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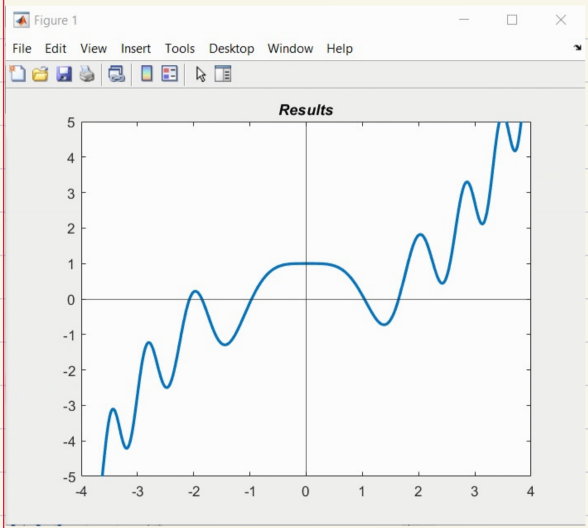
① (a)



By looking at the graph, we see that there are roots in the interval $[1,2]$, we want to evaluate $f(x)$ at $x = 1, 1.5, 2$ for computation purposes.

$f(1) = 0.1$ $\quad f(1.5) = -0.5864$ $\quad f(2) = 1.8$

Since there are two roots and $\text{sign}[f(1)] \neq \text{sign}[f(1.5)]$, $\text{sign}[f(1.5)] \neq \text{sign}[f(2)]$, thus we know that there is exactly one root for each interval of length $\frac{1}{2}$ in $[1,2]$.

Now, our correct interval should be $\begin{cases} ① \ [1, 1.5] \\ ② \ [1.5, 2] \end{cases}$

(b) $\dfrac{\log 2 - \log 10^{-10}}{\log 2} = 34.2193 \approx 35$ iterations

Solution:

2. The graph and matlab code is given below.

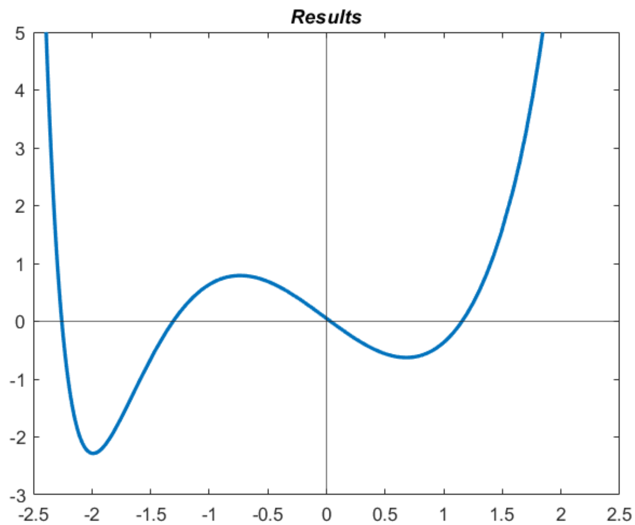


Figure 1: used to compare values of p(x) with tanh(x)

Here is my Matlab code for bisect.m,

Listing 1: bisect.m, Matlab code for computing the root

```
% bisect : find an approximate solution to f(x)=0 using bisection
%
% f (input) : function f(x)
% a,b (input) : interval to search that brackets the root, a<b and f(a)*f(b) <= 0
% tol (input) : tolerance
% maxIterations (input) : maximum number of iterations allowed
% xc (output) : approximate solution
function xc = bisect( f,a,b,tol,maxIterations )
fa=f(a);
fb=f(b);
k = 0;
while (b-a)/2 > tol
    if k > maxIterations
        error("Exceed max iterations")
    end
    c=(a+b)/2;
    fc=f(c);
    if fc == 0
        break
    end
    if sign(fc)*sign(fa)<0 %a and c make the new interval
        b=c;fb=fc;
    else %c and b make the new interval
        a=c;fa=fc;
    end
    k = k+1;
end
```

```
fprintf("bisect: k=%2d, a=%11.4e f(a)=%8.2e b=%11.4e f(b)=%8.2e c=%14.7e\n",k,a,fa,b,fb,c);
xc=(a+b)/2; %new midpoint is best estimate
```

From the graph, we can categorize into 4 intervals: [-3,-2], [-2,-1], [0,1], [1,2], then compute value for each.

Listing 2: Results from bisect.m

```
>> bisect.m
>> bisect(f,-3,-2,10.^-4, 100)
bisect: k=13, a=-2.2568e+00 f(a)=2.20e-03 b=-2.2567e+00 f(b)=-5.86e-04 c=-2.2567139e+00
ans = -2.2568
>> bisect(f,-2,-1,10.^-4, 100)
bisect: k=13, a=-1.3080e+00 f(a)=-2.53e-04 b=-1.3079e+00 f(b)=1.02e-04 c=-1.3079834e+00
ans = -1.3079
>> bisect(f,0,1,10.^-4, 100)
bisect: k=13, a= 3.3203e-02 f(a)=7.39e-05 b= 3.3325e-02 f(b)=-1.08e-04 c= 3.3325195e-02
ans = 0.0333
>> bisect(f,1,2,10^-4, 100)
bisect: k=13, a= 1.1559e+00 f(a)=-9.44e-05 b= 1.1560e+00 f(b)=2.65e-04 c= 1.1558838e+00
ans =  1.1559
```

By using previously mentioned code, I have computed four roots at different location on x-axis.

③ (a) $g(x) = \frac{1}{2}x + \frac{1}{x}$

$g'(x) = \frac{1}{2} + (-x^{-2})$

$g'(\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0 < 1$, thus we expect convergence at $g(\sqrt{2})$

$g(\sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$, $g(x)$ converges to $r = \sqrt{2}$

Now we can compute asympotic convergence rate:

$|g'(r)| = g'(\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0$

(b) $g(x) = \frac{2}{3}x + \frac{2}{3x}$ $\qquad \frac{2}{3}x^{-1} \qquad -\frac{2}{3}x^{-2}$

$g'(x) = \frac{2}{3} - \frac{2}{3}x^{-2}$

$g'(\sqrt{2}) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} < 1$

$g(\sqrt{2}) = \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{3} = \sqrt{2}$ converge to $r = \sqrt{2}$

$|g'(r)| = g'(\sqrt{2}) = \frac{1}{3}$ asympotic convergence rate is $\frac{1}{3}$

(C) $g(x) = \frac{3}{4}x + \frac{1}{2x}$

$g'(x) = \frac{3}{4} - \frac{1}{2}x^{-2}$

$g'(\sqrt{2}) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} < 1$

$g(\sqrt{2}) = \frac{3\sqrt{2}}{4} + \frac{1}{2\sqrt{2}} = \sqrt{2}$ converge to $r = \sqrt{2}$

$|g'(r)| = g'(\sqrt{2}) = \frac{1}{2}$ asympotic convergence rate is $\frac{1}{2}$

(d) $g(x) = \frac{3}{2}x - \frac{1}{x}$

$g'(x) = \frac{3}{2} + x^{-2}$

$g'(\sqrt{2}) = \frac{3}{2} + \frac{1}{2} = 2 > 1$ it diverges

FASTEST SLOWEST

ⓐ > ⓑ > ⓒ > ⓓ

(doesn't converge)

4. The matlab code is given below.

Listing 3: fixedpoint.m, Matlab code for computing the root

```
% Compute a fixed point g(x)=x
% g (input) : function to use
% x0 (input) : initial guess
% tol (input) : convergence tolerance
% maxk (input) : maximum number of k
% xc (output) : approximation to the fixed point
function xc = fixedpoint(g, x0, tol, maxIteartioins   )
 xOld = x0;
 k = 1;

 while k <= maxIteartioins
     xc = g(xOld);
     fprintf("fixedPoint: k=%2d xc=%18.12e |xc-xOld|=%8.2e\n",k,xc,abs(xc-xOld))
     if tol > abs(xc-xOld)      %check if satisfies the condition
         break
     end
     if k == maxIteartioins     % Halts algorithm after input maxIterations
         error('No␣Convergence␣found!')
     end
     k = k + 1;
     xOld = xc;
 end
end
```

By using the code given above, compute root for (a), (b) and (c):

Listing 4: Output of Part (a) using fixedpoint.m to approximate the real root

```
>> f0 = @(x) ((1-x)/3).^(1/3)
>> fixedpoint(f0, 0.6, 10.^ -5, 20)
fixedPoint: k= 1 xc=5.108729549290e-01 |xc-xOld|=8.91e-02
fixedPoint: k= 2 xc=5.463028598322e-01 |xc-xOld|=3.54e-02
fixedPoint: k= 3 xc=5.327804279360e-01 |xc-xOld|=1.35e-02
fixedPoint: k= 4 xc=5.380218632771e-01 |xc-xOld|=5.24e-03
fixedPoint: k= 5 xc=5.360023857297e-01 |xc-xOld|=2.02e-03
fixedPoint: k= 6 xc=5.367822720930e-01 |xc-xOld|=7.80e-04
fixedPoint: k= 7 xc=5.364813625451e-01 |xc-xOld|=3.01e-04
fixedPoint: k= 8 xc=5.365975047761e-01 |xc-xOld|=1.16e-04
fixedPoint: k= 9 xc=5.365526832505e-01 |xc-xOld|=4.48e-05
fixedPoint: k=10 xc=5.365699816285e-01 |xc-xOld|=1.73e-05
fixedPoint: k=11 xc=5.365633056407e-01 |xc-xOld|=6.68e-06

ans = 0.5366
```

Listing 5: Output of Part (b) using fixedpoint.m to approximate the real root

```
>> f1 = @(x)1-3.*x.^3
>> fixedpoint(f1, 0.6,10.^-5, 20)
fixedPoint: k= 1 xc=3.520000000000e-01 |xc-xOld|=2.48e-01
fixedPoint: k= 2 xc=8.691573760000e-01 |xc-xOld|=5.17e-01
fixedPoint: k= 3 xc=-9.697745186738e-01 |xc-xOld|=1.84e+00
fixedPoint: k= 4 xc=3.736110045397e+00 |xc-xOld|=4.71e+00
fixedPoint: k= 5 xc=-1.554516809969e+02 |xc-xOld|=1.59e+02
fixedPoint: k= 6 xc=1.126957560194e+07 |xc-xOld|=1.13e+07
fixedPoint: k= 7 xc=-4.293821031017e+21 |xc-xOld|=4.29e+21
```

```
fixedPoint: k= 8 xc=2.374942346166e+65 |xc-xOld|=2.37e+65
fixedPoint: k= 9 xc=-4.018652636312e+196 |xc-xOld|=4.02e+196
fixedPoint: k=10 xc=               Inf |xc-xOld|=     Inf
fixedPoint: k=11 xc=              -Inf |xc-xOld|=     Inf
fixedPoint: k=12 xc=               Inf |xc-xOld|=     Inf
fixedPoint: k=13 xc=              -Inf |xc-xOld|=     Inf
fixedPoint: k=14 xc=               Inf |xc-xOld|=     Inf
fixedPoint: k=15 xc=              -Inf |xc-xOld|=     Inf
fixedPoint: k=16 xc=               Inf |xc-xOld|=     Inf
fixedPoint: k=17 xc=              -Inf |xc-xOld|=     Inf
fixedPoint: k=18 xc=               Inf |xc-xOld|=     Inf
fixedPoint: k=19 xc=              -Inf |xc-xOld|=     Inf
fixedPoint: k=20 xc=               Inf |xc-xOld|=     Inf
Error using fixedpoint (line 18)
No Convergence found!
```

Listing 6: Output of Part (c) using fixedpoint.m to approximate the real root

```
>> f2 = @(x)(1+6.*x.^3)/(1+9.*x.^2)
>> fixedpoint(f2, 0.6,10.^-5, 20)
fixedPoint: k= 1 xc=5.415094339623e-01 |xc-xOld|=5.85e-02
fixedPoint: k= 2 xc=5.365978036024e-01 |xc-xOld|=4.91e-03
fixedPoint: k= 3 xc=5.365651661047e-01 |xc-xOld|=3.26e-05
fixedPoint: k= 4 xc=5.365651646722e-01 |xc-xOld|=1.43e-09

ans = 0.5366
```

4 (a) $g(x) = \left[\frac{(1-x)}{3}\right]^{\frac{1}{3}}$

$g'(x) = -\frac{1}{3\cdot 3^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}$

$g'(0.6) = -\frac{1}{3^{2}\sqrt{3}\,(0.4)^{\frac{2}{3}}} \approx |-0.426| < 1$, we can expect it to converge

(b) $g(x) = 1 - 3x^3$

$g'(x) = -9x^2$

$g'(0.6) = -9 \cdot 0.36 = |-3.24| > 1$ expect to diverge

(c) $g(x) = (1+6x^3)/(1+9x^2)$

$g'(x) = \frac{54x^4 + 18x^2 - 18x}{(1+9x^2)^2}$

$g'(0.6) = \frac{54\cdot(0.6)^4 + 18\cdot(0.6)^2 - 18\cdot 0.6}{(1+9\cdot 0.36)^2}$

$|g'(0.6)| < 1$

expect it to converge