第十一次作业解答

5.1.1 (1) $f(x)$在$[0,1]$上连续，故R-可积

(2) $f(x)$在$[0,1]$上无界，故R-不可积

(3) $f(x)$在$[0,1]$上单调递增，故R-可积.

5.1.3. 取 $\tilde{D}(x)=\begin{cases}1, & x\in[a,b]\cap\mathbb{Q}\\ -1, & x\in[a,b]\cap\mathbb{Q}^c\end{cases}$ 则$\tilde{D}(x)$不可积，而$|\tilde{D}(x)|$可积

5.1.4 (1) 不妨设$c\in(a,b)$（若$c=a$或b，仅需对下述证明稍加修改）

由$f(x)$在c处连续知$\exists\delta>0$, st $[c-\delta,c+\delta]\subset[a,b]$且$f(x)>\frac{1}{2}f(c)$, $\forall x\in[c-\delta,c+\delta]$

故 $\int_a^b f(x)\,dx \geqslant \int_{c-\delta}^{c+\delta} f(x)\,dx \geqslant 2\delta\cdot\frac{1}{2}f(c)>0$.

(2) 证明同(1)　　(3) 取 $f(x)=\begin{cases}0, & x\in(a,b]\\ 1, & x=a\end{cases}$ 即可.

5.1.6 (1) 由 $|a\sin x+b\cos x|\leq\sqrt{(a^2+b^2)(\sin^2x+\cos^2x)}=\sqrt{a^2+b^2}$

可得 $\int_0^{2\pi}|a\sin x+b\cos x|\,dx\leq 2\pi\sqrt{a^2+b^2}$.

(2) 记 $f(x)=x^m(1-x)^n=m^m n^n\left(\frac{x}{m}\right)^m\left(\frac{1-x}{n}\right)^n \underset{\text{均值}}{\leq} m^m n^n\left(\frac{\frac{x}{m}+\cdots+\frac{x}{m}+\cdots+\frac{1-x}{n}}{m+n}\right)^{m+n}$

$=\frac{m^m n^n}{(m+n)^{m+n}}$　故 $\int_0^1 x^m(1-x)^n\,dx\leq 1\cdot\frac{m^m\cdot n^n}{(m+n)^{m+n}}$.

5.1.8：若$f(x)$在(a,b)上无零点且$f(x)\in C[a,b]$，则不妨设$f(x)>0$, $\forall x\in(a,b)$

再由端点处连续性可得$f(x)\geqslant 0$, $\forall x\in[a,b]$

由5.1.4(2)可知$\int_a^b f(x)\,dx>0$，矛盾！故$f(x)$在(a,b)中至少一零点

5.1.11 (1) $f'(x)=2x\cdot\sin(x^4)$　(3) $f'(x)=2xe^{-x^4}-e^{-x^2}$.

5.1.12 (2) $(f^{-1})'(0)=\frac{1}{f'(1)}=e^{x^2}\big|_{x=1}=e$.　（$y=0$时，$x=1$）

5.1.14 $G'(x)=\dfrac{xf(x)\int_0^x f(t)\,dt-\int_0^x tf(t)\,dt\cdot f(x)}{\left(\int_0^x f(t)\,dt\right)^2}=\dfrac{f(x)}{\left(\int_0^x f(t)\,dt\right)^2}\cdot g(x)$

其中 $g(x)=x\int_0^x f(t)\,dt-\int_0^x tf(t)\,dt$，则 $g'(x)=\int_0^x f(t)\,dt\geqslant 0,\ \forall x\geqslant 0$

从而 $g(x)\geqslant g(0)=0\Rightarrow G'(x)\geqslant 0$，即 $G(x)$ 在 $[0,+\infty)$ 上单调递增.

5.1.15 (2) 原式 $=\dfrac{1}{\alpha+1}x^{\alpha+1}\Big|_0^1=\dfrac{1}{\alpha+1}$

(4) 原式 $=\displaystyle\int_2^3 \frac{\frac{2}{5}}{2x-1}-\frac{\frac{1}{5}}{x+2}\,dx=\frac{1}{5}\ln\left|\frac{2x-1}{x+2}\right|\Bigg|_2^3=\frac{1}{5}\ln\frac{4}{3}$.

5.1.18 (1) 原式 $\overset{洛}{=}\displaystyle\lim_{x\to 0}\frac{\sin x^3}{4x^3}=\frac{1}{4}$

(3) 原式 $=\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+(\frac{0}{n})^2}}+\frac{1}{\sqrt{1+(\frac{1}{n})^2}}+\cdots+\frac{1}{\sqrt{1+(\frac{n-1}{n})^2}}\right)$

$=\displaystyle\int_0^1\frac{1}{\sqrt{1+x^2}}\,dx\ \underset{t=\ln(x+\sqrt{1+x^2})}{\overset{x=\sinh t}{=\!=\!=}}\ \int_0^{\ln(1+\sqrt{2})}\frac{1}{\cosh t}\cosh t\,dt=\ln(1+\sqrt{2})$

5.1.19 (2) 由 $0\leq\displaystyle\int_0^1\frac{x^n}{1+x}\,dx\leqslant\int_0^1 x^n\,dx=\frac{1}{n+1}$ 知 $\displaystyle\lim_{n\to\infty}\int_0^1\frac{x^n}{1+x}\,dx=0$.

5.1.22 (2) 原式 $=\displaystyle\int_{-3}^{-2}-3+\int_{-2}^{-1}-2+\int_{-1}^{0}-1+\int_0^1 0+\int_1^2 1+\int_2^3 2+\int_3^4 3\,dx=0$

(4) 2原式 $\underset{x=-t}{=}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+e^x}\cos^3x\,dx+\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{1}{1+e^{-t}}\cos^3(-t)\,d(-t)$

$=\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3x\,dx=2\int_0^{\frac{\pi}{2}}\cos^3x\,dx\ \overset{点火}{\underset{公式}{=\!=}}\ \frac{4}{3}\Rightarrow$ 原式 $=\dfrac{2}{3}$.

(8) 原式 $\overset{x=a\sin t}{=\!=\!=}\displaystyle\int_0^{\frac{\pi}{2}}\frac{a\cos t}{a(\sin t+\cos t)}\,dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{(\sin t+\cos t)+(\cos t-\sin t)}{\sin t+\cos t}\,dt$

$=\dfrac{\pi}{4}+\dfrac{1}{2}\ln|\sin t+\cos t|\Big|_0^{\frac{\pi}{2}}=\dfrac{\pi}{4}$

(10) 原式 $=\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{a^2\tan^2x+b^2}\,d(\tan x)=\frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2ab}$

(11) 原式 $\overset{x=\sin t}{=\!=\!=}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4t\cos^2t\,dt=2\int_0^{\frac{\pi}{2}}\sin^4t-\sin^6t\,dt\ \overset{点火}{\underset{公式}{=\!=}}\ \frac{\pi}{16}$

(12) 原式 $= 4\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = 4\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{8}\pi$

5.1.23. $\int_0^{\pi} x f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{0} (\pi - t) f(\sin(\pi - t))\,d(\pi - t)$

$= \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_0^{\frac{\pi}{2}} (\pi - x) f(\sin x)\,dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

故 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\pi\arctan(\cos x)\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$

5.1.24 $\int_0^1 \sin(x^2)\,dx \leq \int_0^1 x^2\,dx = \frac{1}{3}$

$\int_0^1 \sin(x^2)\,dx > \int_0^1 \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x\Big|_0^1 = \frac{1}{2} - \frac{\sin 2}{4} > \frac{1}{6}$

5.1.27 由 $(1-\alpha)\int_0^{\alpha} f(x)\,dx \geqslant (1-\alpha)\alpha f(\alpha) \geqslant \alpha\int_{\alpha}^{1} f(x)\,dx$

可得 $\int_0^{\alpha} f(x)\,dx \geqslant \alpha\left(\int_{\alpha}^{1} f(x)\,dx + \int_0^{\alpha} f(x)\,dx\right) = \alpha\int_0^1 f(x)\,dx.$

5.1.28 (1) 由 Taylor 公式知 $\exists\, \xi \in (a, x)$, st $f(x) = f(a) + f'(\xi)(x-a)$, $\forall x \in [a, b]$

故 $|f(x)| = |f'(\xi)|(x-a) \leq M(x-a)$

从而 $\int_a^b |f(x)|\,dx \leq \int_a^b M(x-a)\,dx = \frac{M}{2}(b-a)^2$

(2) 由 Taylor 公式知 $\exists\, \xi_1 \in (a, x)$, $\xi_2 \in (x, b)$, st

$\cancel{\forall x\in[a,\frac{a+b}{2}]}$, $f(x) = f(a) + f'(\xi_1)(x-a) \implies |f(x)| \leq M(x-a)$, $\forall x \in [a, \frac{a+b}{2}]$

$f(x) = f(b) + f'(\xi_2)(x-b) \implies |f(x)| \leq M(b-x)$, $\forall x \in [\frac{a+b}{2}, b]$

从而 $\int_a^b |f(x)|\,dx \leq \int_a^{\frac{a+b}{2}} M(x-a)\,dx + \int_{\frac{a+b}{2}}^{b} M(b-x)\,dx$

$= \frac{M}{4}(b-a)^2.$