \langle \tau \hat{u}_h, w\rangle + (f, w) \end{cases}$$

$$\begin{pmatrix} \underset{}{\overset{N_q\times N_q}{A_{qq}}} & \overset{N_q\times N_u}{A_{qu}} \\ \underset{N_u\times N_q}{A_{uq}} & \underset{N_u\times N_u}{A_{uu}} \end{pmatrix} \begin{pmatrix} \overset{N_q}{q_h} \\ \underset{N_u}{u_h} \end{pmatrix} = \begin{pmatrix} \overset{N_q\times N_{\hat{u}}}{B_{q\hat{u}}} \\ \underset{N_u\times N_{\hat{u}}}{B_{u\hat{u}}} \end{pmatrix} (\hat{u}_h) + \begin{pmatrix} 0 \\ \underset{N_u\times 1}{F_u} \end{pmatrix}$$

(1) $A_{qq} = (\text{Basis\_q}_1, \text{Basis\_q}_2)_K = (\phi_m, \phi_n)_K = J_K (\phi_m, \phi_n)_{\hat{K}}$

$= J_K \boxed{Id}$

Local matrix

$A_{qu} = -(\text{Basis\_u}, \nabla\cdot \text{Basis\_q})$

$A_{uq} = (\nabla\cdot \text{Basis\_q}, \text{Basis\_u}) = -A_{qu}^T$

Compute A_{uq}: $(\text{Basis\_u}, \nabla\cdot \text{Basis\_q})_{N_u\times N_q}$

① $\text{Basis\_u} = \phi_m = \phi_{ij}$

$\text{Basis\_q} = (\phi_n, 0) = (\phi_{pq}, 0)$

$$\Rightarrow \left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial x}\right)_K = J_K \left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial x}\right)_{\hat{K}}$$

② $\text{Basis\_u} = \phi_m = \phi_{ij}$

$\text{Basis\_q} = (0, \phi_n) = (0, \phi_{pq})$

$$\Rightarrow \left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial y}\right) = J_K \left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial y}\right)_{\hat{K}}$$

$$\frac{\partial \phi_m}{\partial x} = \frac{\partial \phi_m}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial \phi_m}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial \phi_m}{\partial y} = \frac{\partial \phi_m}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi_m}{\partial s}\frac{\partial s}{\partial y}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial \phi_m}{\partial x} \\ \frac{\partial \phi_m}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \phi_m}{\partial r} \\ \frac{\partial \phi_m}{\partial s} \end{pmatrix}$$

$$= (A^{-1})^T \begin{pmatrix} \frac{\partial \phi_m}{\partial r} \\ \frac{\partial \phi_m}{\partial s} \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} r \\ s \end{pmatrix} + b$$

$$\Rightarrow \begin{pmatrix} r \\ s \end{pmatrix} = A^{-1}\begin{pmatrix} x \\ y \end{pmatrix} - A^{-1} b$$

$$\boxed{A^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{pmatrix}}$$

Need to precompute this (mesh related)

$$\left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial x}\right)_K = \left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi_{pq}}{\partial s}\frac{\partial s}{\partial x}\right)_K$$

$$= J_K \left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial \phi_{pq}}{\partial s}\frac{\partial s}{\partial x}\right)_{\hat{K}}$$

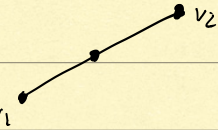
$$= J_K\left[\frac{\partial r}{\partial x}\underbrace{\boxed{\left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial r}\right)}}_{\text{Local matrix}} + \frac{\partial s}{\partial x}\underbrace{\boxed{\left(\phi_{ij}, \frac{\partial \phi_{pq}}{\partial s}\right)}}_{\text{Local matrix}}\right]$$

do numerical Quadrature

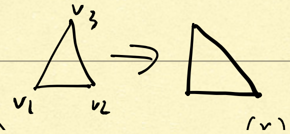
(3) $$A_{uu} = \langle \tau\, \text{Basis\_u}, \text{Basis\_u} \rangle$$

$$= \sum_{i=1}^{3} \langle \tau \phi_{ij}, \phi_{pq} \rangle_{F_i}$$

$$= \sum_{i=1}^{3} \frac{\tau |e_i|}{2} \int^{1} \phi_{ij}\left(\begin{pmatrix} r \\ s \end{pmatrix}\Big|_{e_i}\right)\cdot \phi_{pq}\left(\begin{pmatrix} r \\ s \end{pmatrix}\Big|_{e_i}\right)$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} r+1 \\ s+1 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$



$i=1 \qquad j=1$

$\Rightarrow$ Gauss Quadrature on 1D

$v_1 v_2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$v_1 v_3 = \begin{pmatrix} -1 \\ s \end{pmatrix}$

$v_2 v_3 = \begin{pmatrix} r \\ -r \end{pmatrix}$

④ $B_{q\hat{u}} = -\langle \hat{u}, q \cdot n \rangle_{F_i} \qquad i=1,2,3$

$= -\langle l_i, \phi_m \cdot n_t \rangle_{F_i}$

$= -n_t \langle l_i, \phi_m \rangle_{F_i}$

$t = 1$ or 2

$n = (n_1, n_2)$

$q = (\phi_m, 0)$ or $(0, \phi_m)$

$B_{u\hat{u}} = \langle \tau \hat{u}_h, w \rangle_{F_i}$

$= \tau \langle l_i, \phi_m \rangle_{F_i}$

$\Rightarrow \quad = \tau \dfrac{|e_i|}{2} \displaystyle\int_{-1}^{1} l_i \, \phi_m \left(A \begin{pmatrix} r+1 \\ s+1 \end{pmatrix}\Big|_{e_i} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right) dt$

$\swarrow$

Gauss Quad. ✓

⑤ $F_u = (f, \text{Basis\_u})$

$$\int_{K_t} f \phi_m = J_k \int_{\hat{K}_t} f\left(A \begin{pmatrix} r+1 \\ s+1 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right) \phi_m(r,s) \, dr\, ds$$

$$= J_k \int_{K_2} \tilde{f}(a,b)\,\phi_m(a,b)\,\frac{1-b}{2}\,da\,db$$

See Integrals on Triangle for details

//

Global Equations

$$\langle q_h \cdot n + \tau(u_h - \hat{u}_h), \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0$$

$$\Rightarrow \sum_K \sum_{F_i} \langle q_h \cdot n + \tau(u_h - \hat{u}_h), \mu \rangle_{F_i} = 0$$

$$\langle q_h \cdot n, \mu \rangle = \langle Basis\_q_j \cdot n, \mu_i \rangle$$
$$= -B_{q\hat{u}}^T$$

$$\langle \tau u_h, \mu \rangle = \langle \tau\, Basis\_u_j, \mu_i \rangle$$
$$= B_{u\hat{u}}^T$$

$$\Rightarrow \langle q_h \cdot n + \tau u_h, \mu \rangle_{F_i} = \left(-B_{q\hat{u}}^T,\ B_{u\hat{u}}^T\right)_i \begin{pmatrix} q_h \\ u_h \end{pmatrix}$$

$$\begin{pmatrix} q_h \\ u_h \end{pmatrix} = \sum_i \begin{pmatrix} Q_i \\ U_i \end{pmatrix} \hat{u}_i + \begin{pmatrix} Q_w \\ U_w \end{pmatrix}$$

$$\langle q_h \cdot n + \tau u_h, \mu \rangle_i - \langle \tau \hat{u}, \mu \rangle_i$$

$$= \sum_j \left(-B_{q\hat{u}}^T, B_{u\hat{u}}^T\right)_i \begin{pmatrix} Q_j \\ U_j \end{pmatrix} \hat{u}_j + \left(-B_{qu}^T, B_{u\hat{u}}^T\right)_i \begin{pmatrix} Q_w \\ U_w \end{pmatrix}$$

$$- \tau \, Id \, \hat{u}_i$$

$$= \sum_{j \neq i} \left(-B_{qu}^T \; B_{u\hat{u}}^T\right)_i \begin{pmatrix} Q_j \\ U_j \end{pmatrix} \hat{u}_j$$

coupled unknows

main unknown

$$+ \left[\left(-B_{qu}^T \; B_{u\hat{u}}^T\right)_i \begin{pmatrix} Q_i \\ U_i \end{pmatrix} - \tau \, Id\right] \hat{u}_i$$

$$+ \left(-B_{qu}^T \; B_{uu}^T\right)_i \begin{pmatrix} Q_w \\ U_w \end{pmatrix}$$

move to right hand side.