Mansoura University
Faculty of Computers and Information
Department of Information System

[IS311T] Information Theory

Grade: 3rd grade (IS)

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Outline

- Sample spaces
- Events
- Conditional probability
- Total probability
- Tree diagram
- Bayes' rule
- Random variable
- Expected value & variance

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Sample space

- **A sample space (S)**: The set of all possible outcomes to an experiment.

  Example: Throw a dice, $S = \{1, 2, 3, 4, 5, 6\}$. In this case, each outcome is equally likely (p(any outcome) = $\frac{1}{6}$).

- **A probability measure (P)** on a sample space S: A rule that assigns to each outcome $s \in S$ a probability $p(s) \geq 0$ satisfying:

$$\sum_{s \in S} p(s) = 1$$

  Example: Throw two dice, count the total, $S = \{2, 3, \dots, 12\}$. In this case, the outcomes are not equally likely.

S	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
p(s)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Events

- **An event (E):** A subset of sample space S ($E \subseteq S$).

  The probability of the event E is:

$$p(E) = \sum_{s \in E} p(s)$$

Example: Roll a dice. Let E be the event of getting an odd number.

E = {1, 3, 5}.

$p(E) = p(1) + P(3) + P(5) = \frac{3}{6} = \frac{1}{2}$

Example: Roll two dice. Let E be the event of getting an even total.

$$p(E) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{18}{36} = \frac{1}{2}$$

Combination of events and independent events

$$\overline{E} = \{s \in S | s \notin E\} \quad p(\overline{E}) = 1 - p(E)$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

- If A, B are mutually exclusive, then $p(A \cup B) = p(A) + p(B)$
- A, B are independent events if knowing one gives no information about the probability of the other.

$$p(A \cap B) = p(A)p(B)$$

- If A, B are independent events, then

i. $A \ and \ \bar{B}$ are independent

ii. $\bar{A} \ and \ B$ are independent

iii. $\bar{A} \ and \ \bar{B}$ are independent

Independent events

Proof of (i)

Suppose A, B are independent, then

$$p(A \cap B) = p(A)p(B)$$

$$A = (A \cap B) \cup (A \cap \bar{B})$$

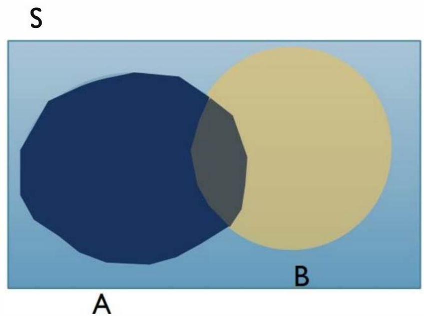
$$p(A) = p(A \cap B) + p(A \cap \bar{B})$$

$$p(A \cap \bar{B}) = p(A) - p(A \cap B)$$

$$p(A \cap \bar{B}) = p(A) - p(A)p(B)$$

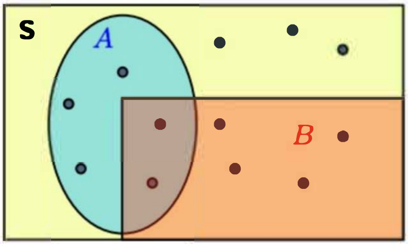
$$p(A \cap \bar{B}) = p(A)(1 - p(B))$$

$$p(A \cap \bar{B}) = p(A)p(\bar{B})$$



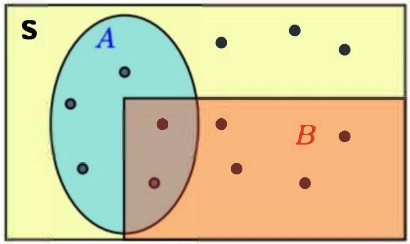
Conditional probabilities

Assume 12 equally likely outcomes



$$p(A) = \frac{5}{12} \quad p(B) = \frac{6}{12}$$

If B occurred



$$p(A|B) = \frac{2}{6} \quad p(B|B) = 1$$

Conditional probabilities

- If A, B are events, then the conditional probability of A given B is

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

- The probability of A if you know that you are in B. Think of B as the new sample space.
- Conditional probability is not symmetric

$$p(A|B) \neq p(B|A)$$

- If A, B are independent, then

$$p(A|B) = p(A)$$

Total probability

- The collection of events $A_1, A_2, \dots, A_n$ is said to partition a sample space S if

  (a) $A_1 \cup A_2, \cup \cdots, \cup A_n = S$

  (b) $A_i \cap A_j = \phi \; \forall \, i, j$

  (c) $A_i \neq \phi \; \forall i$

- In essence, a partition is a collection of non-empty, non-overlapping subsets of a sample space whose union is the sample space itself.
- Suppose B is an event in S

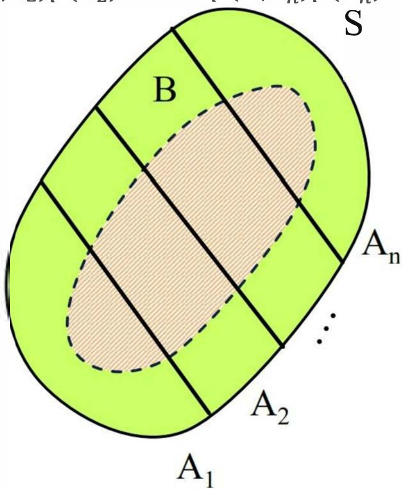
$$\mathrm{p}(B) = \sum_{i=1}^{n} p(B|A_i)p(A_i)$$

Total probability

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$$

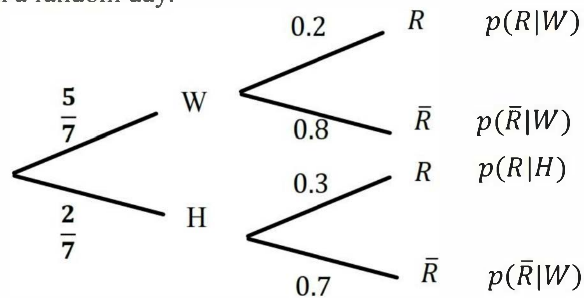
$$p(B) = p(B \cap A_1) + \mathrm{p}(B \cap A_2) + \cdots + (pB \cap A_n)$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \cdots + p(B|A_n)p(A_n)$$



Total probability

- **Example**: Does it rain more on weekends? After a year of observation, we find probability of rain (R) on a workday (W) is 0.2, and probability of rain on a weekend day (H) is 0.3. What is the overall probability of rain on a random day.



$$p(R) = p(R|W)p(W) + p(R|H)p(H)$$

$$p(R) = 0.2 \times \frac{5}{7} + 0.3 \times \frac{2}{7} = 0.2286$$

Bayes' Rule

- Referring to the same figure of total probability

$$p(A_j|B) = \frac{p(B|A_j)p(A_j)}{p(B)}$$

$$p(A_j|B) = \frac{p(B|A_j)p(A_j)}{\sum_{i=1}^{n} p(B|A_i)p(A_i)}$$

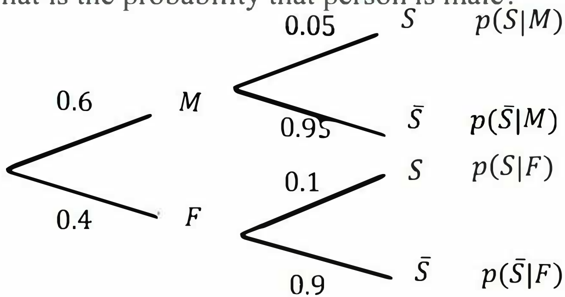
- In the previous example: Suppose on a certain day it rains. What is the probability that this day was a weekend (H) day?

$$p(H|R) = \frac{p(R|H)p(H)}{p(R)}$$

$$p(H|R) = \frac{0.3 \times \frac{2}{7}}{0.2286} = 0.375$$

Bayes' Rule

Example: The probability of having a side effect for a drug is 10% for females. The probability of not having the side effect for males is 95%. 40% of the study was females. Given a person having a side effect what is the probability that person is male?



$$p(M|S) = \frac{p(S|M)p(M)}{p(S|M)p(M) + p(S|F)p(F)}$$

$$p(D|P) = \frac{0.05 \times 0.6}{0.05 \times 0.6 + 0.1 \times 0.4} = 0.4615$$

Random variable

- **Definition**: A random variable (X) is a function from sample space to a number system ($\mathcal{R}$). $X: S \to \mathcal{R}$
- Examples: Rolling a dice. $S = \{1, 2, 3, 4, 5, 6\}$.
  - $X(s) = s$

    $X(1) = 1, X(2) = 2, \dots, X(6) = 6$
  - $X(s)$ = no. of letters in English word for s.

    $X(1) = X(2) = X(6) = 3,$

    $X(4) = X(5) = 4, X(3) = 5$

x	3	4	5
p(x)	1/2	1/3	1/6

- For every random variable (X), there are two important values associated with its probability distribution. These numbers are:
  1. Expected value of X ($E(X)$) or the mean (μ)
  2. Variance of X ($Var(X)$)

Random variable

- Expectation: when the experiment is repeated many times, the expected value is the average number that is obtained. The value where the probability distribution balances (center of mass)

$$\mu = E(X) = \sum xp(x)$$

- Variance: measures the spread out of the values from mean

$$\sigma^2 = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$$

- Examples:
  - $E(X) = 3 \times \frac{1}{2} + 4 \times \frac{1}{3} + 5 \times \frac{1}{6} = \frac{11}{3}$
  - $Var(X) = \left(\frac{2}{3}\right)^2 \times \frac{1}{2} + \left(\frac{1}{3}\right)^2 \times \frac{1}{3} + \left(\frac{4}{3}\right)^2 \times \frac{1}{6} = \frac{5}{9}$

THANK YOU