第十次作业解答

4.1.1 (2) 原式 $=\int (e^{2x}-e^{x}+1)\,dx=\frac{1}{2}e^{2x}-e^{x}+x+C$

(4) 原式 $=\int \frac{1}{\cos^2\theta}-1\,dx=\tan\theta-x+C$

(6) 原式 $=\int \frac{1+\cos^2 x}{2\cos^2 x}\,dx=\cancel{2}\frac{1}{2}\tan\theta+\frac{1}{2}x+C$

4.1.2 (3) 原式 $=\int \frac{1}{1+\sin x+\cos x}\,d(1+\sin x+\cos x)=\ln|1+\sin x+\cos x|+C$

(5) 原式 $=-\frac{1}{2}\int\sqrt{1-x^2}\,d(1-x^2)=-\frac{\cancel{4}1}{3}(1-x^2)^{\frac{3}{2}}+C$

(6) 原式 $=2\int\frac{1}{1+x}\,d(\sqrt{x})=2\arctan(\sqrt{x})+C$

(8) 原式 $=\int\frac{1}{1+x\ln x}\,d(1+x\ln x)=\ln|1+x\ln x|+C$

4.1.3 (1) 原式 $\xlongequal{\sqrt{e^x-2}=t}\int t\cdot\frac{2t}{t^2+2}\,dt=2t-2\sqrt{2}\arctan\frac{t}{\sqrt{2}}+C$

$=2\sqrt{e^x-2}-2\sqrt{2}\arctan\left(\sqrt{\frac{1}{2}e^x-1}\right)+C$

(4) 原式 $\xlongequal[\text{设 } a>0]{x=a\sin t}\int\frac{a^2\sin^2 t}{a\cos t}a\cos t\,dt=\int\frac{1}{2}a^2(1-\cos 2t)\,dt$

$=\frac{1}{2}a^2t-\frac{1}{4}a^2\sin 2t+C=\frac{1}{2}a^2\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+C$

(6) 原式 $\xlongequal{\sqrt{1+x^2}=t}\int\frac{\sqrt{t^2-1}\ \frac{1}{2}\ln(t^2-1)}{t^3}\frac{1}{2}\frac{2t}{\sqrt{t^2-1}}\,dt=\int\frac{1}{2}\cdot\frac{\ln(t^2-1)}{t^2}\,dt=\int\frac{\ln(t+1)+\ln(t-1)}{2}\,d\left(\frac{-1}{t}\right)$

而 $\int\ln(t+a)\,d\left(-\frac{1}{t}\right)=-\frac{\ln(t+a)}{t}+\int\frac{1}{t(t+a)}\,dt=\frac{1}{a}\ln\frac{t}{t+a}-\frac{\ln(t+a)}{t}$

故原式 $=\frac{1}{2}\ln\frac{t-1}{t+1}-\frac{\ln(t^2-1)}{2t}+C=\ln\left(\sqrt{1+x^2}-1\right)-\ln x-\frac{\ln x}{\sqrt{1+x^2}}+C$

(8) 原式 $\xlongequal{x=a\tan t}\int\frac{1}{a^2\tan^2 t\,\frac{a}{\cos t}}a\frac{1}{\cos^2 t}\,dt=\int\frac{\cos t}{a^2\sin^2 t}\,dt=-\frac{1}{a^2}\frac{1}{\sin t}+C$

$=-\frac{1}{a^2}\frac{\sqrt{x^2+a^2}}{x}+C$

(11) 原式 $\xlongequal{x=\frac{1}{\sin t}}\int\frac{\frac{1}{\sin t}-1}{\frac{1}{\sin^2 t}\ \frac{\cos t}{\sin t}}\frac{-\cos t}{\sin^2 t}\,dt=\int\sin t-1\,dt=-\cos t-t+C$

$=-\frac{\sqrt{x^2-1}}{x}-\arcsin\frac{1}{x}+C$

(12) 原式 $\xlongequal{x=\frac{1}{s}} \int \frac{s^8}{1+\frac{1}{s^2}}\left(-\frac{1}{s^2}\right) ds = \int \frac{-s^8-s^6+s^6+s^4-s^4-s^2+s^2+1-1}{s^2+1} ds$

$= -\cancel{s^6} - \frac{1}{7}s^7 + \frac{1}{5}s^5 - \frac{1}{3}s^3 + s - \arctan s + C = -\frac{1}{7}x^{-7} + \frac{1}{5}x^{-5} - \frac{1}{3}x^{-3} + x^{-1} - \arctan\frac{1}{x} + C.$

4.1.4 (2) 原式 $= \begin{cases} \int x^2 dx = \frac{1}{3}x^3 + C & (x \le -1 \text{ 或 } x \ge 1) \\ \int 1\, dx = x + C' & (-1 \le x \le 1) \end{cases}$

由原函数在 $x=\pm 1$ 处连续知 原式 $= \begin{cases} \frac{1}{3}x^3 + \frac{2}{3} + C & x \ge 1 \\ x + C & -1 \le x \le 1 \\ \frac{1}{3}x^3 - \frac{2}{3} + C & x \le -1. \end{cases}$

4.1.5 (2) 原式 $= \int \ln x \, d\left(\frac{1}{3}x^3\right) = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

(6) 原式 $= \int x^2 de^x = x^2 e^x - 2\int e^x x\, dx = x^2 e^x - 2\int x\, de^x$

$= x^2 e^x - 2\left(x e^x - \int e^x dx\right) = (x^2 - 2x + 2)e^x + C$

(8) 原式 $= \int (\arctan x)^2 d\left(\frac{1}{2}x^2\right) = \frac{1}{2}x^2(\arctan x)^2 - \int \frac{1}{2}x^2 \, 2\arctan x \frac{1}{1+x^2} dx$

$= \frac{1}{2}x^2(\arctan x)^2 - \int \arctan x \, dx + \int \arctan x \frac{1}{1+x^2} dx$

$= \frac{1}{2}x^2(\arctan x)^2 - x\arctan x + \frac{1}{2}\ln(1+x^2) + \frac{1}{2}(\arctan x)^2 + C$

(10) 原式 $= x\ln(x+\sqrt{x^2+1}) - \int x \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} dx = x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}} dx$

$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + C$

4.1.6 (1) $a_n = \int \sin^n x\, dx = -\int \sin^{n-1}x \, d\cos x = -\sin^{n-1}x\cos x + \int \cos^2 x\,(n-1)\sin^{n-2}x\, dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x - \sin^n x\, dx = -\sin^{n-1}x\cos x + (n-1)a_{n-2} - (n-1)a_n$

故 $a_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}a_{n-2}.$

(2) $b_n=\int x^n e^x dx=\int x^n d(e^x)=x^n e^x-\int n e^x x^{n-1}dx=x^n e^x-nb_{n-1}$

4.1.7 (6) 原式 $=\int 2x\,d(\sqrt{e^x-2})=2x\sqrt{e^x-2}-2\int\sqrt{e^x-2}\,dx$

$\xlongequal{4.1.3(1)} 2x\sqrt{e^x-2}-4\sqrt{e^x-2}+4\sqrt{2}\arctan\left(\sqrt{\frac{1}{2}e^x-1}\right)+C$

(8) 原式 $=-\int\frac{\cot x}{\cot x+1}d\cot x=\int -1+\frac{1}{\cot x+1}d\cot x=-\cot x+\ln|\cot x+1|+C$

(9) 原式 $\xlongequal{x=\cos^2\theta}\int\frac{\sin\theta}{1-\cos\theta}2\cos\theta(-\sin\theta)\,d\theta=\int -2\cos\theta-2\cos^2\theta\,d\theta$

$=-2\sin\theta-\frac{1}{2}\sin 2\theta-\theta\overset{+C}{}=-2\sqrt{1-x}-\sqrt{x}\sqrt{1-x}-\overset{(\arcsin\sqrt{1-x})}{\arccos\sqrt{x}}+C$

(10) 原式 $\xlongequal[x=\frac{t^2+1}{1-t^2}]{\sqrt{\frac{x-1}{x+1}}=t}\int t\left(\frac{1-t^2}{1+t^2}\right)^2\frac{4t}{(1-t^2)^2}dt=4\int\frac{t^2+1-1}{(1+t^2)^2}dt$

$=4\int\frac{1}{1+t^2}-\frac{1}{(1+t^2)^2}dt\xlongequal[t=\tan s]{后者用}2\arctan \overset{t}{\cancel{s}}-\frac{2t}{1+t^2}+C$

$=2\arctan\sqrt{\frac{x-1}{x+1}}\pm\frac{\sqrt{x^2-1}}{x}$

(12) 原式 $=\int\frac{x-x\sin x}{\cos^2 x}dx=\int x\,d\tan x-\int x\,d\sec x$

$=x\tan x-\underline{\int\tan x\,dx}-x\sec x+\underline{\int\sec x\,dx}\xlongequal{合并}x(\tan x-\sec x)+\ln|1+\sin x|+C$

(20) 原式 $=-\int xe^x d\left(\frac{1}{1+x}\right)=-\frac{x}{1+x}e^x+\int\frac{1}{1+x}(1+x)e^x dx=\frac{e^x}{1+x}$.

4.2.1 (3) 原式 $=\int\frac{(x+1)(x^2-x+1)}{x(x+1)(x-1)}dx=\int 1+\frac{1}{x-1}-\frac{1}{x}dx=x+\ln|x-1|-\ln|x|+C$

(4) 原式 $=\int\frac{-\frac{1}{2}x-\frac{1}{2}}{x^2+1}+\frac{1}{x}+\frac{-\frac{1}{2}}{x+1}dx=-\frac{1}{4}\ln(x^2+1)-\frac{1}{2}\arctan x+\ln|x|$

$-\frac{1}{2}\ln|x+1|+C$

(7) 原式 $=\int\frac{x^4-1}{x^8+1}d\left(\frac{1}{2}x^2\right)\xlongequal{x^2=t}\int\frac{t^2-1}{t^4+1}\cdot\frac{1}{2}dt.$

$=\frac{1}{2}\int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=\frac{1}{2}\int\frac{1}{(t+\frac{1}{t})^2-2}d\left(t+\frac{1}{t}\right)$

$\xlongequal{t+\frac{1}{t}=s} \frac{1}{2}\int \frac{1}{(s+\sqrt{2})(s-\sqrt{2})} ds = \frac{1}{4\sqrt{2}}\ln\left|\frac{s-\sqrt{2}}{s+\sqrt{2}}\right| + C = \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\frac{1}{x^2}-\sqrt{2}}{x^2+\frac{1}{x^2}+\sqrt{2}}\right) + C$

(8) 原式 $= \int \frac{x^8}{(x^8+1)^2} d\left(\frac{1}{8}x^8\right) \xlongequal{x^8=t} \frac{1}{8}\int \frac{t+1-1}{(t+1)^2} dt$

$= \frac{1}{8}\left(\ln|t+1| + \frac{1}{t+1}\right) + C = \frac{1}{8}\left[\ln(x^8+1) + \frac{1}{x^8+1}\right] + C$

4.2.2 (1) 原式 $\xlongequal{t=\tan\frac{x}{2}} \int \frac{1+\frac{2t}{t^2+1}}{\frac{2t}{t^2+1}\left(1+\frac{1-t^2}{t^2+1}\right)} \frac{2}{t^2+1} dt = \int \frac{1}{2}t + \frac{1}{2t} + 1\ dt$

$= \frac{1}{4}t^2 + \frac{1}{2}\ln|t| + t + C = \frac{1}{4}\tan^2\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + C$

(4) 原式 $\xlongequal{t=\tan x} \int \frac{\frac{t^2}{t^2+1}}{t+1} \frac{1}{t^2+1} dt = \int \frac{1}{4}\left[\frac{1}{1+t} - \frac{t-1}{1+t^2} + \frac{2t-2}{(1+t^2)^2}\right] dt$.

$= \frac{1}{4}\ln|1+t| - \frac{1}{8}\ln(1+t^2) + \frac{1}{4}\arctan t - \frac{1}{4}\frac{1}{1+t^2} - \frac{1}{4}\left(\arctan t + \frac{t}{1+t^2}\right) + C$

$= \cancel{\frac{1}{4}\ln|1+\tan x| - \frac{1}{8}} \frac{1}{4}\ln|\sin x+\cos x| - \frac{1}{4}(\cos^2 x + \cos x\sin x) + C$.

(9) 原式 $\xlongequal{t=\tan\frac{x}{2}} \int \frac{1}{2\frac{2t}{t^2+1}+2\frac{1-t^2}{t^2+1}\frac{2t}{t^2+1}} \cdot \frac{2}{1+t^2} dt = \int \frac{1}{4}t + \frac{1}{4t} dt$

$= \frac{1}{8}t^2 + \frac{1}{4}\ln|t| + C = \frac{1}{8}\tan^2\frac{x}{2} + \frac{1}{4}\ln\left|\tan\frac{x}{2}\right| + C$.

(10) 设 $I = \int \frac{\cos x}{a\sin x+b\cos x} dx$, $J = \int \frac{\sin x}{a\sin x+b\cos x} dx$

则 $aJ + bI = \int 1\, dx = x + C_1$

$aI - bJ = \int \frac{a\cos x - b\sin x}{a\sin x+b\cos x} dx = \ln|a\sin x+b\cos x| + C_2$

故 $I = \frac{1}{a^2+b^2}\left(a\ln|a\sin x+b\cos x| + bx\right) + C$.

原式 $=$