Q1. Squared BGM :

S_t : geometric Brownian Motion w $m = \sigma = 1$, $S_0 = 1$.

find $E[S_2^2]$: $dS_t = S_t(m\,dt + \sigma\,dB_t) = S_t(dt + dB_t)$

$\rightarrow S_t = S_0\, e^{((1-\frac{1}{2})t + B_t)} = e^{t/2 + B_t}$

$$E(S_2^2) = E\left((e^{1+B_2})^2\right) = E\left(e^{2+2B_2}\right) \quad : B_2 \sim N(0,2)$$

$$= \int_{-\infty}^{\infty} \frac{e^{2+2x} \cdot e^{-\frac{1}{2}\frac{x^2}{2}}}{\sqrt{2 \cdot 2\pi}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-x^2/4 + 2x + 2}}{\sqrt{\pi}}$$

$$= \frac{e^6}{2}\int_{-\infty}^{\infty} \frac{e^{-(\frac{x}{2}-2)^2}}{\sqrt{\pi}}\,dx = \frac{e^6}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{x-4}{2}\right)^2}}{\sqrt{2\pi}}\,dx$$

$$x = \sqrt{2}t \quad : \quad \frac{e^6}{\cancel{\sqrt{2}}} \int_{-\infty}^{\infty} \frac{e^{-(t-2\sqrt{2})^2/2}}{\sqrt{2\pi}} \cdot \cancel{\sqrt{2}}\,dt$$

Normal distn w $m = 2\sqrt{2}$ $\Rightarrow$ 1

$\therefore$ e^6 remains

Q2: Make your Martingale

$X_1, X_2, \ldots \sim Exp(\lambda)$ IID.

$M_0 = 1$, $M_n = M_{n-1} \cdot \frac{1}{2} e^{\lambda/2 X_n}$. $F_n = \sigma(X_1, \ldots X_n)$

$0 < a < p < b$ for some real values a & b,

$\{M_n^p\}_{n \geq 0}$ is a sub-martingale. find a & b.

$\rightarrow$

$$E(M_t^p \mid F_s) \geq M_s^p \qquad t, s \in \mathbb{N}$$

$$M_t = M_s \times \prod_{i=s+1}^{t} \left(\tfrac{1}{2} e^{\lambda/2\, X_i}\right) = M_s \left(\tfrac{1}{2}\right)^{t-s} e^{\lambda/2 \left(\sum_{i=s+1}^{t} X_i\right)}$$

$$E(M_t^p \mid F_s) = \frac{M_s^p}{2^{p(t-s)}} E\left(e^{p\lambda/2 (\sum X_i)}\right) \geq M_s^p \qquad \text{(All } X_i \text{ indep of } F_s\text{)}$$

$$\therefore \underbrace{E\left(e^{p\lambda/2 \cdot \left(\sum_{i=s+1}^{t} X_i\right)}\right)}_{} \geq 2^{p(t-s)}$$

$(t-s)$ $p(t-s)$

$$= E\left(e^{pX/2 \cdot t}\right) \geq 2$$

$$\int_0^\infty e^{pX/2 x} \lambda e^{-\lambda x} dx \geq 2^p$$

$$= \lambda \int_0^\infty e^{\lambda x (p/2 - 1)} dx \geq 2^p$$

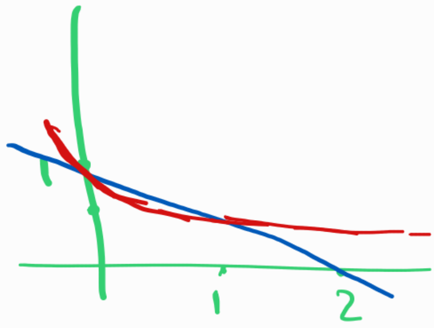
if $p < 2$ $\qquad$ $p \geq 2$ not possible

$$= \frac{1}{1 - p/2} \geq 2^p$$

$$\Rightarrow \quad 2^{-p} \geq 1 - \frac{p}{2}$$

$p \geq 0 \Rightarrow$
if $0 \leq p \leq 1$
$2^{-p} < 1 - \frac{p}{2}$

$\therefore p \in (1, 2)$



$\therefore a + b = 3$

Q3. W_t, X_t s.t $\frac{1}{X_t}$ satisfies $d\left(\frac{1}{X_t}\right) = \frac{1}{X_t}(2dt - dW_t)$

$dX_t = X_t(a\,dt + b\,dW_t)$ find a & b.

$\rightarrow \frac{1}{X_t} \sim$ geometric BM $\qquad \frac{1}{X_t} = \frac{1}{X_0} e^{\left(\left(2 - \frac{1}{2}\right)t - B_t\right)}$

$$X_t = X_0 \cdot e^{-\left(\left(2 - 1/2\right)t - B_t\right)}$$

$$= X_0\, e^{\left(-\frac{3}{2}t + B_t\right)} \qquad \therefore b = 1$$

$$= X_0\, e^{\left(\left(-1 - \frac{1}{2}\right)t + B_t\right)}$$

$$\therefore dX_t = X_t(-dt + dW_t)$$

Q4. S_t : $dS_t = 3S_t dt + 4S_t dW_t$

find a, b where $X_t = \ln(S_t)$ & $dX_t = a\,dt + b\,dW_t$

M1 : $S_t \sim$ Geo.BM $\Rightarrow$ $S_t = S_0 e^{(-5t + 4W_t)}$

$\rightarrow$ $X_t = \ln(S_0) - 5t + 4W_t$

$dX_t = -5dt + 4dW_t$

M2 (from fundamental principles): $f(s) = \ln(s)$. $f'(s) = \frac{1}{s}$, $f''(s) = -\frac{1}{s^2}$

$dX_t = df(S_t) = f'(S_t)dS_t + \frac{1}{2}f''(S_t)\, d[S,S]_t$.

$f'(S_t) = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}(16 S_t^2 dt)$

$= 3dt - 8dt + 4dW_t = -5dt + 4dW_t$

Q5. W_t std Brownian Motion, $T_4 = \inf(t > 0 : |W_t| > 4)$

$Var(T_4) = ?$

① $E(T_4)$: using Martingales? :

W_t is a martingale ✓ →

T_4: stopping time mentioned (hitting −4 or 4)

is $W_t^2 - t$ a martingale?

$E(W_t^2 - t \mid F_s)$: $W_t^2 = (W_s + W_{t-s})^2$

$= W_s^2 + 2W_s \cdot W_{t-s} + W_{t-s}^2$

$\therefore E(W_t^2 - t \mid F_s) =$

$E(W_s^2 + 2W_s.\cancel{W_{t-s}} + W_{t-s}^2 - s - (t-s))$

$= W_s^2 - s + \underbrace{E(W_{t-s}^2) - (t-s)}_{0}$

H.P

$\rightarrow$ Now, $W_s^2 - s$ is bounded until stopping time

$\therefore E(W_T^2 - T) = E(W_0^2 - 0) = 0$

$\therefore E(W_T^2) = E(T) = 16$.

$\rightarrow$ $df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$

$M_t = f(B_t) - f(B_0) - \frac{1}{2}\int_0^t f''(B_s)\,ds$

B_t^2: $B_t^2 - 0 - \frac{1}{2}\int_0^t 2ds = B_t^2 - t$

[illegible] e.g. $3t + B_t$ is a martingale

Bt^3 : $Bt^3 - 0 - \frac{1}{2}\int_0 6B_s ds \rightarrow B_t^3 - 3\int ...$

Bt^4 : $B_t^4 - \frac{1}{2}\int_0^t 12B_s^2\, ds \;= \underline{B_t^4 - 6B_t^2 t + 3t^2}$

$d(tB_t^2) = 2tB_t\, dB_t + (B_t^2 + t)\,dt$

$\int B_s^2\, ds = tB_t^2 - \frac{t^2}{2} - 2\underbrace{\int_0^t sB_s\, dB_s}_{}$ stochastic integral (martingale)

so $E(\underset{\sim}{W_T^4} - 6W_T^2 T + 3T^2) = 0$

$4^4 - 6\cdot 4^2 E(T) + 3E(T^2) = 0$

$\Rightarrow 4^4 - 6\cdot 4^2 \cdot 4^2 + 3E(T^2) = 0$

$\frac{5\cdot 4^4}{3} = E(T^2)$ $\quad \therefore E(T^2) - E(T)^2 =$

$\frac{5\cdot 4^4}{3} - 4^4 = \frac{2\cdot 4^4}{3} = \frac{512}{3}$

Q6: $1^{1000} + \dots + n^{1000}$ is poly in n w leading term an^{1001}.

Find a.

$\sum_{i=1}^{n} i^r$ $\qquad \sum i^2 = \frac{n(n+1)(2n+1)}{6} \sim \frac{1}{3} \qquad \sum i \sim \frac{1}{2}$

$\sum i^3$? :

$f(n) = \sum_{i=1}^{n} i^{1000}$ $\qquad f(n) = a_0 x^{1001} + a_1 x^{1000} + a_2 x^{999} + \dots$

$f(n) - f(n-1) = n^{1000}$

$= a(n^{1001} - (n-1)^{1001}) + a_1(n^{1000} - (n-1)^{1000}) + \dots \rightarrow 0$

$a(1001\, n^{1000}) = n^{1000}$ $\qquad a = \frac{1}{1001}$ $\qquad$ not gonna contribute to n^{1000}

Q7. $A: 3\times n$ for some $n \geq 2$, $Ax = e_1$ has no soln, $Ax = e_2$ has a unique soln. Find n.

$A = [A_1, A_2, \dots A_n]$ $\qquad \sum x_i A_i = e_1$ has no soln.

$\rightarrow$ Simple rank nullity to get $N(A) = \{0\} \rightarrow R(A) = 2$.

Q8. $y = \sqrt{r^2 - x^2}$ over interval $r^2 < x < r$, r fixed between 0 & 1.

S_r : surface area obtained by rotating this curve above x-axis. $S = \sup_{0<r<1} S_r$.

... is $\cos\theta$.

Slope is

& $h = r\sin\theta$

Area at x:

$\pi h \, dx \cdot \frac{r}{\sqrt{r^2 - x^2}}$

$= \pi r \, dx$

$\sin\theta = \frac{\sqrt{r^2 - x^2}}{r}$ $\quad h = r \cdot \sqrt{r^2 - x^2}$

$\int_{r^2}^{r} 2\pi r \, dx = 2\pi r (r - r^2)$

$\frac{d}{dr}(\pi r^2 - \pi r^3)$

$2r\pi = 3\pi r^2$

$\frac{2}{3} = r$

$S = 2\pi \cdot \frac{2}{3}\left(\frac{2}{3} - \frac{4}{9}\right)$

$= 2\pi \cdot \frac{4}{9}\left(\frac{1}{3}\right)$

Q9. Make your martingale.

$N(t)$: poisson process, rate λ. $S(t) = e^{N(t)\log(c) - \lambda t}$. c s.t $S(t)$ is martingale

$\rightarrow E(S(t) \mid F_u) = S_u = c^{N(u)} e^{-\lambda u}$ $\quad t > u$

$\rightarrow S_t = c^{N(t)} e^{-\lambda t} = c^{N(u) + N(t-u)} e^{-\lambda(u + t - u)}$

$= S_u \cdot c^{N(t-u)} e^{-\lambda(t-u)}$

($N_t - N_s$ has poisson distn w para $\lambda(t-s)$) & indep of random var N_r, $r \leq s$)

$E(S_t \mid F_u) = S_u e^{-\lambda(t-u)} E\left(c^{N(t-u)}\right)$

$= S_u e^{-\lambda(t-u)} \cdot \sum_{k=0}^{\infty} \frac{c^k (\lambda(t-u))^k e^{-\lambda(t-u)}}{k!}$

$= S_u e^{-\lambda(t-u)} \left(e^{\lambda(t-u)(c-1)} \sum_{k=0}^{\infty} \frac{(\lambda c(t-u))^k e^{-\lambda(t-u)c}}{k!} \right)$

$= S_u e^{-\lambda(t-u)} e^{\lambda(t-u)(c-1)} = S_u e^{\lambda(t-u)(c-2)} = S_u$

$\therefore c = 2$

Q10. Largest Inscribed Circle:

Area of largest circle that can be inscribed inside Δ : (16, 16, 24)

$\rightarrow$ only 1 circle that touches all 3 sides inside Δ.

We know $rs = \Delta = \frac{ah}{2}$

$S = 12 + 8 + 8 = 28$

$a = 24,\ h = \sqrt{16^2 - 12^2} = 4\sqrt{4^2 - 3^2} = 4\sqrt{7}$

$28 r = 12 \times 4\sqrt{7}$

$r = \frac{12}{\sqrt{7}}$

$\pi r^2 = \pi \times \frac{12^2}{7} = \frac{144\pi}{7}$

Q11. Rotation Matrix Ranked

$R_{\alpha,\beta,\gamma}$: rotatn matrix in $\mathbb{R}^3$ by α, β, γ radians ccw x, y, z axes

$\text{Rank}(R_{\alpha,\beta,\gamma})^2 + \text{Null}(R_{\alpha,\beta,\gamma})^2$

easy : Rank has to be 3 : 9

Q12. W_t . SBM . Var of $X = \int_0^2 \sqrt{t}\, e^{\frac{W_t^2}{8}} dW_t$ $E(X) = 0$ ✓

$\therefore \text{Var}(X) = \langle x, x \rangle \quad dx = \sqrt{t} e^{W_t^2/8} dW_t$

$dx \cdot dx = t e^{W_t^2/4} dt$ (tonelli)

$\text{Var}(X) = E\left(\int_0^2 t e^{W_t^2/4} dt\right) = \int_0^2 t\, E\left(e^{W_t^2/4}\right) dt$

Now $E\left(e^{W_t^2/4}\right) = \int_{-\infty}^{\infty} \frac{e^{x^2/4} \cdot e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx$ $\quad \frac{1}{4} - \frac{1}{t2} < 0 \Rightarrow t < 2$ ✓

$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}\left(\frac{1}{t} - \frac{1}{2}\right)}}{\sqrt{2\pi t}} dx = \frac{3}{\sqrt{t}} \int_{-\infty}^{\infty} \frac{e^{-x^2/2 3^2} dx}{3\sqrt{2\pi}} = \frac{3}{\sqrt{t}} = \sqrt{\frac{2}{2-t}}$ $\quad 3 = \frac{\sqrt{2t}}{\sqrt{2-t}}$

$\therefore \text{Var}(X) = \int_0^2 \frac{\sqrt{2}\, t\, dt}{\sqrt{2-t}} \Rightarrow 2 - t = x \quad \int_0^2 \frac{\sqrt{2}(2-x)\, dx}{\sqrt{x}}$

$= \sqrt{2} \int_0^2 \left(\frac{2}{\sqrt{x}} - \sqrt{x}\right) dx = 4\sqrt{2}\, x^{1/2} \Big]_0^2 - \frac{2}{3} \cdot \sqrt{2}\, x^{3/2} \Big]_0^2$

$= 8 - \frac{8}{3} = \frac{16}{3}$

Q13. W_t : SBM. Compute $\text{Var}\left(\int_0^t W_s\, dW_s\right)$.

$\rightarrow X = \int_0^t W_s\, dW_s \Rightarrow \text{Var}(X) = \int_0^t W_s^2 ds$

$\rightarrow E\left(\int_0^t W_s^2 ds\right) = \int_0^t E(W_s^2)\, ds$ (Tonelli, integrand +ve)

$$= \int_0^t \frac{x^2 e^{-x^2/2s}}{\sqrt{2\pi s}} ds$$

. MGF of $W_s \sim \sqrt{s}Z$ is $E(e^{\lambda\sqrt{s}Z})$

$= e^{s\lambda^2/2} = 1 + \frac{s\lambda^2}{2} + \cdots$

$\therefore E(W_s^2) = s$

$$= \int_0^t s\,ds = \frac{t^2}{2}$$

Q14: Project Matrix Ranked.

v. $P_v = \frac{vv^T}{\|v\|^2}$. $\mathrm{Rank}(P_v)^2 + \mathrm{Nullity}(P_v)^2$ as a funcn of n

$\rightarrow$ $P_v = \frac{[v_1 v, v_2 v, \ldots v_n v]}{\|v\|^2}$ $\rightarrow$ Rank has to be 1 since $C(A) = \mathrm{span}\{v\}$

$\rightarrow$ nullity $n-1$

$\therefore (n-1)^2 + 1 = n^2 - 2n + 2$

Q15. Idempotent Eigenvalues: 10×10 (non-defective) matrix A: $\mathrm{null}(A) = 4$. λ: only possible nonzero eigenvalue of A & γ: dim (eigenspace). Find $\lambda^2 + \gamma^2$.

$\rightarrow$ $A^2 = A \Rightarrow$ $Av = \lambda v \Rightarrow A^2 v = \lambda^2 v$: $Av = \lambda^2 v$ $\Rightarrow$ $\lambda = 0$ or 1. Since matrix non-defective, rank(A) =

now $\mathrm{null}(A) = 4 \Rightarrow \mathrm{rank}(A) = 6$.

$\dim(E_1) = 6 \Rightarrow 1 + 6^2 = 37$

Q16: $11^{2360} \% 50$

$$\boxed{\Rightarrow (1+10)^{2360} \% 50 = \left(1 + 2360\cdot 10 + \binom{2360}{2}\cdot 10^2 + \cdots\right) \% 50 = 1}$$

Q17. W_t: SBM. $\mathrm{Var}\left(\int_0^t W_s ds\right)$ as a funcn of t.

$\rightarrow$ $dx = W_s ds$ $\quad d(tW_t) = W_t dt + t dW_t \Rightarrow tW_t - \int_0^t s\,dW_s = \int_0^t W_t dt$.

$E\left(\left(t^2 W_t^2 - 2tW_t \int_0^t s\,dW_s + \left(\int_0^t s\,dW_s\right)^2\right)\right)$

$= t^3 - \underbrace{2tE\left(\int_0^t W_t \times s\,dW_s\right)}_{\text{to calculate only this}} + \underbrace{E\left(\left(\int_0^t s\,dW_s\right)^2\right)}$ $\rightarrow \mathrm{Var}\left(\int_0^t s\,dW_s\right)$

$= \int_0^t s^2 ds = \frac{t^3}{3}$

$t^3 - 2t\int_0^t sE(W_t dW_s) + \frac{t^3}{3}$

$= t^3 - 2t\left(\int_0^t s\cdot ds\right) + \frac{t^3}{3} = t^3 - 2t\cdot\frac{t^2}{2} + \frac{t^3}{3} =$

fundamental principles

(i did using ...
but faster soln is Ito isometry

($\int W_s\,ds = sW_s\Big|_0^t - \int_0^t s\,dW_s = \int_0^t (t-s)\,dW_s$) so var $= \int_0^t (t-s)^2 ds = \frac{t^3}{3}$)

Q18. W_t SBM. $P(W_1>0, W_2>0)$

→ $W_2 = W_{0,1} + W_{1,2}$ $\quad W_{0,1} \ge 0$

if $W_{0,1} = x$, $W_{1,2} \ge -x$

$$\int_0^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-x}^\infty \frac{e^{-t^2/2}\,dt}{\sqrt{2\pi}}\,dx = \int_0^\infty \int_{-x}^\infty \frac{e^{-x^2/2 - t^2/2}(dt\,dx)}{2\pi}$$

change of variables to r, θ

$x = r\cos\theta$, $y = r\sin\theta$, Jacobian $= r$

$$\Rightarrow \theta : \left(-\frac{\pi}{4}, \frac{\pi}{2}\right) \quad \int_0^\infty \int_{-\pi/4}^{\pi/2} \frac{e^{-r^2/2}\, r\,dr\,d\theta}{2\pi} = \frac{1}{2}\int_0^\infty e^{-r^2/2}\, r\,dr \times \frac{3}{4}$$

$$= \frac{3}{8} e^{-r^2/2}\Big]_0^\infty = \frac{3}{8}$$

Q19: W_t SBM, $T_a = \inf\{t>0 : |W_t| > a\}$ for $a>0$. $E(e^{-\lambda T_a})$

→ Making martingales for T_a^n ?

$$\Rightarrow d(W_t^{2n}) = 2n W_t^{2n-1} dW_t + (n)(2n-1) W_t^{2n-2} dt$$

$$\therefore W_t^{2n} - \int_0^t (n)(2n-1) W_s^{2n-2} ds \quad \text{is a martingale}$$

$$d(t W_s^{2n-2}) = \underbrace{W_s^{2n-2} dt}_{} + (2n-2) W_s^{2n-3} dW_t + t(n-1)(2n-3) W_s^{2n-4} dt$$

$$W_t^{2n} - \left(t W_s^{2n-2} - \int_0^t s(n-1)(2n-3) W_s^{2n-4} ds\right) n(2n-1) \quad \text{is a martingale}$$

→ New approach:

$G(t) = e^{\theta W_t - \theta^2 t/2}$ is a martingale (known formula)

$$E\left(e^{\theta W_t - \theta^2 t/2} \mid F_s\right) = E\left(G(s) \times e^{\theta(W_t - W_s) - \frac{\theta^2}{2}(t-s)} \mid F_s\right)$$

$$= G(s) \times e^{-\frac{\theta^2}{2}(t-s)} E\left(e^{\theta(W_{t-s})}\right)) \rightarrow \text{mgf of } W_{t-s}$$

$$= G(s)\, e^{-\frac{\theta^2}{2}\times(t-s)} \times e^{(t-s)\theta^2/2} = G(s)$$. Hence proved.

now using this,

$$E\left(e^{\theta W_T - \theta^2 T/2}\right) = 1$$

$$\therefore E\left(e^{-\theta^2 T/2}\right) E\left(e^{\theta W_T}\right) = 1$$

independent since W_T symmetric wrt T

$W_T = -a$ or $W_T = a$ w p $1/2$

$$\therefore E\left(e^{-\theta^2 T/2}\right)\left(e^{-a\theta} + e^{a\theta}\right) = 2$$

$$\frac{\theta^2}{2} = \lambda \qquad \theta = \sqrt{2\lambda}$$

$$\therefore E\left(e^{-\lambda T}\right) = \frac{2}{e^{-a\sqrt{2\lambda}} + e^{a\sqrt{2\lambda}}}$$

$\Rightarrow$ This gives me answer for all moments of stopping times!! just crank out the taylor series & look for coefficients of λ^n.

Q20. $dS_t = 3S_t\,dt + 4S_t\,dW_t$. $X_t = \ln(S_t)$ $\quad x = f(y)$ $\quad dx = f'(y)\,dy + \frac{1}{2} f''(y)(dS_t)^2$

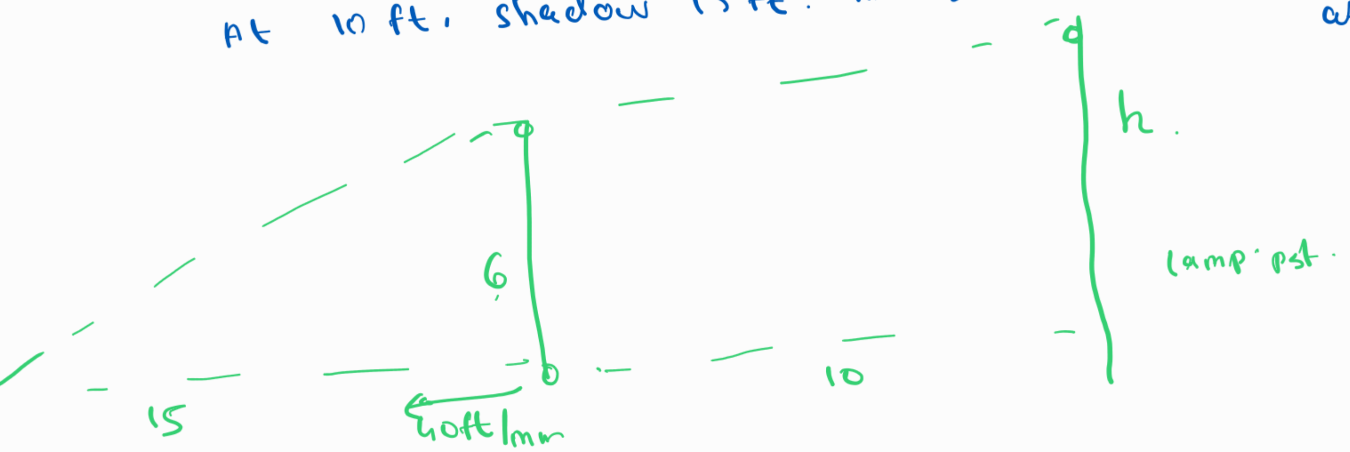
(und)

$$\therefore dX_t = d(\ln(S_t)) + \frac{1}{2}(\ln(S_t))''(dS_t)^2$$

$$= \frac{dS_t}{S_t} - \frac{\cancel{16}\,8\,\cancel{S_t^2}\,dt}{\cancel{2}\cancel{S_t^2}}$$

$$= -5\,dt + 4\,dW_t$$

Q21. 6 feet tall, walking at 40 ft/min. At 10 ft, shadow 15 ft. find rate of increase of shadow length at 40ft.



$$\frac{6}{h} = \frac{15}{25} \quad \frac{3}{5} \Rightarrow h = 10$$

rate at which shadow

$\frac{6}{10} = \frac{s}{(s+d)} \Rightarrow \frac{6}{4} = \frac{s}{d} \Rightarrow \frac{3}{2}d = s$

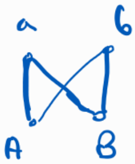
$\therefore \frac{3}{2}\frac{d(d)}{dt} = \frac{ds}{dt}$

$= 60$

Q22. 5 companies, 5 ppl, each applicant - 2 companies, each company 2 applicants. Number of ways?

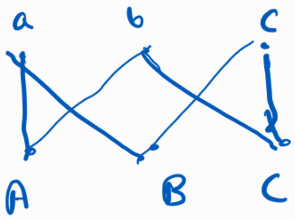
$\therefore C_4 + C_6$ or C_{10}

Now number of ways if C_4



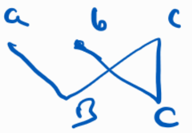
$\rightarrow 4$

$\rightarrow$ for C_6 $\rightarrow$



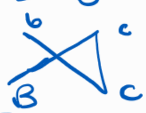
$\Rightarrow 21$

$(A-\phi)$



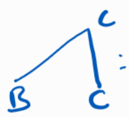
$\rightarrow 2\times2\times3 = 12$

$(A-a)$:

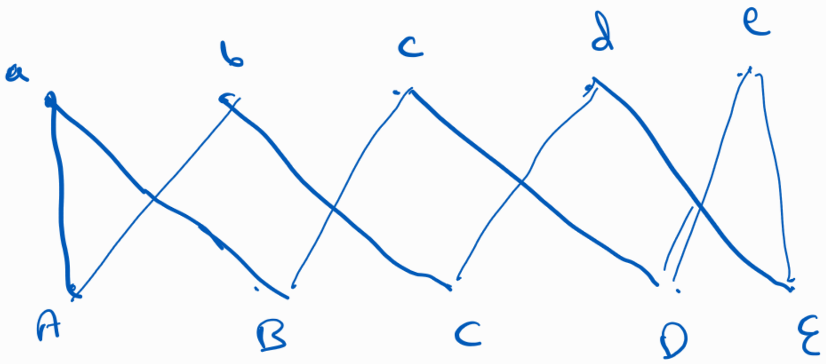


$\Rightarrow 2\times3 = 6$

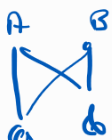
$(A-(a,b))$:



: 3



$\Rightarrow C_4 + C_6$ or C_{10}



a

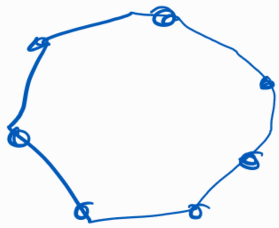
$\binom{5}{2} \times \binom{5}{2} \times (6)$

$= 600$

A-a

A

C10 :



(reflectn same)

$\frac{3! \times 2!}{2} = 3! = 6$

A

C10

$\frac{5! \times 4!}{2} = 60 \times 24$

$= 1440$

$\Rightarrow 1440 + 600$

$= 2040$