Random Matrix Fields Generator

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1. Introduction

Many interesting quantum systems depend on external parameters. Examples include systems that are influenced by an external field like an electric field and particles in a periodic environment such as a periodic lattice structure, the Bloch parameters (k) being the external parameters. Another case is systems that have fast and slow degrees of freedom, where the Hamiltonian that describes the fast degrees of freedom depends on the slow ones as parameters as in the Born-Oppenheimer approximation, where the nuclear positions are regarded as parameters for the electronic dynamics. Many of the questions that can be asked regarding such systems are local in nature, for instance how does the spectrum of the system depends upon the parameters.

More recently people noticed the existence of global properties that in order to be calculated one should know the Hamiltonian for all parameters values; these are topological properties. Topological properties are invariant under continuous changes in the Hamiltonian as long as energy gaps remain open. Chern numbers were one of the first topological indices to be examined in that manner, in connection with quantized Hall conductance [TKNN]. People hope to use such topological properties to create systems that can sustain disorder and noise created by the environment, thus making delicate systems more durable.

Why random matrices? Eugene Wigner used random matrices in his work to model the complex spectrum of a heavy atom nucleus. The system was too complicated for making exact prediction of the energy levels of each nucleus [Mehta]. Wigner and collaborators found a resemblance between the energy level spacings and the spacings between the eigenvalues in a random matrix, and proposed to describe the system statistically with a random matrix model.

Symmetry arguments suggest using random matrices with distributions that are invariant under unitary transformations in systems with broken time-reversal symmetry. The requirement for being invariant under unitary transformations was found to lead to universal properties, for example two matrices with different element distributions will have similarly distributed physical properties (e.g. energy gap distribution).

Since we wish to study complex systems that depend on external parameters we need a family of random matrices that depend on parameters, in other words a random matrix field. In the nineties people researched how do physical properties of the system varies as a result of changing the parameter, for example by altering the external electrical field how will the energy change. They found that the way the energy varies in relation to the parameter is universal, this universality was later recreated successfully using random matrices [AA]. In 1995 [WW] studied the statistics of Chern numbers of a specific model of random matrix field. They made arguments supported by numerical calculations that the Chern index distribution is universal.

Beyond that, not a lot is known regarding the behavior of topological indices of random matrix fields; we assume that such distribution may teach us something about topological indices in physical complex systems. We hope as well to find universality in such systems, as this will help us better understand the behavior of topological indices in many different physical systems.

In this project we created a code that provides its user with a random matrix field on the 2-sphere S^2, with user-supplied correlations. The project's origins came from the need of such a system in Or Swartzberg's masters thesis, his research includes numerical calculations of Chern numbers using Monte-Carlo simulation; where he aspires to learn about the statistics of Chern numbers for different ensembles of random matrix fields. The mathematics behind such a system, the statistical tests and the code are described in the following pages.

2. Theoretical background

We consider a random matrix field, $\mathcal{H}(\boldsymbol{p})$ on S^2. The variable $\boldsymbol{p}$ is a point on the surface of the sphere, that can be presented in spherical coordinates as $\boldsymbol{p} = (\theta, \phi)$ (the radius is constant, $r = 1$) or in Cartesian coordinates at each point as follows

$$\boldsymbol{p} = \begin{pmatrix} cos\phi sin\theta \\ sin\phi sin\theta \\ cos\theta \end{pmatrix} \tag{1}$$

We wish to model Hamiltonian systems, therefore $\mathcal{H}$ must be Hermitian $\mathcal{H}(\boldsymbol{p}) = \mathcal{H}^\dagger(\boldsymbol{p})$. In addition, we want to make $\mathcal{H}(\boldsymbol{p})$ statistically invariant under the transformation $\mathcal{H}(\boldsymbol{p}) \to U^\dagger \mathcal{H}(\boldsymbol{p}) U$ for any fixed unitary matrix U, and select the matrix elements to be centered Gaussian random variables [Gat]. The non-zero correlations between the matrix elements are

$$\left\langle \mathcal{H}_{ij}(\boldsymbol{p}) \mathcal{H}_{ij}(\boldsymbol{p}')^* \right\rangle = \left\langle \mathcal{H}_{ii}(\boldsymbol{p}) \mathcal{H}_{ii}(\boldsymbol{p}') \right\rangle = h(\boldsymbol{p}, \boldsymbol{p}'), \tag{2}$$

where h is a function to be specified. We next assume that the distribution is isotropic and homogeneous

$$\begin{aligned} h(\boldsymbol{p}, \boldsymbol{p}') &= C(cos\gamma) \\ cos\gamma &= \boldsymbol{p} \cdot \boldsymbol{p}' \end{aligned} \tag{3}$$

$C(cos\gamma)$ will be called the real space (RS) correlation function. $\mathcal{H}$ is therefore a matrix of random functions, where each element is a Gaussian random field [WU]. Since the diagonal elements of $\mathcal{H}$ are real, and all the off-diagonal elements are complex ($\mathcal{H}$ is Hermitian), we can write for a matrix of size $d \times d$

$$\begin{aligned} \mathcal{H}_{ii} &= f_i(\boldsymbol{p}) \text{ , } i = 1, ..., d \\ \mathcal{H}_{ij} &= \frac{g_{ij}(\boldsymbol{p}) + \mathrm{i}\, k_{ij}(\boldsymbol{p})}{\sqrt{2}} \text{ , } 1 \le i < j \le d \end{aligned} \tag{4}$$

where $f_i(\boldsymbol{p}), g_{ij}(\boldsymbol{p}), k_{ij}(\boldsymbol{p})$ are independent random functions centered around zero with correlation $C(cos\gamma)$. Since we work in S^2 we chose to write these random functions as a linear combination of Spherical Harmonics $Y_l^m(\boldsymbol{p})$

$$f(\boldsymbol{p}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_l^m(\boldsymbol{p}), \tag{5}$$

where a_{lm} are centered Gaussian random variable, with correlations

$$\langle a_{lm} a^*_{l'm'} \rangle \equiv c_l \delta_{ll'} \delta_{mm'} \tag{6}$$

We call the c_l the harmonic space (HS) correlation. It is related to the RS correlation by

$$C(cos\gamma) = \left\langle f(\boldsymbol{p}) f(\boldsymbol{p}')^* \right\rangle = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} c_l \mathcal{P}_l(cos\gamma) \tag{7}$$

Where $P_l(cos\gamma)$ are the Legendre polynomials of order l. Reversing (7) we can get an expression for the HS correlation

$$\begin{aligned} c_l &= 2\pi \int_{-1}^{1} C(x) \mathcal{P}_l(x) dx \\ x &= cos\gamma \end{aligned} \tag{8}$$

a_{lm} is complex, when $m \neq 0$ and can be written as a sum of two real independent random variables $a_{lm} = b_{lm} + id_{lm}$, $b_{lm}, d_{lm} \in \mathbb{R}$, with variance

$$\left\langle b_{lm}^2 \right\rangle = \left\langle d_{lm}^2 \right\rangle = \sqrt{\frac{c_l}{2}} \tag{9}$$

Since we want $f(\boldsymbol{p})$ to be real, the a_{lm} must obey

$$a_{lm}^* = (-1)^m a_{l-m} \tag{10}$$

Clearly this condition compels a_{l0} to be real so that

$$\left\langle a_{l0}^2 \right\rangle = c_l \tag{11}$$

3. Methods

The software that had been used to generate the code was "Mathematica" (code is available in appendix), three main functions were built and two were checked statistically.

The idea is that a user provides a correlation function and a list of points $\{\boldsymbol{p}_1, ..., \boldsymbol{p}_n\}$.The code then generates a list of matrices ($output = \{\mathcal{H}(\boldsymbol{p}_1), ..., \mathcal{H}(\boldsymbol{p}_n)\}$), corresponding to the points handed by the user.

The first function, "Clcalc" (14), calculates the transformation from the RS correlation $C(x)$ to the HS correlations c_l given the relation specified in (8). Since the other functions receive c_l as an input rather than $C(x)$, a user who prefers the RS correlation $C(x)$ will find "Clcalc" useful.
However, in practice, we were not able to find a $C(x)$ that will generate c_l which are all positive. Therefore "Clcalc" was not tested, and is presented for the sake of future users who would prefer to use RS correlations instead of HS correlations.

The second function, "randHS" (15) calculates the random functions $f(\boldsymbol{p})$ as mentioned in (5). The inputs of "randHS" are the HS correlations c_l, and the list of points the user provides. The random variables a_{lm} are calculated using RandomVariate and NormalDistribution.
Theoretically (5) takes $l \to \infty$ but in practice we can't take c_l to be an infinitely long list, therefore a length parameter $lmax$ is used in this function and signifies the largest l taken in (5).

The last function, "randMat" (16) builds the random matrices $\mathcal{H}(\boldsymbol{p})$. "randMat" inputs consist of the size of the matrix (Msize) as well as two lists, one of the HS correlation c_l and the other of the points the user wishes to calculate. In order for the matrix to be Hermitian the diagonal part and the rest of the matrix are created in a different manner like specified in (4). The off-diagonal part is divided into two sections, upper triangle and lower triangle, $\mathcal{H}$ is Hermitian, as a result one is the conjugate of the other. The matrix elements are generated by calls to RandHS.

4. Statistical tests

The functions were validated using a few statistical checks, for further explanation of the functions please see code section "statistical checks" in the appendix.
The first test inspected whether the functions were truly distributed as a Gaussian with the correct variance. The test drew one thousand random functions for a chosen correlation ($c_l = \{1, 1, 1, 1\}$ is shown in Figure 1, two more examples are given in the appendix), to each were assigned 300 points that had been chosen randomly using a uniform distribution. On the sphere two histograms of a single point distribution were produced, each with different points and functions, leaving just c_l unchanged. We naturally expect the histograms to be Gaussian around the mean value zero with variance matching the term in (7). Further, a Gaussian fit was taken on top of the histogram.

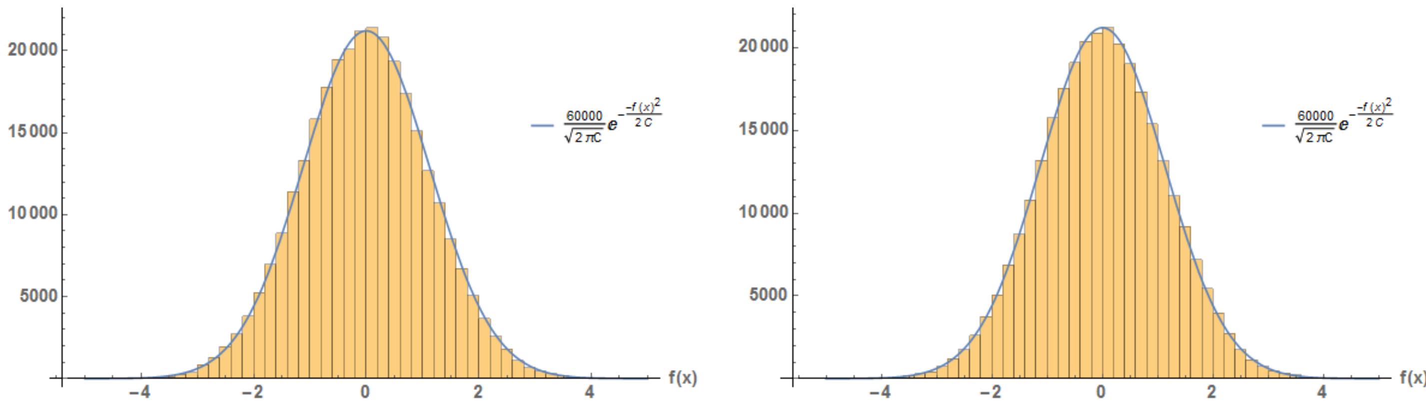

Figure 1: These histograms represent a single points distribution for 300,000 random samples. Each histogram is built from one thousand random functions $f(x)$, each function has 300 points (different points were taken for each histogram) that had been randomly chosen using a uniform distribution. These random functions were generated by "randHS" using $c_l = \{1, 1, 1, 1\}$. The blue line is a Gaussian distribution with mean value zero and variance C that was calculated using (7), for this case it is $\frac{4}{\pi}$. For more histograms with different correlations please see appendix.

Figure 1 presents two histograms of single points distributions, that should be Gaussians with variance given by (7) with $p = p'$. Each of the histograms has the same c_l but was given different points for generating the random functions. Figure 1 clarifies that the Gaussian random process taking place in "randHS" generates the expected single point distribution, the mean value is indeed zero, the variance in this case is $\frac{4}{\pi}$ and it sits beautifully with the target Gaussian.

The second test checked to see if "randHS" succeeds to reconstruct the connection between the random functions and their correlation $C(x)$, as it is presented in (7). The average of n random functions $\{f_i, ..., f_n\}$ with $c_l = \{2.784, 1.998, 0.95, 0.191\}$ was taken at two different points $\boldsymbol{p}$ and $\boldsymbol{p}'$, i.e. $\langle f_i(\boldsymbol{p}) f_i(\boldsymbol{p}') \rangle$ for $1 < i < n$ when three different cases where checked; $n = 100, 1000, 10000$. A total of 150 pairs of $\boldsymbol{p}$ and $\boldsymbol{p}'$ were taken, all the points were randomly chosen using UniformDistribution function; represents a continuous uniform statistical distribution giving values between 0 and π for the polar angle θ and between 0 and 2π for the azimuthal angle ϕ. These averages are compared to the analytical value of the RS correlation $C(x)$ that was calculated using (7).

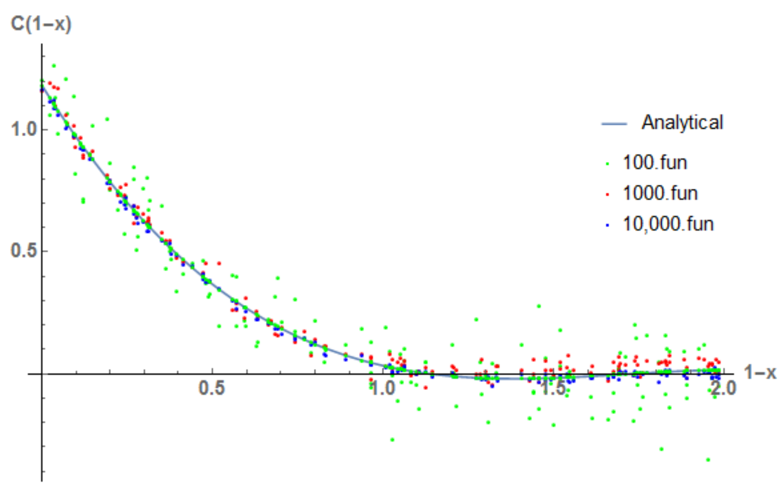

Figure 2: Comparing the analytical RS correlation, given by $c_l = \{2.784, 1.998, 0.95, 0.191\}$ (line plot) to the statistical measurements taken for 150 pairs of points $\boldsymbol{p}$ and $\boldsymbol{p}'$. Green dots represent the average of one hundred functions per point, red represents one thousand and blue ten thousand. The analytical RS correlation is calculated using (7) for the c_l mentioned before.

We expect from a statistical test as such to have a converges rate of $\frac{1}{\sqrt{N}}$ where N is the number of points provided. From Figure 2 we conclude it is indeed so, the fluctuations of the correlation value $C(1-x)$ lessen for larger N. This test demonstrates that "randHS" generates the desired correlation function.

The third test verified the functions given by (4) in "randMat".
Two hundred pairs of points ($\boldsymbol{p}$,$\boldsymbol{p}'$) were randomly chosen using a uniform distribution, for each pair of points an average of n $d \times d$ sized matrices were taken; $\frac{1}{n}\sum_{k=1}^{n}\mathcal{H}_k(\boldsymbol{p})\,\mathcal{H}_k(\boldsymbol{p}')^*$. As in (4) we will split the average calculation into diagonal and off-diagonal elements

$$\frac{1}{n}\sum_{k=1}^{n}\mathcal{H}_{ii}^{k}(\boldsymbol{p})\,\mathcal{H}_{ii}^{k}(\boldsymbol{p}')\ ,\ i=1,...,d \tag{12}$$

$$\frac{1}{n}\sum_{k=1}^{n}\mathcal{H}_{ij}^{k}(\boldsymbol{p})\,\mathcal{H}_{ij}^{k}(\boldsymbol{p}')^*\ ,\ 1\le i<j\le d \tag{13}$$

In Figure 3 these terms, (12) and (13), are compared to the analytical RS correlation calculated using (7).

Figure 3 displays the output of the final test taken, it compares the same parameters as the second test yet it checks that the matrices act as expected in (2). It checks whether the averages converge to the analytical RS correlation given, testing diagonal and off-diagonal correlations independently. Observing Figure 3 and others (see appendix figures 6 and 7) the code appears to work as intended.

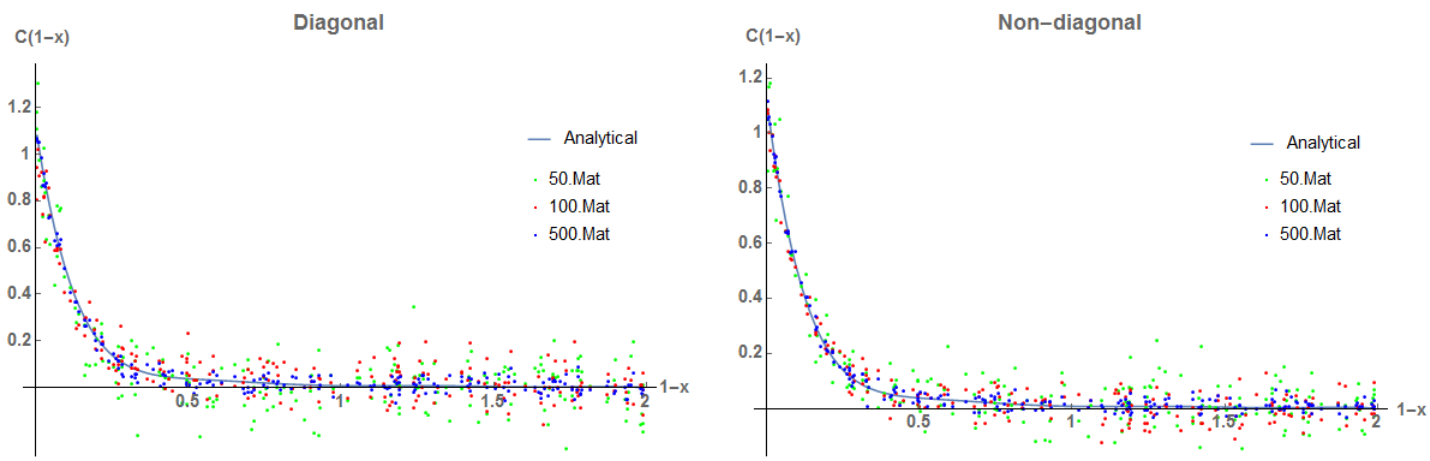

Figure 3: Comparison of the analytical RS correlation (line plot) to the diagonal (12) and off-diagonal (13) part of $\mathcal{H}$. $\mathcal{H}$ is a 3×3 sized matrix and was created using $c_l = \{0.955, 0.767, 0.564, 0.387, 0.246, 0.144, 0.0743, 0.03, 0.005\}$, each matrix has 400 points that had been randomly chosen using a uniform distribution, a total of 200 pairs $(\boldsymbol{p},\boldsymbol{p}')$. Green dots represent the average of fifty matrices per point, red represents one hundred and blue five hundred. The analytical RS correlation is calculated using (7).

5. Conclusions

In this work we managed to create a code in Mathematica that generates random matrix fields using known algorithms. We relied on $\mathcal{H}(\boldsymbol{p})$ being Hamiltonian and invariant under unitary transformation, as well as it having an isotropic and homogeneous distribution.
A linear combination of spherical harmonic functions $Y_l^m(\boldsymbol{p})$ was used in order to model $\mathcal{H}$ on a sphere (5), with coefficients that are independent centered Gaussian random variables (6).
The program was later checked statistically to determine its functionality, indicating that the random matrix fields are indeed generated as expected and are good to use. In the future we propose that further study will be done on the subject of finding good RS correlation functions that could be used in this code.

Appendix:

Code:

Main functions:

$$\begin{aligned}&\texttt{Clcalc}[\texttt{Cf\_},\texttt{lmax\_}]:=\\[1em]&\texttt{Block}[\{\texttt{cl}\},\texttt{cl}=\texttt{Table}[2\pi\,\texttt{Integrate}[\texttt{LegendreP}[\texttt{k},\texttt{x}]\,\texttt{Cf},\{\texttt{x},-1,1\}],\{\texttt{k},0,\texttt{lmax}\}];\\&\texttt{If}[\texttt{AllTrue}[\texttt{cl},\#\geq 0\&],\texttt{cl},\text{"}\texttt{Error}-\texttt{cl has negative values}\text{"}]]\end{aligned}\tag{14}$$

$$\begin{aligned}&\texttt{randHS}[\texttt{cl\_},\texttt{points\_}]:=\\[1em]&\texttt{Block}[\{\texttt{f},\texttt{alm}\},\texttt{lmax}=\texttt{Length}[\texttt{cl}]-1;\\&\texttt{alm}=\texttt{Table}[\texttt{If}[\texttt{m}==0,\texttt{RandomVariate}[\texttt{NormalDistribution}[0,1]],\\&\texttt{RandomVariate}[\texttt{NormalDistribution}[0,\sqrt{\tfrac{1}{2}}]+\texttt{i}\,\texttt{RandomVariate}[\texttt{NormalDistribution}[0,\sqrt{\tfrac{1}{2}}]]],\{\texttt{i},0,\texttt{lmax}\},\{\texttt{m},0,\texttt{i}\}];\\&\texttt{f}=\sum_{\texttt{l}=0}^{\texttt{lmax}}\sqrt{\texttt{cl}[[\texttt{l}+1]]}\sum_{\texttt{m}=0}^{\texttt{l}}\texttt{alm}[[\texttt{l}+1,\texttt{m}+1]]\texttt{SphericalHarmonics}[\texttt{l},\texttt{m},\#[[1]],\#[[2]]]+\\&\sum_{\texttt{l}=0}^{\texttt{lmax}}\sqrt{\texttt{cl}[[\texttt{l}+1]]}\sum_{\texttt{m}=1}^{\texttt{l}}\texttt{Conjugate}[\texttt{alm}[[\texttt{l}+1,\texttt{m}+1]]\texttt{SphericalHarmonics}[\texttt{l},\texttt{m},\#[[1]],\#[[2]]]]\&/@\texttt{points}//\texttt{Chop}]\end{aligned}\tag{15}$$

$$\begin{aligned}&\texttt{randMat}[\texttt{cl\_},\texttt{points\_},\texttt{Msize\_}]:=\\[1em]&\texttt{Block}[\{\texttt{mv, fv1, fv2}\},\ \texttt{Npoints}=\texttt{Length}[\texttt{points}];\ \texttt{mv}=\texttt{Table}[0,\ \texttt{Npoints},\ \texttt{Msize},\ \texttt{Msize}];\\&\texttt{Do}[\texttt{If}\ [\texttt{j}<\texttt{k},\ \texttt{fv1}=\texttt{randHS}[\texttt{cl},\texttt{points}];\ \texttt{fv2}=\texttt{randHS}[\texttt{cl, points}];\ \texttt{Do}[\texttt{mv}[[\texttt{i},\texttt{j},\texttt{k}]]=\frac{\texttt{fv1}[[\texttt{i}]]+\texttt{i}\,\texttt{fv2}[[\texttt{i}]]}{\sqrt{2}};\\&\texttt{mv}[[\texttt{i},\texttt{k},\texttt{j}]]=\frac{\texttt{fv1}[[\texttt{i}]]-\texttt{i}\,\texttt{fv2}[[\texttt{i}]]}{\sqrt{2}};,\ \{\texttt{i, Npoints}\}]];\\&\texttt{If}\ [\texttt{j}==\texttt{k},\ \texttt{fv3}=\texttt{randHS}[\texttt{cl, points}];\ \texttt{Do}[\texttt{mv}[[\texttt{i},\texttt{j},\texttt{k}]]=\texttt{fv3}[[\texttt{i}]],\ \{\texttt{i, Npoints}\}]];,\ \{\texttt{j, Msize}\},\{\texttt{k, Msize}\}];\ \texttt{mv}]\end{aligned}\tag{16}$$

Statistical checks:

Statistical tests:

```
(* I need to Calculate the the Sum -
 from the corelations I have (cl) and legendre polynomials,
and the <> from Mean value of 2 functions
 (a good place to start would be the same θ and ϕ - the same p1 and p2).
  (*  C[x]= <f(p1)f*(p2)> =∑_{l=0}^k cl (2l+1)/(4π) Pl[cosγ]   *)
```

Same point p1=p2

For same points with cl: for p1=p2 the Legendre polynomials are equal to 1 so we get $\sum_{l=0}^{k} cl\ \frac{2l+1}{4\pi}$

If cl is given then we use cl to check the code, if C(x) is given then we use the function Clcalc[C,lmax] to get cl.

```
randtestcl[cl_, points_, Fnumber_] := Mean[Flatten[randFamilyHS[cl, points, Fnumber]^2]]
(*gives the avarages <f(p)f*(p)>*)
sumtestcl[cl_] := Sum[(2 i + 1)/(4 π) cl[[i + 1]], {i, 0, Length[cl] - 1}]
 (*does the summation test C(x) =∑_{l=0}^k cl (2l+1)/(4π) *)
```

Functions:

SPFcheck tests (given Fnumber functions and their points and cl) if the summation $C(1) = \sum_{l=0}^{k} cl\ \frac{2l+1}{4\pi}$ test confirms with the averages $< f(p)\ f^*(p) >$, i.e. if $C(1) = < f(p)\ f^*(p) >$

```
SPFcheck[cl_, points_, Fnumber_] :=
 Block[{sum = sumtestcl[cl], mean = randtestcl[cl, points, Fnumber]},
  Print[{N[sum] "Sum Value", mean "Mean Value", Fnumber "Number of functions"}]]
```

Different points p1≠ p2

For different points the sum $C(\cos\gamma) = \sum_{l=0}^{k} cl \frac{2l+1}{4\pi} Pl[\cos\gamma]$ must be calculated. normal[points] is given points and it calculates the Cartesian coordinates of the point so it can be used in sumtestclp1p2 that calculates C(x) using summation over the Legendre polynomials like so $C(\cos\gamma) = \sum_{l=0}^{k} cl \frac{2l+1}{4\pi} Pl[\cos\gamma]$.

```
normal[points_] := Table[{Cos[points[[j, 2]]] Sin[points[[j, 1]]],
   Sin[points[[j, 2]]] Sin[points[[j, 1]]], Cos[points[[j, 1]]]} , {j, 1, Length[points]}]
  (*points needs to be a list of 2 lists like so - points={{θ1,ϕ1},{θ2,ϕ2}},
or an even list so 2 points can be taken each time*)
sumtestclp1p2[cl_, points_] := (lmax = Length[cl] - 1;
  If[Length[Flatten[points]] > 2, Table[
    Sum[(2 i + 1)/(4 π) cl[[i + 1]] LegendreP[i, normal[points][[1 + j]].normal[points][[2 + j]]],
     {i, 0, lmax}], {j, 0, Length[points] - 2, 2}], Sum[(2 i + 1)/(4 π) cl[[i + 1]]
     LegendreP[i, normal[points][[1]].normal[points][[1]]], {i, 0, lmax}]]) (*The
   output will be a list of sumtest for each couple of points = {cl12,cl34,cl56,...}*)
```

Matrices:

MMcheckp1p2 takes as an input a list of matrices (Mat), points and a list of HS correlations (cl). It checks to see if $C[x] = \langle H_{ij}(p1)\, H_{ij}^{*}(p2) \rangle$, C(x) is calculated using sumtestclp1p2.

```
MMcheckp1p2[Mat_, points_, cl_] := (lmax = Length[cl] - 1;
  Msize = Length[Mat[[1, 1]]];
  Mnumber = Length[Mat];
  Npoints = Length[Mat[[1]]];
  M1M2 =
   Table[Mat[[i, j + 1]] Conjugate[Mat[[i, j + 2]]], {j, 0, Npoints - 2, 2}, {i, Mnumber}];
  meanup = {};
  Do[g = {};
   Do[If[j < k, AppendTo[g, M1M2[[n, i, j, k]]], 0], {j, Msize}, {k, Msize}, {i, Mnumber}];
   AppendTo[meanup, Mean[g]], {n, Npoints/2}];
  meandiag = {};
  Do[diag = {};
   Do[AppendTo[diag, Tr[Mat[[i, 1 + j]] Mat[[i, j + 2]]] / Msize], {i, Mnumber}];
   AppendTo[meandiag, Mean[diag]], {j, 0, Npoints - 2, 2}];
  Print[{N[sumtestclp1p2[cl, points]], "Sum value", meanup ,
    "upper mean value", meandiag , "diagonal mean value", Msize "matrix size",
    Mnumber "number of matrices", Mnumber (Msize + (Msize^2 - Msize)/2) "number of functions"}])
  (*Statistical check for many matrices*)
```

Mathematical developments:

$$\left\langle \mathcal{H}_{ij}\left(\boldsymbol{p}\right)\overline{\mathcal{H}_{ij}\left(\boldsymbol{p}'\right)}\right\rangle = \left\langle \left(\frac{g_{ij}\left(\boldsymbol{p}\right)+\mathrm{i}\,k_{ij}\left(\boldsymbol{p}\right)}{\sqrt{2}}\right)\left(\frac{g_{ij}\left(\boldsymbol{p}'\right)-\mathrm{i}\,k_{ij}\left(\boldsymbol{p}'\right)}{\sqrt{2}}\right)\right\rangle = \tag{17}$$
$$= \frac{1}{2}\left\langle g_{ij}\left(\boldsymbol{p}\right)g_{ij}\left(\boldsymbol{p}'\right)+\mathrm{i}\,k_{ij}\left(\boldsymbol{p}\right)g_{ij}\left(\boldsymbol{p}'\right)-\mathrm{i}\,k_{ij}\left(\boldsymbol{p}'\right)g_{ij}\left(\boldsymbol{p}\right)+k_{ij}\left(\boldsymbol{p}\right)k_{ij}\left(\boldsymbol{p}'\right)\right\rangle$$
$$g_{ij}\,and\,k_{ij}\,are\,independent\,from\,one\,another$$
$$\Rightarrow \left\langle \mathcal{H}_{ij}\left(\boldsymbol{p}\right)\overline{\mathcal{H}_{ij}\left(\boldsymbol{p}'\right)}\right\rangle = \frac{1}{2}\left\langle g_{ij}\left(\boldsymbol{p}\right)g_{ij}\left(\boldsymbol{p}'\right)\right\rangle + \frac{1}{2}\left\langle k_{ij}\left(\boldsymbol{p}\right)k_{ij}\left(\boldsymbol{p}'\right)\right\rangle \overset{(2),(3)}{=} C\left(cos\gamma\right)$$

$$C\left(cos\gamma\right) = \left\langle f\left(\boldsymbol{p}\right)f\left(\boldsymbol{p}'\right)^{*}\right\rangle = \sum_{l,l'=0}^{\infty}\sum_{m,m'=-l}^{l}\left\langle a_{lm}a_{l'm'}\right\rangle Y_{l}^{m}\left(\boldsymbol{p}\right)Y_{l'}^{m'}\left(\boldsymbol{p}'\right) \overset{(6)}{=} \sum_{l=0}^{\infty}c_{l}\sum_{m=-l}^{l}Y_{l}^{m}\left(\boldsymbol{p}\right)Y_{l}^{m}\left(\boldsymbol{p}'\right) \tag{18}$$
$$(*)\,\mathcal{P}_{l}\left(\vec{x}\cdot\vec{y}\right) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{l}^{m}\left(\vec{x}\right)Y_{l}^{m}\left(\vec{y}\right)$$
$$\Rightarrow C\left(cos\gamma\right) \overset{(*)}{=} \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}c_{l}\mathcal{P}_{l}\left(cos\gamma\right)$$

$$(**)f_{n}\left(x\right) = \sum_{l=0}^{\infty}a_{l}\mathcal{P}_{l}\left(x\right) \rightarrow a_{l} = \frac{2l+1}{2}\int_{-1}^{1}f\left(x\right)\mathcal{P}_{l}\left(x\right)dx$$
$$(20)+(**):\;\frac{2l+1}{4\pi}c_{l} = \frac{2l+1}{2}\int_{-1}^{1}C\left(x\right)\mathcal{P}_{l}\left(x\right)dx \tag{19}$$
$$\Rightarrow c_{l} = 2\pi\int_{-1}^{1}f\left(\tilde{\gamma}\right)\mathcal{P}_{l}\left(\tilde{\gamma}\right)d\tilde{\gamma}$$

$$f^{*} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}^{*}Y_{l}^{m*} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}^{*}\left(-1\right)^{-m}Y_{l}^{-m} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{l-m}^{*}\left(-1\right)^{m}Y_{l}^{m} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}Y_{l}^{m} \tag{20}$$
$$\Rightarrow \left(-1\right)^{m}a_{l-m} = a_{lm}^{*}$$

Additional figures:

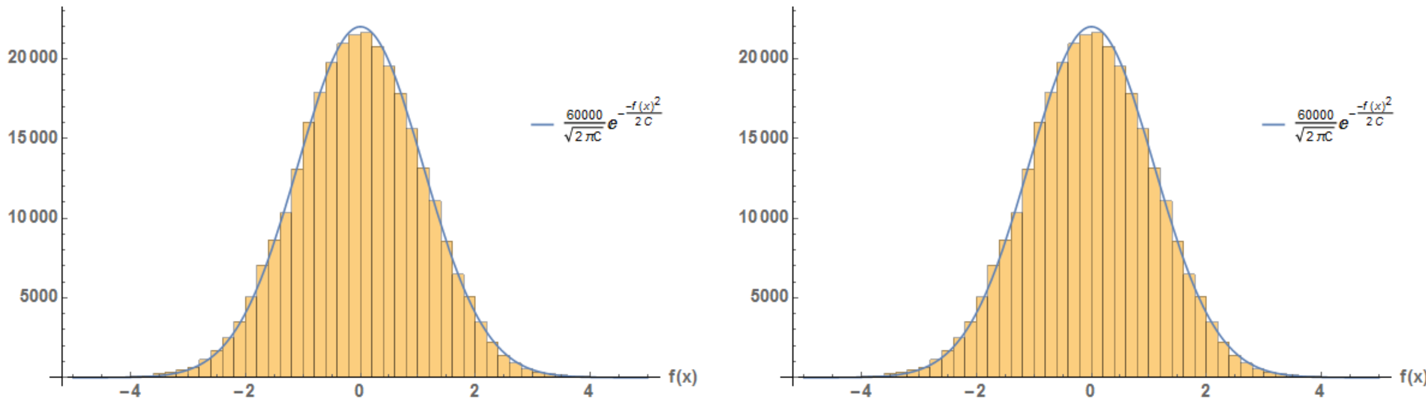

Figure 4: These histograms represent a single points distribution for 300,000 random samples. Each histogram is built from one thousand random functions $f(x)$, each function has 300 points (different points were taken for each historgam) that had been randomly chosen using a uniform distribution. These random functions were generated by "randHS" using $c_l = \{2.784, 1.998, 0.95, 0.191\}$. The blue line is a gaussian distribution with mean value zero and variance C that was calculated using (7), for this case it is 1.18315.

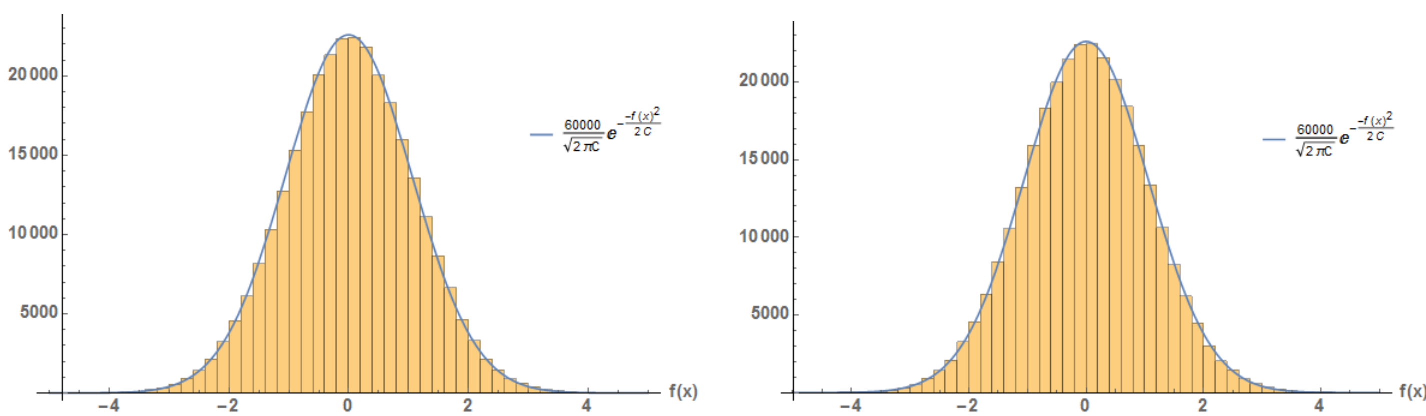

Figure 5: These histograms represent a single points distribution for 300,000 random samples. Each histogram is built from one thousand random functions $f(x)$, each function has 300 points (different points were taken for each historgam) that had been randomly chosen using a uniform distribution. These random functions were generated by "randHS" using $c_l = \{0.955, 0.767, 0.564, 0.387, 0.246, 0.144, 0.074, 0.03, 0.005\}$. The blue line is a Gaussian distribution with mean value zero and variance C that was calculated using (7), for this case it is 1.12168.

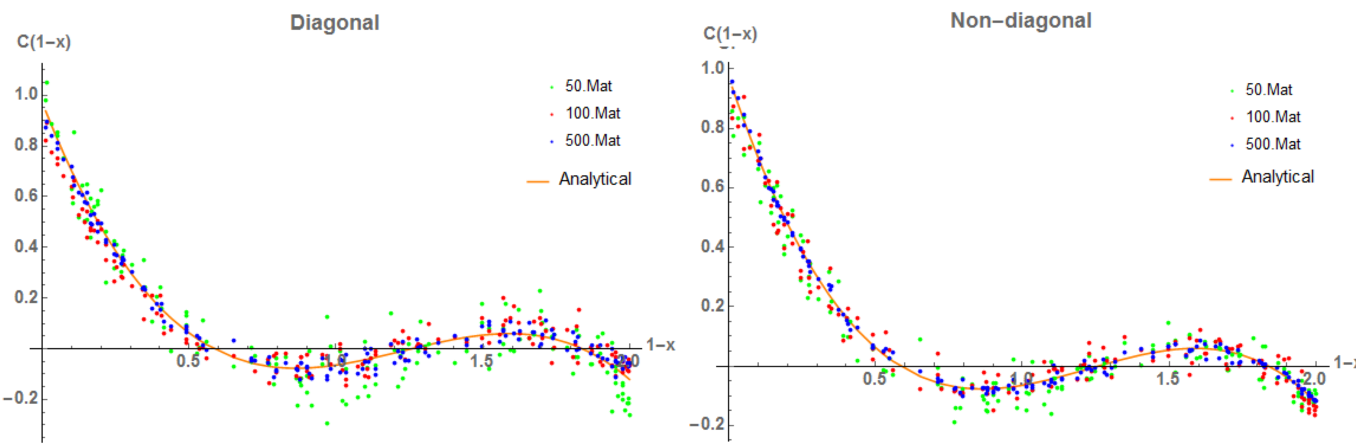

Figure 6: Comparing the analytical RS correlation (line plot) to the diagonal (12) and off-diagonal (13) part of $\mathcal{H}$. $\mathcal{H}$ is a 3×3 sized matrix and was created using $c_l = \{1.245, 1.088, 0.82, 0.516\}$, each matrix has 300 points that had been randomly chosen using a uniform distribution, a total of 150 pairs $(\boldsymbol{p}, \boldsymbol{p}')$. Green dots represent the average of fifty matrices per point, red represents one hundred and blue five hundred. The analytical RS correlation is calculated using (7).

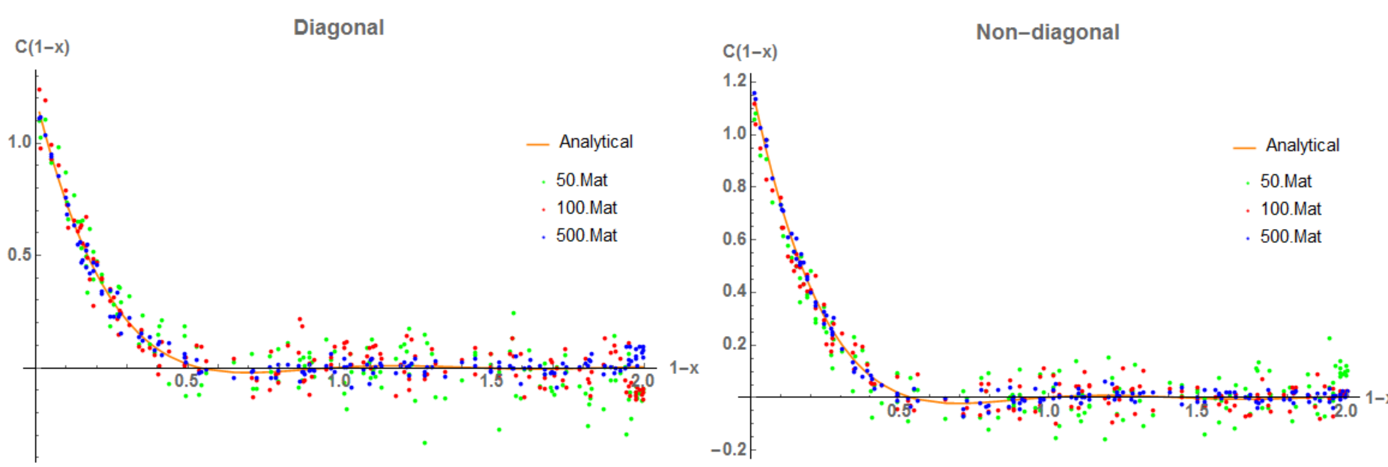

Figure 7: Comparing the analytical RS correlation (line plot) to the diagonal (12) and off-diagonal (13) part of $\mathcal{H}$. $\mathcal{H}$ is a 3×3 sized matrix and was created using $c_l = \{1.245, 1.088, 0.82, 0.516, 0.247, 0.059\}$ (two more terms more taken than in Figure 6), each matrix has 300 points that had been randomly chosen using a uniform distribution, a total of 150 pairs $(\boldsymbol{p}, \boldsymbol{p}')$. Green dots represent the average of fifty matrices per point, red represents one hundred and blue five hundred. The analytical RS correlation is calculated using (7).

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