习题5.4

1.(1) $\int_0^{+\infty} xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_0^{+\infty} = \frac{1}{2}$，收敛.

(4) 当 $x>1$ 时，$\frac{\arctan x}{x} > \frac{\pi}{4}\cdot\frac{1}{x}$. 而 $\int_1^{+\infty}\frac{1}{x}dx = +\infty$ 发散.

故 $\int_1^{+\infty}\frac{\arctan x}{x}dx$ 发散

(5) $\int_0^{+\infty} e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x+\cos x)\Big|_0^{+\infty} = \frac{1}{2}$，收敛.

(10) $\int_0^1 \ln\frac{1}{1-x^2}dx = [(1-x)\ln(1-x)-(1-x)-(1+x)\ln(1+x)]\Big|_0^1 = 2-2\ln 2$，收敛

(12) 记 $I_n = \int_0^1 (\ln x)^n dx$，$I_1 = -1$.

$I_{n+1} = \int_0^1 (\ln x)^{n+1}dx = x(\ln x)^{n+1}\Big|_0^1 - \int_0^1 (n+1)(\ln x)^n dx$

$= -(n+1)I_n$. 递归可得，$I_n = (-1)^n\cdot n!$. 收敛.

3.(1) 当 $x\to1^+$ 时，$\frac{1}{x\sqrt{x-1}} \sim \frac{1}{\sqrt{x-1}}$，当 $x\to+\infty$ 时，$\frac{1}{x\sqrt{x-1}} \sim x^{-\frac{3}{2}}$.

故 $\int_1^{+\infty}\frac{1}{x\sqrt{x-1}}dx$ 收敛

$\int_1^{+\infty}\frac{1}{x\sqrt{x-1}}dx \overset{t=\sqrt{x-1}}{=\!=} 2\int_0^{+\infty}\frac{1}{1+t^2}dt = 2\times\frac{\pi}{2} = \pi$.

(2) 当 $\alpha\le1$ 时. $\int_1^{+\infty}\frac{1}{x^\alpha}dx$ 发散.

当 $\alpha\ge1$ 时. $\int_0^1\frac{1}{x^\alpha}dx$ 发散.

故对任意实数 α，$\int_0^{+\infty}\frac{1}{x^\alpha}dx$ 发散.

4.(2) $\int_{-1}^1 x^{-\frac{1}{3}}dx = \int_{-1}^0 x^{-\frac{1}{3}}dx + \int_0^1 x^{-\frac{1}{3}}dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_{-1}^0 + \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1$

$= 0$

习题6.1

1.(2) $\frac{dy}{dx} = e^{x-y} \Rightarrow e^y dy = e^x dx \Rightarrow e^y = e^x + C \Rightarrow y = \ln(e^x+C)$

(3) $x\frac{dy}{dx}=y^2-y \Rightarrow \frac{dy}{y^2-y}=\frac{dx}{x} \Rightarrow \ln\left|\frac{y-1}{y}\right|=\ln|x|+C$

$\Rightarrow y=\frac{1}{1\pm C|x|}$，$y=0, y=1$ 也是方程的解.

2.(2) $\frac{dy}{dx}=\frac{y}{x}+\frac{x}{y}$. 令 $u=\frac{y}{x}$. $\frac{dy}{dx}=u+x\frac{du}{dx}=u+\frac{1}{u}$

$\Rightarrow udu=\frac{1}{x}dx \Rightarrow \frac{1}{2}u^2=\ln|x|+C \Rightarrow y^2=2x^2(\ln|x|+C)$.

(4) $(x^2+3y^2)dx-2xydy=0 \Rightarrow \frac{dy}{dx}=\frac{x}{2y}+\frac{3y}{x}$. 令 $u=\frac{y}{x}$

$\frac{dy}{dx}=u+x\frac{du}{dx}=\frac{1}{2u}+\frac{3}{2}u \Rightarrow \frac{2u}{u^2+1}du=\frac{1}{x}dx$

$\Rightarrow \ln(u^2+1)=\ln|x|+C \Rightarrow y^2=x^2(C|x|-1)$.

3.(1) 令 $\begin{cases}x+y+3=0\\ x-y+1=0\end{cases} \Rightarrow \begin{cases}x=-2\\ y=-1\end{cases}$，令 $\begin{cases}u=x+2\\ v=y+1\end{cases}$. $t=\frac{v}{u}$.

则 $\frac{dy}{dx}=\frac{x+y+3}{x-y+1} \Rightarrow t+u\frac{dt}{du}=\frac{1+t}{1-t} \Rightarrow \frac{1-t}{1+t^2}dt=\frac{1}{u}du$

$\Rightarrow \arctan t-\frac{1}{2}\ln(t^2+1)=\ln|u|+C$

$\Rightarrow \arctan\frac{y+1}{x+2}-\frac{1}{2}\ln\left[\left(\frac{y+1}{x+2}\right)^2+1\right]=\ln|x+2|+C$

4.(1) $\frac{dy}{dx}-\frac{2x}{1+x^2}y=1+x^2$. 一阶线性ODE.

代入公式. $y=e^{-\int\left(\frac{-2x}{1+x^2}\right)dx}\left[C+\int(1+x^2)e^{\int\left(\frac{-2x}{1+x^2}\right)dx}dx\right]=(1+x^2)(C+x)$

(3) $\frac{dx}{dy}=\frac{1}{y}x+y^2$. 一阶线性ODE.

$x=e^{-\int-\frac{1}{y}dy}\left[C+\int y^2e^{\int-\frac{1}{y}dy}dy\right]=|y|(C+\int|y|dy)$.

$\Rightarrow x=y(C+\frac{y^2}{2})$

5.(2) 一阶线性ODE, 代入公式得 $y=\dfrac{1-\cos x+C}{x}$

由 $y(\pi)=1$ 可知 $C=\pi-2$. 故 $y=\dfrac{\pi-1+\cos x}{x}$

6.(2) 令 $u=y-x$. 则 $\dfrac{du}{dx}=\dfrac{dy}{dx}-1=\cos u-1$

$\dfrac{du}{\cos u-1}=dx \Rightarrow \dfrac{\cos\frac{y-x}{2}}{\sin\frac{y-x}{2}}=x+C$.

$u=2k\pi$. 即 $y=x+2k\pi$ 也是方程的解.

8. $y=f(x)$ 在点 $(x_0, f(x_0))$ 处的切线方程为 $y=f'(x_0)(x-x_0)+f(x_0)$

与坐标轴的交点为 $(0, f(x_0)-x_0f'(x_0))$, $(x_0-\dfrac{f(x_0)}{f'(x_0)}, 0)$

$$\begin{cases} 2x_0=x_0-\dfrac{f(x_0)}{f'(x_0)} \\ 2f(x_0)=f(x_0)-x_0f'(x_0) \end{cases} \Rightarrow f(x)+xf'(x)=0$$

$f(x)=\dfrac{C}{x}$. 又因为 $f(2)=3$.

故 $C=6$, 也即 $f(x)=\dfrac{6}{x}$.

12. (3) $\lambda^2=\lambda \Rightarrow \lambda_1=0$, $\lambda_2=1$.

设一个特解为 $y'=C_1x+C_2$. $\Rightarrow$ $C_1=-1$. $C_2=-1$

也即 $y'=-x-1$, $y=-\dfrac{x^2}{2}-x$

通解为 $y=C_1e^x+C_2-\dfrac{x^2}{2}-x$.

(4) 不显含 x 的方程, 令 $p=y'$. 则 $p\dfrac{dp}{dy}=y''$

$p\dfrac{dp}{dy}+p^2=2e^{-y}$. 再令 $u=p^2$. 则 $\dfrac{1}{2}\dfrac{du}{dy}+u=2e^{-y}$

$\Rightarrow u=Ce^{-2y}+4e^{-y}$ 即 $p=\pm\sqrt{Ce^{-2y}+4e^{-y}}$

两端求积分, 得 $\pm\displaystyle\int\frac{dy}{\sqrt{Ce^{-2y}+4e^{-y}}}=x+C_1$

即 $\pm\dfrac{1}{2}\sqrt{C+4e^{y}}=x+C_1$, 也即 $y=\ln[C_1+(x+C_2)^2]$.

习题6.2

1.(3) $y_2(x)=y_1(x)\int\frac{1}{y_1^2(x)}e^{-\int_{x_0}^{x}p(t)dt}dx$

$=x\int\frac{1}{x^2}(1+x^2)dx$

$=x(x-\frac{1}{x})$

$=x^2-1$

通解为 $y(x)=C_1x+C_2(x^2-1)$

2.(1) 注意到 $y_1(x)=x^2$ 是一个特解.

$y_2(x)=y_1(x)\int\frac{1}{y_1^2(x)}e^{-\int_{x_0}^{x}p(t)dt}dx$

$=x^2\int\frac{1}{x^4}\cdot x^2dx$.

$=-x$.

通解为. $y(x)=C_1x^2+C_2x$.