ص ۱

hw1 کنترل صنعتی

۸۱۰۱۹۹۵۷۰ شیرین جمشیدی

۱)

الف)

نقطه تعادل : $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\dot{x}_1 = 0 \rightarrow \boxed{x_2 = 0}$ (I) ; $\dot{x}_2 = 0 \rightarrow -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} = 0 \xrightarrow{\text{(I)}} \boxed{x_1 + \sqrt{u+1} = 0} \rightarrow$ $\boxed{\text{نقطه تعادل: } x_0 = \begin{pmatrix} -\sqrt{u+1} \\ 0 \end{pmatrix}}$ (الف)

ب)

$\dot{X}_{(t)} = AX_{(t)} + BU_{(t)}$; $f_1 = \dot{x}_1$, $f_2 = \dot{x}_2$, $x_0 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$$A = \frac{\partial f}{\partial x}\Big|_{x_0, u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 + \frac{2x_2^4}{x_1^3} & -\frac{4x_2^3}{x_1^2} \end{bmatrix}_{x_0, u_0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\Big|_{x_0, u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}(u+1)^{-1/2} \end{bmatrix}_{x_0, u_0} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}$$

$$\boxed{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}}_{B} u}$$ (ب)

; $y = CX + Du \rightarrow \boxed{y = \underbrace{(-4 \quad 0)}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{6}_{D} u}$ (ب)

$$C = \frac{\partial y}{\partial x}\Big|_{x_0, u_0} = \left(\frac{\partial y}{\partial x_1} \quad \frac{\partial y}{\partial x_2}\right) = (2x_1 \quad 0)_{x_0, u_0} = (-4 \quad 0) \quad , \quad D = \frac{\partial y}{\partial u}\Big|_{x_0, u_0} = 2u = 6$$

2)

$\xrightarrow{\mathcal{L}} sX_{(s)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X_{(s)} + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} u$; $G_{(s)} = C(sI - A)^{-1}B + D = \frac{Y_{(s)}}{U_{(s)}}$

$Y_{(s)} = (-4 \quad 0) X_{(s)} + 6u$

$G_{(s)} = (-4 \quad 0)\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} + 6$; $\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2-1}\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}$

$$G_{(s)} = \frac{1}{s^2-1}(-4 \quad 0)\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} + 6 = \frac{1}{s^2-1}(-4s \quad -4)\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} + 6$$

$$= \frac{-1}{s^2-1} + 6 = \frac{6s^2-7}{s^2-1} \longrightarrow \boxed{G_{(s)} = \frac{6s^2-7}{s^2-1}}$$ (2)

(حل ۲)

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الف)

(الف)

$$\dddot{y}+12\ddot{y}+2\dot{y}+16y=2u$$

$$x_1=y\ ,\ x_2=\dot{x}_1=\dot{y}\ ,\ x_3=\dot{x}_2=\ddot{y}\ ,\ \dot{x}_3=\dddot{y}=2u-12x_3-2x_2-16x_1 \Rightarrow \boxed{\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \dot{x}_3=-12x_3-2x_2-16x_1+2u\\ y=x_1\end{cases}}$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\underbrace{\begin{pmatrix}0&1&0\\0&0&1\\-16&-2&-12\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\underbrace{\begin{pmatrix}0\\0\\2\end{pmatrix}}_{B}u\ ,\quad y=\underbrace{\begin{pmatrix}1&0&0\end{pmatrix}}_{C}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\underbrace{0}_{D}u$$

ب)

$$G_{(s)}=C(sI-A)^{-1}B+D\ ;\ (sI-A)^{-1}=\begin{pmatrix}s&-1&0\\0&s&-1\\16&2&s+12\end{pmatrix}^{-1}=\frac{1}{s^3+12s^2+2s+16}\begin{pmatrix}s^2+12s+2&s+12&1\\-16&s^2+12s&s\\-16s&-2s-16&s^2\end{pmatrix}$$

$$G_{(s)}=\frac{1}{|sI-A|}\begin{pmatrix}s^2+12s+2&s+12&1\end{pmatrix}\begin{pmatrix}0\\0\\2\end{pmatrix}\Rightarrow\boxed{G_{(s)}=\frac{2}{s^3+12s^2+2s+16}}$$

(ب)

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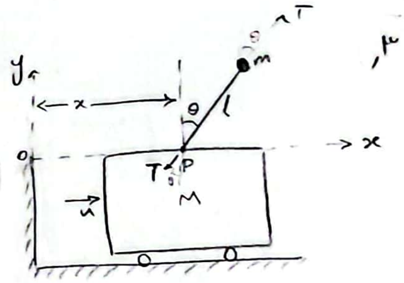

الف)

$$\begin{cases}T\sin\theta=m\ddot{x}+\underbrace{l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta}_{\frac{d^2}{dt^2}(l\sin\theta)}\\ u-T\sin\theta=M\ddot{x}\\ T\cos\theta-mg=-l\ddot{\theta}\sin\theta-l\dot{\theta}^2\cos\theta\end{cases}\quad \boxed{u=(M+m)\ddot{x}+l(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)}$$

$$\boxed{-l\ddot{\theta}\sin\theta-l\dot{\theta}^2\cos\theta=(u-M\ddot{x})\cot\theta-mg}$$

ب)

$$z_1=x\ ,\ x_1=\dot{x}\rightarrow\dot{x}_1=\ddot{x}\ ,\ \underbrace{x_2=\theta}_{y_2\equiv\text{خروجی}},\ x_3=\dot{\theta}\rightarrow\dot{x}_3=\ddot{\theta}\qquad\begin{pmatrix}z_1\\x_1\\x_2\\x_3\end{pmatrix}$$

$$\left.\begin{aligned}&T\cos x_2-mg=-l\dot{x}_3\sin x_2-lx_3^2\cos x_2\\ &u-T\sin x_2=M\dot{x}_1\rightarrow T=\frac{u-M\dot{x}_1}{\sin x_2}\end{aligned}\right\}\rightarrow(u-M\dot{x}_1)\cot x_2-mg=-l\dot{x}_3\sin x_2-lx_3^2\cos x_2$$

$$T\sin x_2=m\dot{x}_1+l\dot{x}_3\cos x_2-lx_3^2\sin x_2$$

$$\begin{cases}(u-M\dot{x}_1)\cot x_2-mg=-l\dot{x}_3\sin x_2-lx_3^2\cos x_2\\ u=(M+m)\dot{x}_1+l\dot{x}_3\cos x_2-lx_3^2\sin x_2\end{cases}$$

$$u-\frac{M}{M+m}(u-l\dot{x}_3\cos x_2+lx_3^2\sin x_2)\cot x_2-mg=-l\dot{x}_3\sin x_2-lx_3^2\cos x_2$$

$$\rightarrow u-\frac{M}{M+m}(u+lx_3^2\sin x_2)\cot x_2-mg+lx_3^2\cos x_2=-\left(l\sin x_2+l\frac{M}{M+m}\frac{\cos^2x_2}{\sin x_2}\right)\dot{x}_3$$

$$\boxed{\dot{x}_3=\frac{\frac{M}{M+m}(u+lx_3^2\sin x_2)\cot x_2+mg-lx_3^2\cos x_2-u}{l\sin x_2+l\frac{M}{m+M}\cot x_2\cos x_2}}\qquad\boxed{\dot{z}_1=x_1\ ,\ \dot{x}_2=x_3}$$

ص ۳

ادامه ۳،

$$\dot{x}_1 = \frac{u - l\dot{x}_3\cos x_2 + l x_3^2 \sin x_2}{M+m} \qquad \boxed{\dot{x}_1 = \frac{u}{M+m} + \frac{\frac{M}{M+m}(u + l x_3^2 \sin x_2)\cot x_2 + mg - l x_3^2 \cos x_2 - u}{(M+m)\tan x_2 + M\cot x_2} + \frac{l x_3^2 \sin x_2}{M+m}}$$

ج)

$\theta = x = 0 \;,\; \dot{\theta} = \dot{x} = 0 \longrightarrow$ نقطه‌ی تعادل مد نظر، شرایط با $u=0$ می‌باشد.

د) $f_1 = \dot{z}_1 \;,\; f_2 = \dot{x}_1 \;,\; f_3 = \dot{x}_2 \;,\; f_4 = \dot{x}_3$

$$\dot{X} = AX + BU \;,\; A = \frac{\partial f}{\partial x}\Big|_{x_0, u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial z_1} & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \\ \frac{\partial f_4}{\partial z_1} & \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} \end{bmatrix}_{x_0, u_0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{bmatrix}$$

$$\begin{cases} a = \frac{\partial f_2}{\partial x_2}\Big|_{x_0,u_0} = \frac{-m}{M} g \;,\; d = \frac{\partial f_4}{\partial x_3}\Big|_{x_0,u_0} = 0 \\ b = \frac{\partial f_2}{\partial x_3}\Big|_{x_0,u_0} = 0 \;,\; c = \frac{\partial f_4}{\partial x_2}\Big|_{x_0,u_0} = mg \end{cases}$$

$$B = \frac{\partial f}{\partial u}\Big|_{x_0,u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_4}{\partial u} \end{bmatrix}_{x_0,u_0} = \begin{bmatrix} 0 \\ \frac{1}{M+m} + \frac{\frac{M}{M+m}\cot x_2 - 1}{(M+m)\tan x_2 + M\cot x_2} \\ 0 \\ \frac{\frac{M}{M+m}\cot x_2 - 1}{l\sin x_2 + l\frac{M}{M+m}\cot x_2 \cos x_2} \end{bmatrix}_{x_0,u_0} = \begin{bmatrix} 0 \\ \frac{1}{M+m} + \frac{\frac{M}{M+m}}{M} \\ 0 \\ \frac{\frac{M}{m+M}}{l\frac{M}{m+M}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{M+m} \\ 0 \\ \frac{1}{l} \end{bmatrix}$$

(ادامه در پایین صفحه)

۲، (الف)

الف)

نقطه تعادل
$$\begin{cases} \dot{x} = 0 \rightarrow 6x - 2x^2 - xy^2 = 0 \rightarrow x = 0 \text{ or } 6 - 2x - y^2 = 0 \\ \dot{y} = 0 \rightarrow 6y - xy - y^2 = 0 \rightarrow y = 0 \text{ or } 6 - x - y = 0 \end{cases} \longrightarrow \boxed{\text{نقاط تعادل: } \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix}}$$

ب)

$$\dot{Z} = AZ + \overset{\text{ندارد}}{BU} \longrightarrow A = \frac{\partial f}{\partial z}\Big|_{z_0,u_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}_{z_0,u_0} = \begin{bmatrix} 6 - 4x - y^2 & -2xy \\ -y & 6 - x - 2y \end{bmatrix}_{z_0,u_0} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

در حول $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(ب)

$$\boxed{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

ادامه ۳ ج)

$$y = \overbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}}^{C} \overbrace{\begin{pmatrix} z_1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{X} + \overset{D}{0}\, u \qquad G(s) = C(sI - A)^{-1}B + D$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & \frac{m}{M}g & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & -mg & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{mg}{mgMs - Ms^3} & \frac{mg}{mgMs^2 - Ms^4} \\ 0 & \frac{1}{s} & \frac{mg}{mgM - Ms^2} & \frac{mg}{mgMs - Ms^3} \\ 0 & 0 & \frac{s}{s^2 - mg} & \frac{1}{s^2 - mg} \\ 0 & 0 & -\frac{mg}{mg - s^2} & \frac{s}{s^2 - mg} \end{bmatrix}$$

(ج)

$$G(s) = \left(\frac{1}{s} \quad \frac{1}{s^2} \quad \left(\frac{mg}{mgMs - Ms^3} + \frac{s}{s^2 - mg}\right) \quad \left(\frac{mg}{mgMs^2 - Ms^4} + \frac{s}{s^2 - mg}\right)\right) \begin{pmatrix} 0 \\ \frac{2}{M+m} \\ 0 \\ \frac{1}{l} \end{pmatrix} = \boxed{\frac{2}{s^2(M+m)} + \frac{1}{l}\left(\frac{mg}{mgMs^2 - Ms^4} + \frac{s}{s^2 - mg}\right) = G(s)}$$

ص ④

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(الف)

1. $q_{i1} - q_{o1} = A_1 \dot{h}_1 \;\rightarrow\; u_1 - k_1\sqrt{h_1} = A_1 \dot{h}_1$

2. $q_{o1} + q_{i2} - q_{o2} = A_2 \dot{h}_2 \rightarrow k_1\sqrt{h_1} + u_2 - k_2\sqrt{h_2} = A_2 \dot{h}_2 \;\rightarrow$

(الف)

$$\boxed{\begin{cases} \dot{h}_1 = -\dfrac{k_1}{A_1}\sqrt{h_1} + \dfrac{u_1}{A_1} \\[2mm] \dot{h}_2 = -\dfrac{k_2}{A_2}\sqrt{h_2} + \dfrac{k_1}{A_2}\sqrt{h_1} + \dfrac{u_2}{A_2} \\[2mm] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \end{cases}}$$

ب) $f_1 = \dot{h}_1 \,,\; f_2 = \dot{h}_2$

$$\dot{H} = AH + BU \rightarrow A = \left.\frac{\partial f}{\partial h}\right|_{\substack{h_s \\ q_s}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial h_1} & \dfrac{\partial f_1}{\partial h_2} \\[2mm] \dfrac{\partial f_2}{\partial h_1} & \dfrac{\partial f_2}{\partial h_2} \end{bmatrix}_{\substack{h_s \\ q_s}} = \begin{bmatrix} -\dfrac{k_1}{2A_1}(h_1)^{-1/2} & 0 \\[2mm] \dfrac{k_1}{2A_2}(h_1)^{-1/2} & -\dfrac{k_2}{2A_2}(h_2)^{-1/2} \end{bmatrix}_{\substack{h_s \\ q_s}}$$

$$= \begin{bmatrix} -\dfrac{k_1}{2A_1\sqrt{h_{1s}}} & 0 \\[2mm] \dfrac{k_1}{2A_2\sqrt{h_{1s}}} & -\dfrac{k_2}{2A_2\sqrt{h_{2s}}} \end{bmatrix} \;;\; B = \left.\frac{\partial f}{\partial u}\right|_{\substack{h_s \\ q_s}} = \begin{bmatrix} \dfrac{1}{A_1} & 0 \\[2mm] 0 & \dfrac{1}{A_2} \end{bmatrix}$$

$$y = CH + DU \rightarrow C = \left.\frac{\partial y}{\partial h}\right|_{\substack{h_s \\ q_s}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,\; D = \left.\frac{\partial y}{\partial u}\right|_{\substack{h_s \\ q_s}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(ب)

$$\Rightarrow \boxed{\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} \dfrac{-k_1}{2A_1}(h_{1s})^{-1/2} & 0 \\[2mm] \dfrac{k_1}{2A_2}(h_{1s})^{-1/2} & \dfrac{-k_2}{2A_2}(h_{2s})^{-1/2} \end{bmatrix}}_{A} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \underbrace{\begin{bmatrix} \dfrac{1}{A_1} & 0 \\[2mm] 0 & \dfrac{1}{A_2} \end{bmatrix}}_{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}} \;;\; \boxed{y = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}$$

ج) $G_{(s)} = C(sI - A)^{-1}B + D^{\circ} \;\rightsquigarrow\; (sI-A)^{-1} = \begin{bmatrix} s + \dfrac{k_1}{2A_1}(h_{1s})^{-1/2} & 0 \\[2mm] \dfrac{-k_1}{2A_2}(h_{1s})^{-1/2} & s + \dfrac{k_2}{2A_2}(h_{2s})^{-1/2} \end{bmatrix}^{-1}$

$$= \frac{1}{\left(s + \dfrac{k_1}{2A_1\sqrt{h_{1s}}}\right)\left(s + \dfrac{k_2}{2A_2\sqrt{h_{2s}}}\right)} \begin{bmatrix} s + \dfrac{k_2}{2A_2\sqrt{h_{2s}}} & 0 \\[2mm] \dfrac{k_1}{2A_2\sqrt{h_{1s}}} & s + \dfrac{k_1}{2A_1\sqrt{h_{1s}}} \end{bmatrix} \rightsquigarrow \overset{I}{C}(sI-A)^{-1} = (sI-A)^{-1}$$

(ج)

$$(sI-A)^{-1}B = \boxed{\frac{4A_1A_2}{\left(2sA_1 + \dfrac{k_1}{\sqrt{h_{1s}}}\right)\left(2sA_2 + \dfrac{k_2}{\sqrt{h_{2s}}}\right)} \begin{bmatrix} \dfrac{s}{A_1} + \dfrac{k_2}{2A_1A_2\sqrt{h_{2s}}} & 0 \\[2mm] \dfrac{k_1}{2A_2^2\sqrt{h_{2s}}} & \dfrac{s}{A_2} + \dfrac{k_1}{2A_1A_2\sqrt{h_{1s}}} \end{bmatrix} = G_{(s)}}$$