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9.18　1.2　17 证明下列数列收敛

例(1) $a_n=(1-\frac{1}{2})(1-\frac{1}{2^2})\cdots(1-\frac{1}{2^n})$

$a_n\downarrow$ 有下界 0　从而 a_n 有极限

$\lim\limits_{n\to\infty} a_n=?$　~~$(1+\frac{1}{2})a_n=1-\frac{1}{2^{n+1}}$~~

~~$\lim\limits_{n\to\infty} a_n=\frac{2}{3}$~~

(3) $a_n=\alpha_0+\alpha_1 q+\cdots+\alpha_n q^n$

$|\alpha_k|<M$　$|q|<1$　设 ~~$\forall m, q>0$~~

~~a_n-a_m~~　$\forall\varepsilon>0$　$\exists N=\left[\log_q\frac{\varepsilon(1-q)}{M}\right]$

$\forall m, t>0$

$|a_{m+t}-a_m|=|\alpha_{m+1}q^{m+1}+\cdots+\alpha_{m+t}q^{t}|$

$\le M(q^{m+1}+\cdots+q^{m+t})$

$=Mq^{m+1}\frac{1-q^t}{1-q}<\frac{Mq^{m+1}}{1-q}$

证明　$\le\frac{M}{1-q}\frac{\varepsilon(1-q)}{M}=\varepsilon$

18 (1) $a_n=\frac{n}{c^n}$ 收敛　$c>1$

$a_n>0$　$\frac{a_{n+1}}{a_n}=\frac{n+1}{nc}$　$n>\frac{1}{c-1}$ 时　$a_n>a_{n+1}$

$\therefore n>\frac{1}{c-1}$ 时　a_n 单减，以 0 为下界 a_n 收敛

$\lim\limits_{n\to\infty} a_n=\frac{1}{c^{n+1}-c^n}=0$.

3、 $a>0$ $a_0>0$ $a_{n+1}=\frac{1}{2}(a_n+\frac{a}{a_n})$

$a_n+\frac{a}{a_n}\geq 2\sqrt{a}$

当 $n\geq 0$

$\therefore a_{n+1}\geq 2\sqrt{a}$

$\therefore a_n\geq\sqrt{a}$ 当 $n\geq 1$ $\therefore a_n^2\geq a$ $(n\geq 1)$

$\therefore a_n$ 有下界 又 $\frac{a_{n+1}}{a_n}=\frac{1}{2}+\frac{a}{2a_n^2}\leq 1$ $(n\geq 1)$

$a_n\downarrow$

$\{a_n\}$ 收

5、 $a_n=\underbrace{\sin\ \sin\cdots\sin}_{n个}1$ ~~从而 $a_2=\sin a_1<a_1$~~ ~~从而 $a_2=$~~

$0<a_1=\sin 1<1$ ~~又 $a_{n+1}=\sin a_n<a_n$~~

~~又 $a_1<1$ $\therefore \forall a_n<1$~~ ~~数学归纳法当 n 成立时~~

~~且~~ ~~$0<a_{n+1}=\sin a_n<1$~~

$\therefore a_{n+1}\leq$ ~~从而 $\forall n$ $0<a_n<1$~~

又

$\therefore$ ~~$\sin a$~~ $a_{n+1}=\sin a_n<a_n$ 成立

$\therefore a_n\downarrow$ 又 $a_n\geq 0$ $\therefore a_n$ ~~成~~有下界

$\therefore \lim a_n$ 存在 又 $a_{n+1}=\sin a_n$ 两边令 $n\to\infty$

$\therefore a=\sin a$ $\therefore a=0$

1. $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$ $\{a_n\}$ $\{b_n\}$ 为正

求证若 b_n 收 则 a_n 收

错1: $a_{n+1} \le \frac{a_n}{b_n} b_{n+1} \le b_{n+1}$

$0 \le a_n \le b_n$ 两边令 $n \to \infty$ ∴ a_n 有极限

错2. $\frac{a_{n+1}}{\cancel{b_n} a_n} \le \frac{b_{n+1}}{b_n}$ 两边取 $\lim_n$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \le 1$ ∴ $a_{n+1} < a_n$ ∴ a_n 单调↓

又 $a_n > 0$ ∴ a_n 收

正: $\frac{a_{n+1}}{b_{n+1}} < \frac{a_n}{b_n}$

$C_n = \frac{a_n}{b_n}$ $C_n \downarrow$ 有下界 ∴ $\lim_{n\to\infty} C_n$ 存在

$C_n \cdot b_n = a_n$ 存在

25. $a_{n+1} = \frac{1}{a_n} + a_n$

$a_{n+1} > a_n$ ∴ $a_n \cancel{\ge} \ge 1$ 等价

∴ $a_{n+1} \le a_n + 1 \le \le \cdot a_1 + n \cancel{-1} = n+1$ $\frac{1}{a_n} > \frac{1}{n}$

又 $a_{n+1} - a_n = \frac{1}{a_n}$

$a_n - a_{n-1} = \frac{1}{a_{n-1}}$

⋮

∴ $a_n = \cancel{a} 1 + \sum_{i=1}^{n-1} \frac{1}{a_i} \ge \cancel{a} 1 + \frac{1}{2} + \frac{1}{n-1}$

13.(1) $\lim_{x\to 1^-}(x^5-5x+2+\frac{1}{x})$

$=-1+5+2+1=5$

(2) $\lim_{x\to 1}\frac{x^n-1}{x-1}=\lim_{x\to 1}(1+\cdots+x^{n-1})$

$=n$

6. $\lim_{n\to\infty}\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}=\lim_{n\to\infty}\frac{\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}$

$\cancel{=\frac{}{2^n\sin\frac{x}{2^n}}}$ $=\lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}}=\frac{\sin x}{x}$

7. $\lim_{n\to\infty}(\sin\frac{\alpha}{n^2}+\cdots+\sin\frac{n\alpha}{n^2})=\frac{\alpha}{2}$

仅用证 $\lim_{n\to\infty}\left|(\sin\frac{\alpha}{n^2}+\cdots+\sin\frac{n\alpha}{n^2})-\frac{\alpha}{2}\right|=$

$=\lim_{n\to\infty}\left|\left[(\sin\frac{\alpha}{n^2}-\frac{\alpha}{n^2})+\cdots(\sin\frac{n\alpha}{n^2}-\frac{n\alpha}{n^2})\right]\right|$ $\frac{\alpha^2}{2n^2}$

$\cancel{\leq\lim_{n\to\infty}\frac{\alpha}{n^6}+\cdots+\frac{}{n^6}}$

$<\frac{1}{2}\left[\frac{\alpha^2}{n^4}+\cdots+\frac{n^2\alpha^2}{n^4}\right]=\frac{\alpha^2}{2}\frac{1+2^2+\cdots+n^2}{n^4}<\frac{2n^3}{n^4}<\frac{1}{n}$

$x\to$

$x-\sin x<\frac{x^2}{2}$

$f(x)=x-\sin x-\sin\frac{x^2}{2}$

$f(0)=0$

$f'(x)=1-\cos x-x$

$f'(0)=0$

$f''(x)=\sin x-1<0$

$f(x)<0$

1.(3) $a_1>1$ $a_{n+1}=2-\frac{1}{a_n}$

$a_{n+1}-a_n=2-\frac{1}{a_n}-\frac{1}{a_{n+1}}\le 0$ $a_n\downarrow$

又$a_1>1$ ∴ ∃ $a_2=2-\frac{1}{a_1}>2-1>1$

下设n时 $a_n>1$ 下证 $a_{n+1}=2-\frac{1}{a_n}>1$

∴ ∀$a_n\ge 1$ ∴ $a_n\downarrow$ 有下界

∴ $\lim a_n$ 存在 设为 x

$x=2-\frac{1}{x}$ ∴ $x=1$

(4) $a_1=3$ $a_{n+1}=\frac{1}{1+a_n}$ $a_1>0$ ∴ $a_2>0$

若n时 ⇒ $a_{n+1}>0$:

$a_n>0$ 时 $a_{n+1}-a_n=\frac{-a_n}{1+a_n}<0$

~~又a~~ $a_n\downarrow$ 有下界

1.3

(2) $\lim\limits_{x\to0}\frac{\cos x-\cos3x}{x^2}$

$=\lim\limits_{x\to0}\cancel{1-\cos x}\ \frac{\cos x-1}{x^2}+\frac{1-\cos 3x}{x^2}$

$=\frac{-1}{2}+\frac{9}{2}=4$

(4) $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2}$

$=\lim\limits_{x\to\infty}\left(1+\frac{2}{x^2-1}\right)^{x^2}$

$=e^2$

10.(1) $0\le\lim\limits_{x\to\infty}\left|\frac{\arctan x}{x}\right|\le\frac{\pi}{2x}=0$

(3) $\lim\limits_{x\to2}\frac{x^3-2x^2}{x-2}=x^2\big|_{x=2}=4$

$a_n = \sqrt[n]{n!}$ $\quad b_n = n\sin\frac{n\pi}{2}$

24. $\sqrt[n]{n!}$

$\ln a_n = \frac{\ln 1 + \cdots + \ln n}{n}$ $\qquad$ why 趋向么 $\lim\limits_{n\to\infty} a_n = +\infty$

$\lim\limits_{n\to\infty} \ln a_n = \ln n + 1$

$b_n = n \quad n = 4k+1$

$\quad\; 0 \quad n$ 为~~xx~~ 偶

$\quad -n \quad n = 4k-1,$

9.22 13. (1) $\frac{x-1}{x+1} = 1$

(4) $\lim\limits_{x\to 0^+} x^{\frac{1}{q}} = 0$

$\forall \varepsilon > 0 \quad \exists \delta = \varepsilon^q \quad x \in (0, \varepsilon^q)$

$|x^{\frac{1}{q}} - 0| < (\varepsilon^q)^{\frac{1}{q}} = \varepsilon$

当 $f(x)$ 在 $+\infty$ 时 都定义 对 $\forall \{a_n\}$

(4) $\forall \lim\limits_{n\to\infty} f(a_n) = l \qquad \lim\limits_{x\to\infty} f(x) = l$

$\forall \varepsilon > 0 \quad \lim\limits_{x\to\infty} f(x) = l \qquad \exists M > 0$

s.t. $|f(x) - l| < \varepsilon$

又 $\lim\limits_{n\to\infty} a_n = \infty \quad \therefore \exists N \quad n > N \quad a_n > M$

$|f(a_n) - l| < \varepsilon$ 对 $\forall n > N$ 时之

$\therefore \lim\limits_{n\to\infty} f(a_n) = l$

反之 $\{a_n\}$ $\lim\limits_{n\to\infty} f(a_n) = l$, 同一个 不还 $\lim\limits_{x\to\infty} f(x) = l$ 取 $M = 1 \to a$

$M = 2 \to a$

$\forall \varepsilon \quad \exists M \quad |f(x) - l| < \varepsilon$ 对 $x > M$ 时之

$\exists \varepsilon \quad \forall M \quad |f(x) - l| > \varepsilon$ $\underset{\exists x_0}{x > M}$ 对 $\forall \varepsilon$ $\qquad M = n \to a_n$

$x\to 0$ 比较无穷小量之阶

16. (1) $\frac{\tan x-\sin x}{x^3} = \frac{\frac{\sin x}{\cos x}-\sin x}{x^3}$

$= \frac{1}{2}$

(2) x^3+x^2 与 $\sin^2 x$

$\lim\limits_{x\to 0} \frac{x^3+x^2}{\sin^2 x} = 1$

18. (1) $\lim\limits_{x\to 0} \frac{\sin mx}{\sin nx}$

$= \frac{\sin mx}{mx} \frac{nx}{\sin nx} \frac{mx}{nx}$

$= \frac{m}{n}$

(3) $\frac{\sqrt[n]{1+x\sin x}-1}{\arctan x}$

(5) $\lim\limits_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{\sin 2x}$

证明因为收敛

17. (2) $a_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \cdots + \frac{1}{3^n+1}$

$a_{n+1} - a_n = \frac{1}{3^{n+1}+1} > 0$ 单↑

$a_n < \frac{1}{3} + \cdots + \frac{1}{3^n}$

$= \frac{1}{2}(1 - \frac{1}{3^{n+1}})$

(4) $a_n = \frac{\cos 1}{1\cdot 2} + \cdots + \frac{\cos n}{n(n+1)}$ $\forall \varepsilon > 0$

$|a_m - a_n| =$

22 $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

$a_n = (1+\frac{1}{2n+1})^{2n}$

$a_n = (\frac{1+n}{2+n})^n$

$= \frac{1}{(\frac{2+n}{1+n})^n} = \frac{1}{(1+\frac{1}{n+1})^n} = \frac{1}{e}$