**Math 3029/6109, Semester 1, 2019**

**Probability Modelling with Applications**

**Mid Semester Practice Examination**

.

This exam contains 4 problems, each worth 25 points. Answer all questions.

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**Problem 1.** An urn contains 4 red balls and 3 white balls, and we draw three balls from the urn without replacement.

(a) (6 points) What is the probability that all 3 balls are red?

A                B

(b) (9 points) What is the conditional probability that all 3 balls are red, given that at least 2 are red?

(c) (10 points) What is the conditional probability that all 3 balls are red, given that the third ball drawn is red?

*Solution.* (a): Let $X$ be the total number of balls drawn of each color, so that $X$ is a $\mathbb{Z}_+^2$-valued random variable. Since we draw the balls without replacement, $X$ has the hypergeometric distribution $\mathcal{H}_{3,(4,3)}$. The relevant probability is equal to

what is the r.v. & its distribution

$$\mathbb{P}(X = (3,0)) = \mathcal{H}_{3,(4,3)}(\{(3,0)\}) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4 \cdot 1}{\frac{7\cdot 6\times 5}{6}} = \frac{4}{35}.$$

(b): Since

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$\mathbb{P}(X = (2,1)) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{6 \cdot 3}{35} = \frac{18}{35},$$

the relevant probability is

$$\mathbb{P}(X = (3,0) \mid X \in \{(2,1),(3,0)\}) = \frac{\mathbb{P}(X = (3,0))}{\mathbb{P}(X = (3,0)) + \mathbb{P}(X = (2,1))} = \frac{4/35}{4/35 + 18/35} = \frac{4}{22} = \frac{2}{11}.$$

(c): Let $Y$ be the ordered tuple of colors drawn, so that $Y$ is an $\{R, W\}^3$-valued random variable. The relevant probability is

$\Omega = \{R, W\}^3$

$$\begin{aligned}&\mathbb{P}(Y = (R,R,R) \mid Y \in \{(R,R,R),(R,W,R),(W,R,R),(W,W,R)\}) \\ &= \frac{\mathbb{P}(Y = (R,R,R))}{\mathbb{P}(Y = (R,R,R)) + \mathbb{P}(Y = (R,W,R)) + \mathbb{P}(Y = (W,R,R)) + \mathbb{P}(Y = (W,W,R))}.\end{aligned}$$

Write $Y = (Y_1, Y_2, Y_3)$. Then Lecture 9 Thm 1.7

$$\mathbb{P}(Y = (R,R,R)) = \mathbb{P}(Y_1 = R)\mathbb{P}(Y_2 = R \mid Y_1 = R)\mathbb{P}(Y_3 = R \mid Y_1 = Y_2 = R) = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5} = \frac{24}{7\cdot 6\cdot 5}.$$

Similarly, one has

$$\mathbb{P}(Y = (R,W,R)) = \mathbb{P}(Y_1 = R)\mathbb{P}(Y_2 = W \mid Y_1 = R)\mathbb{P}(Y_3 = R \mid Y_1 = R, Y_2 = W) = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5} = \frac{36}{7\cdot 6\cdot 5},$$

$$\mathbb{P}(Y = (W,R,R)) = \mathbb{P}(Y_1 = W)\mathbb{P}(Y_2 = R \mid Y_1 = W)\mathbb{P}(Y_3 = R \mid Y_1 = W, Y_2 = R) = \frac{3}{7}\cdot\frac{4}{6}\cdot\frac{3}{5} = \frac{36}{7\cdot 6\cdot 5},$$

$$\mathbb{P}(Y = (W,W,R)) = \mathbb{P}(Y_1 = W)\mathbb{P}(Y_2 = W \mid Y_1 = W)\mathbb{P}(Y_3 = R \mid Y_1 = Y_2 = W) = \frac{3}{7}\cdot\frac{2}{6}\cdot\frac{4}{5} = \frac{24}{7\cdot 6\cdot 5}.$$

Hence the relevant probability is

$$\frac{24}{24 + 36 + 36 + 24} = \frac{2}{2+3+3+2} = \frac{1}{5}.$$

□

**Problem 2.** Let $X$, $Y$ and $Z$ be independent real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) (10 points) Suppose that the distribution of $X$ has a Lebesgue density function $\rho : \mathbb{R} \to [0, \infty)$ given by

$$\rho(x) = \begin{cases} 0 & x \in (-\infty, \mu - s], \\ \frac{1}{2s}(1 + \cos(\frac{x-\mu}{s}\pi)) & x \in (\mu - s, \mu + s), \\ 0 & x \in [\mu + s, \infty), \end{cases}$$

$f_{a,b}(s) = \frac{1}{B(a,b)} s^{a-1}(1-s)^{b-1}$

$F_X(c) = \int_{-\infty}^{c} \rho(x)dx$

for certain parameters $\mu \in \mathbb{R}$ and $s > 0$. Determine the cumulative distribution function $F_X$ of $X$.

(b) (15 points) Suppose that $X$ has the beta distribution $\boldsymbol{\beta}_{5,1}$, $Y$ the exponential distribution $\mathcal{E}_\alpha$ for some $\alpha > 0$, and $Z$ the gamma distribution $\Gamma_{\alpha,6}$. Determine

pdf $= \alpha e^{-\alpha x}$, $x > 0$

$\Gamma_{\alpha,r}(x) = e^{-\alpha x}\frac{\alpha^r x^{r-1}}{(r-1)!}$

$$\frac{\mathbb{P}(Y \leq 1 \text{ and } X \leq Y)}{\mathbb{P}(Z \leq 1)}.$$

$dx = \frac{s}{\pi} d(\frac{x-\mu}{s}\pi)$

*Solution.* (a): Note that

$$F_X(c) = \int_{-\infty}^{c} \rho(x)\mathrm{d}x \qquad (c \in \mathbb{R}).$$

$y = \frac{x-\mu}{s}\pi \in (-\pi, \frac{(c-\mu)\pi}{s})$

$x \in (\mu - s, c)$

Hence we obtain $F(c) = 0$ for $c \in (-\infty, \mu - s]$, and

$$\begin{aligned} F_X(c) = \int_{-\infty}^{c} \rho(x)\mathrm{d}x &= \frac{1}{2s}\int_{\mu-s}^{c}(1 + \cos(\frac{x-\mu}{s}\pi))\mathrm{d}x = \frac{1}{2s}\int_{\mu-s}^{c}\frac{s}{\pi}(1+\cos(\frac{x-\mu}{s}\pi))d(\frac{x-\mu}{s}\pi) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi(c-\mu)/s}(1 + \cos(x))\mathrm{d}x = (x + \sin(x))\restriction_{-\pi}^{\pi(c-\mu)/s} = \frac{1}{2\pi}\int_{-\pi}(1+\cos y)\,dy \\ &= \frac{1}{2}(\frac{c-\mu}{s} + \frac{1}{\pi}\sin(\pi\frac{c-\mu}{s}) + 1) \leftarrow \mu + s \end{aligned}$$

for $c \in (\mu - s, \mu + s]$. In particular, $F_X(\mu + s) = 1$, and then $F_X(c) = 1$ for all $c > \mu + s$.

(b): The density $\rho$ of the random variable $(X, Y)$ is the product of the densities $\rho_X$ of $X$ and $\rho_Y$ of $Y$, since $X$ and $Y$ are independent. One has

$= 5x^{5-1}(1-x)^{1-1} = 1$

$$\rho_X(x) = 5x^4 \qquad (x \in (0, 1))$$

$B(r, n-r+1) = \frac{(r-1)!(n-r)!}{n!}$

$B(5,1) = \frac{0!(5-1)!}{5!} = \frac{1}{5}$

$r = 1$, $n - r + 1 = 5$

$n = 5$

and

$$\rho_Y(y) = \alpha e^{-\alpha y} \qquad (y > 0).$$

So

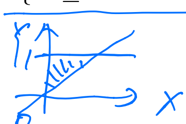
$$\rho(x, y) = 5\alpha x^4 e^{-\alpha y} \qquad (x \in (0, 1), y > 0).$$

Also, the density of $Z$ is given by

$$\rho_Z(z) = \frac{\alpha^6}{\Gamma(6)} z^5 e^{-\alpha z} = \frac{\alpha^6}{5!} z^5 e^{-\alpha z} \qquad (z > 0).$$

The probability of $\{Y \leq 1 \text{ and } X \leq Y\}$ is equal to the integral of $\rho$ over the region

$$D := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y \leq 1\},$$




which is equal to

$$\begin{aligned} \iint_D \rho(x, y)\mathrm{d}x\mathrm{d}y &= 5\alpha\int_0^1 e^{-\alpha y}\int_0^y x^4\mathrm{d}x\mathrm{d}y = \alpha\int_0^1 y^5 e^{-\alpha y}\mathrm{d}y \\ &= \frac{5!}{\alpha^5}\int_0^1 \frac{\alpha^6}{5!} y^5 e^{-\alpha y}\mathrm{d}y = \frac{5!}{\alpha^5}\mathbb{P}(Z \leq 1). \end{aligned}$$

$\mathbb{P}(Z \leq 1) = \int_0^1 \frac{\alpha^6}{5!} z^5 e^{-\alpha z} dz$

So

$$\frac{\mathbb{P}(Y \leq 1 \text{ and } X \leq Y)}{\mathbb{P}(Z \leq 1)} = \frac{120}{\alpha^5}.$$

□

**Problem 3.** Let $X$ and $Y$ be independent standard normal random variables.

$N(0,1)$

r.v. $X$ pdf $aX$ $F_{aX}(c) = F_X(\frac{c}{a})$



$aN_{0,1} = N_{0,a^2}$

(a) (10 points) Determine the probability density of $X + 2Y$.

(b) (15 points) Determine the probability that $|X| \geq \sqrt{3}|Y|$.

$2Y \leq c \iff Y \leq \frac{c}{2}$

*Solution.* (a): Let $\rho_Y$ be the probability density of $Y$. Then $2Y$ has density $\rho_{2Y} : \mathbb{R} \to [0, \infty)$ given by

$$\rho_{2Y}(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x/2)^2}{2}} = \frac{1}{\sqrt{2\pi 4}} e^{-x^2/8} \qquad (x \in \mathbb{R}),$$

which is the density of the normal distribution $\mathcal{N}_{0,4}$. So $Y$ has the $\mathcal{N}_{0,4}$ distribution, and basic results about sums of independent normally distributed random variables from the lectures show that $X + 2Y$ has the $\mathcal{N}_{0,5}$ distribution. The latter has density $\rho_{X+2Y} : \mathbb{R} \to [0, \infty)$ given by

$N_{0,a} + N_{0,b} = N_{0,a+b}$

$$\rho_{X+2Y}(x) = \frac{1}{\sqrt{10\pi}} e^{-x^2/10} \qquad (x \in \mathbb{R}).$$

(b): Since $X$ and $Y$ are independent, the density function $\rho_{(X,Y)} : \mathbb{R}^2 \to [0, \infty)$ of $(X, Y)$ is given by

jointly distributed

$$\rho_{(X,Y)}(x, y) = \rho_X(x)\rho_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \qquad ((x, y) \in \mathbb{R}^2).$$

The probability of the event $\{|X| \geq \sqrt{3}|Y|\}$ is given by the integral of $\rho_{(X,Y)}$ over $D := \{(x, y) \in \mathbb{R}^2 \mid |x| \geq \sqrt{3}|y|\}$:

$$\mathbb{P}(|X| \geq \sqrt{3}Y) = \frac{1}{2\pi} \iint_D e^{-(x^2+y^2)/2} \mathrm{d}x\mathrm{d}y.$$

To compute this integral we switch to polar coordinates $(r, \theta) \in (0, \infty) \times (-\pi, \pi]$ such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ for $(x, y) \neq (0, 0)$. Then $x^2 + y^2 = r^2$, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$, and $D \setminus \{0\}$ is given in the $(r, \theta)$-plane by

$$\{(r, \theta) \in (0, \infty) \times (-\pi, \pi] \mid |\cos(\theta)| \geq \sqrt{3}|\sin(\theta)|\} = \{(r, \theta) \in (0, \infty) \times (-\pi, \pi] \mid \theta \in (-\pi, -\tfrac{5}{6}\pi] \cup [-\tfrac{1}{6}\pi, \tfrac{1}{6}\pi] \cup [\tfrac{5}{6}\pi, \pi]\}.$$

It follows that the probability of $\{|X| \geq \sqrt{3}Y\}$ is

$$\iint_D \rho_{(X,Y)}(x, y)\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi}\Big(\int_{-\pi}^{-5\pi/6} \mathrm{d}\theta + \int_{-\pi/6}^{\pi/6} \mathrm{d}\theta + \int_{\pi-\pi/6}^{\pi} \mathrm{d}\theta\Big) \int_0^\infty re^{-r^2/2}\mathrm{d}r = -\frac{1}{3}e^{-r^2/2}\Big|_{r=0}^{\infty} = \frac{1}{3}. \qquad \square$$

**Problem 4** (25 points)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Call a subset $N$ of $\Omega$ a *null set* if there exists an $\overline{N} \in \mathcal{F}$ with $N \subseteq \overline{N}$ and $\mathbb{P}(\overline{N}) = 0$. Let $\mathcal{N}$ be the collection of all null sets, and let $\overline{\mathcal{F}}$ be the collection of subsets $A$ of $\Omega$ for which there exist $B, C \in \mathcal{F}$ with $B \subseteq A \subseteq C$ and $\mathbb{P}(C \setminus B) = 0$. Show that $\overline{\mathcal{F}}$ is a $\sigma$-algebra and that $\overline{\mathcal{F}} = \sigma(\mathcal{F} \cup \mathcal{N})$.

$\bar{\mathcal{F}} := \{A \subseteq \Omega \mid \exists B, C \in \mathcal{F}$ s.t. $B \subseteq A \subseteq C$ and $P(C \setminus B) = 0\}$

*Solution.* First note that any $A \in \mathcal{F}$ is contained in $\overline{\mathcal{F}}$. Indeed, we trivially have $A \subseteq A \subseteq A$ and $\mathbb{P}(A \setminus A) = \mathbb{P}(\emptyset) = 0$. So $\mathcal{F} \subseteq \overline{\mathcal{F}}$.

$\mathcal{F} \subseteq \bar{\mathcal{F}}$

In particular, one has $\Omega \in \mathcal{F} \subseteq \overline{\mathcal{F}}$.

by def of $\bar{\mathcal{F}}$

Now let $A \in \overline{\mathcal{F}}$ be given, and let $B, C \in \mathcal{F}$ be such that $B \subseteq A \subseteq C$ and $\mathbb{P}(C \setminus B) = \mathbb{P}(C \cap B^c) = 0$. Then $B^c, C^c \in \mathcal{F}$, $C^c \subseteq A^c \subseteq B^c$ and $\mathbb{P}(B^c \setminus C^c) = \mathbb{P}(B^c \cap C) = 0$. So $A^c \in \overline{\mathcal{F}}$.

$P(B^c \setminus C^c) = P(B^c \cap (C^c)^c) = P(B^c \cap C) = P(C - B) = 0$

Finally, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\overline{\mathcal{F}}$ and $(B_n)_{n\in\mathbb{N}}, (C_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be such that $B_n \subseteq A_n \subseteq C_n$ and $\mathbb{P}(C_n \setminus B_n) = 0$ for all $n \in \mathbb{N}$. Then $x \in C_m \setminus B_m$ for any $m \in \mathbb{N}$ and $x \in C_m \setminus (\cup_{n\in\mathbb{N}} B_n)$. So

$$(\cup_{n\in\mathbb{N}} C_n) \setminus (\cup_{n\in\mathbb{N}} B_n) \subseteq (\cup_{n\in\mathbb{N}} C_n \setminus B_n).$$

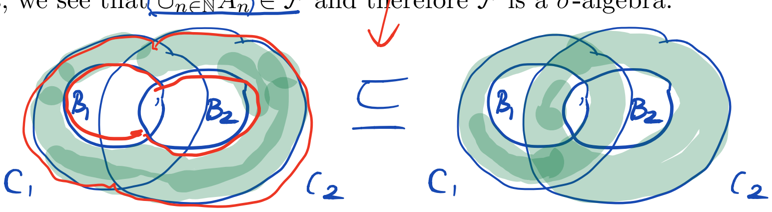
$C_n \setminus B_n \in \mathcal{F}$

Also, $\cup_{n\in\mathbb{N}} B_n, \cup_{n\in\mathbb{N}} C_n \in \mathcal{F}$ and $\cup_{n\in\mathbb{N}} B_n \subseteq \cup_{n\in\mathbb{N}} A_n \subseteq \cup_{n\in\mathbb{N}} C_n$. Because each $C_n \setminus B_n$ is measurable, so is their union, and we find

by subadditivity

$$0 \leq \mathbb{P}((\cup_{n\in\mathbb{N}} C_n) \setminus (\cup_{n\in\mathbb{N}} B_n)) \leq \mathbb{P}(\cup_{n\in\mathbb{N}} C_n \setminus B_n) \leq \sum_{n=1}^{\infty} \underbrace{\mathbb{P}(C_n \setminus B_n)}_{=0} = 0.$$

Combining all this, we see that $\cup_{n\in\mathbb{N}} A_n \in \overline{\mathcal{F}}$ and therefore $\overline{\mathcal{F}}$ is a $\sigma$-algebra.



$$\mathbb{P}(A \cap \emptyset^c) = \mathbb{P}(A \cap \Omega)$$
$$\mathbb{P}(A \setminus \emptyset) = \mathbb{P}(A)$$

$\emptyset \subseteq A \subseteq \Omega$

$\emptyset \subseteq \emptyset$



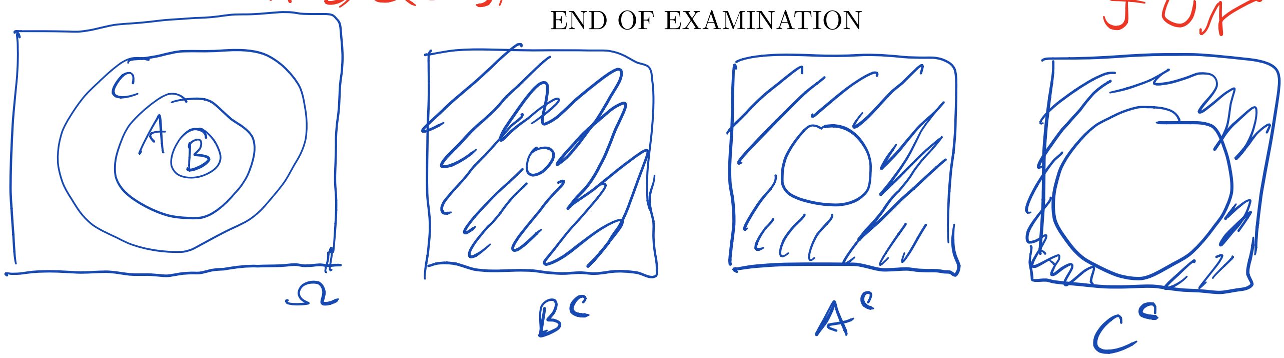
Now, if $N \in \mathcal{N}$ is given, then there exists an $N' \in \mathcal{F}$ containing $N$ and satisfying $\mathbb{P}(N') = 0$. We then find $\emptyset \subseteq N \subseteq N'$ and $\mathbb{P}(N' \setminus \emptyset) = \mathbb{P}(N') = 0$. Since the empty set is contained in $\overset{\mathcal{F}}{\cancel{\Sigma}}$, we find $N \in \overline{\mathcal{F}}$ and thus $\mathcal{N} \subseteq \overline{\mathcal{F}}$. Since we have already shown that $\overline{\mathcal{F}}$ is a $\sigma$-algebra and that $\mathcal{F} \subseteq \overline{\mathcal{F}}$, it follows that $\sigma(\mathcal{F} \cup \mathcal{N}) \subseteq \overline{\mathcal{F}}$.

Conversely, if $A \in \overline{\mathcal{F}}$ is given, then there exist sets $B, C \in \mathcal{F}$ such that $B \subseteq A \subseteq C$ and $\mathbb{P}(C \setminus B) = 0$. Since $A \setminus B$ is contained in $C \setminus B$, we have $A \setminus B \in \mathcal{N}$ and thus $A = B \cup (A \setminus B)$ is the union of an element in $\mathcal{F}$ and an element in $\mathcal{N}$. By the arbitrariness of $A$, $\overline{\mathcal{F}} \subseteq \sigma(\mathcal{F} \cup \mathcal{N})$. ☐

$\in \sigma(\mathcal{F} \cup \mathcal{N})$ smallest $\sigma$-algebra containing $\mathcal{F} \cup \mathcal{N}$

$B \subseteq A \subseteq C$. $(A - B) \subseteq (C - B)$

END OF EXAMINATION




$B \subseteq A \subseteq C$

$B^c \supseteq A^c \supseteq C^c$