5.10(j)

5 divides 11^n -6

For $\boldsymbol{n} \in \mathbb{N}_{\mathbf{0}}, \mathbf{5}\ \boldsymbol{divides}\ \mathbf{11}^{\boldsymbol{n}} - \mathbf{6}$

Base case:

n =0

When n = 0, 11^0 = 1, 1 - 6 = -5, 5 divides -5.

True for n = 0

Induction steps:

Assume P(n) is divisible by 5: 11^n – 6 is divisible by 5

11^n – 6 = 5k for an integer k, 11^n = 5k + 6

P(n+1) = 11^{n+1}= 11^n* 11 = 11 * (5k+6)

11^{n+1} – 6 = 11*(5k+6) – 6 = 55k + 66 − 6 = 55k + 60 = 5(11k + 12)

(multiple of 5)

Hence, 11^{n+1} – 1 is divisible by 5

By the principle induction, P(n) is true for $\boldsymbol{n} \in \mathbb{N}_{\mathbf{0}}$

5.12(i)

$n! \geq n^n e^{-n}\ for\ n\ \geq 1$

Base case: n = 1

n! = 1, $n^n e^{-n}$ = $\frac{1}{e}$

1 ≥ $\frac{1}{e}$, True for n = 1

Induction steps:

Assume $n! \geq n^n e^{-n}$

Prove $P(n) \rightarrow P(n+1), which\ is\ (n+1)! \geq (\frac{n+1}{e})^{n+1}$

$(n+1)! = (n+1) * n!$

Since $n! \geq n^n e^{-n}$,

$(n+1)!\ \geq (n+1) *\ (\dfrac{n}{e})^n$

$(n+1)!\ \geq (n+1)\ *\ (\frac{n}{n+1})^n *\ (\frac{n+1}{e})^n$

$(n+1)!\ \geq$ e $* (\frac{n}{n+1})^n * \left(\frac{n+1}{e}\right)^{n+1}$

Hint $((1 + \frac{1}{n})^n \leq e)$

Since $(\frac{n}{n+1})^n = (\frac{1}{1+\frac{1}{n}})^n = \frac{1}{(1+\frac{1}{n})^n}$.

From hint, $\frac{1}{(1+\frac{1}{n})^n} \geq \frac{1}{e}$, hence e $* \frac{1}{\left(1+\frac{1}{n}\right)^n}\ \geq\ 1$

In this case e $* (\frac{n}{n+1})^n * \left(\frac{n+1}{e}\right)^{n+1} \geq \left(\frac{n+1}{e}\right)^{n+1}$,

since $(n+1)! \geq$ e $* (\frac{n}{n+1})^n * \left(\frac{n+1}{e}\right)^{n+1}$ and e $* (\frac{n}{n+1})^n * \left(\frac{n+1}{e}\right)^{n+1} \geq \left(\frac{n+1}{e}\right)^{n+1}$

It proves that by induction, $(n+1)! \geq \left(\frac{n+1}{e}\right)^{n+1}$ is true for all n ≥ 1

5.18(a)

Base case: n =1

H(1) = 1 = (1+1) *1 – 1. True for n = 1

Inductive steps:

Assume $H(n)$ is true, prove $H(n) \rightarrow H(n+1)$

For $H(n)$, $H_1 + H_2 + H_3 + .. + H_n = (n+1)H_n - n$

If we prove that when H(n+1), both sides will increase by the same value, we can state that $H(n) \rightarrow H(n+1)$

When H(n+1), the increment is:

LHS: $= \big(H(1) + H(2) + .. + H(n+1)\big) - \ (H(1) + H(2) + .. + H(n))$

$= H(n+1)$

$= 1 \ + \ \frac{1}{2} + \ldots + \frac{1}{\mathrm{n+1}}$

RHS: $= (n+2)H(n+1) - n - 1 - (n+1)H_n + n$

$= (n+2) * \left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n+1}\right) - \ (n+1) * \left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n}\right) - 1$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n} + \frac{n+2}{n+1}\right) - 1$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n} + \frac{1}{n+1}\right)$

$= \mathrm{H(n+1)}$

LHS equals RHS

We prove that $H(n) \rightarrow H(n+1)$

So the statement is true for all n ≥ 1

5.60

(a) the perimeter highlighted by the thick line in the graph is 42

(b) The base case when P(n), n =1, P(1) =4, which is true for base case

Inductive steps:

We want to prove $P(n) \rightarrow P(n+1)$ using direct prove

When P(n) has an even number of total lines, there are four possibilities if we add another square:

1. Added square is completely separated from other squares, in other words, no lines are overlapped. We simply add 4 to the total number, which would still make it an even number.
2. Added square has one line overlapped with other squares. At this time, we would have to first subtract 1 because of overlap, then we have to add 3 into total number because the rest of the lines are individual lines that should be counted. Overall, we add 2 to the total number, which makes it an even number.
3. Added square has two lines overlapped with other squares, we have to first subtract 2 and then add 2 lines. Overall, the total size doesn't change, which makes it an even number.
4. Added square has three lines overlapped with other squares. We have to first subtract 3 and then add 1 line. Overall, we have to subtract 2 from the total number, which makes it an even number.

For above four possibilities, the total number will always end up with an even number, so by using a direct proof, when P(n) is true, P(n+1) has to be true.

In this case, we prove that the total number will always be even.

6.6

(a)

Base case:

n = 1, H(n) = 1

$$\frac{1}{2}H(n)^2 + 1 = \frac{3}{2}$$

1 < $\frac{3}{2}$, *base case is true*

Inductive steps:

We want to prove $H(n) \rightarrow H(n+1)$

Assume H(n) is true. If both sides increase by the same amount or the left hand side increase by a smaller amount, then we know that H(n+1) is true.

LHS: H(n+1) – H(n) = $\frac{H(n+1)}{n+1}$

RHS: H(n+1) – H(n) $= \frac{1}{2}H(n+1)^2 + 1 - \frac{1}{2}H(n)^2 - 1$

$= \frac{1}{2}(H(n+1) + H(n))(H(n+1) - H(n))$

$= \frac{1}{2}(H(n+1) + H(n))(\frac{H(n+1)}{n+1})$

$= \frac{1}{2}\left(2 + 1 + \frac{2}{3} + .. + \frac{2}{n} + \frac{1}{n+1}\right) * \frac{1}{n+1}$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n} + \frac{1}{2n+2}\right) * \frac{1}{n+1}$

$$\left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n} + \frac{1}{2n+2}\right) * \frac{1}{n+1} < \frac{H(n+1)}{n+1}$$

Right hand side increment < left hand side increment

The increment of left hand side is bigger than the right hand side. In this case,

We are unable to prove $H(n) \rightarrow H(n+1)$.

(b)

Base case:

n = 1, 1 < 3/2, so the base case is true.

Assume H(n) is true, we want to prove the implication $H(n) \to H(n+1)$

Same as above, the increment of LHS is $\frac{H(n+1)}{n+1}$

Increment of RHS:

RHS: $\frac{1}{2}H(n+1)^2 - \frac{1}{2}H(n)^2 + \frac{1}{2}(\frac{1}{(n+1)^2})$

$= \frac{1}{2}\big(H(n+1) + H(n)\big)\big(H(n+1) - H(n)\big) + \frac{1}{2}(\frac{1}{(n+1)^2})$

$= \frac{1}{2}(H(n+1) + H(n))(\frac{H(n+1)}{n+1}) + \frac{1}{2}(\frac{1}{(n+1)^2})$

$= \frac{1}{2}\left(2 + 1 + \frac{2}{3} + .. + \frac{2}{n} + \frac{1}{n+1}\right) *$

$= H(n+1) * \frac{1}{n+1}$

$= \frac{H(n+1)}{n+1}$

From above, we see that both sides increased by the same value for every term added, and since the base case, 1< 3/2 is true, $H(n) \to H(n+1)$ will always be true because LHS will never be larger than RHS.

It is a stronger claim because we can now prove that RHS always grows at the same rate as LHS rather than growing slower than LHS in (a).

6.45(a)

Base case:

n = 2. P(n) is true. If there are only two cities, a and b, there are direct flights between a and b. If there is a special city, we could go directly to the special city from the current location (either a to b or b to a).

Inductive steps:

Assume P(n) is true, prove $P(n) \rightarrow P(n+1)$

When there are n+1 cities, we know that there is a one-way flight between every pairs of cities. If we want to go to that special city, we could either take the one-way flight, or transfer once at another city. (since there is a one-way flight between every pairs of cities, we are sure that any other cities has their own one-way flights to the special city)

Thus, P(n+1) is true, and $P(n) \rightarrow P(n+1)$ is true. The statement is now proved by induction, there must at least one special city that can be reached directly or via one stop.