Q1. Squared BGM :

S_t : geometric Brownian Motion w $m = \sigma = 1$, $S_0 = 1$.

find $E[S_2^2]$. : $dS_t = S_t(m\,dt + \sigma\,dB_t) = S_t(dt + dB_t)$

$\rightarrow S_t = S_0\, e^{((1-\frac{1}{2})t + B_t)} = e^{t/2 + B_t}$

$$E(S_2^2) = E\left((e^{1+B_2})^2\right) = E\left(e^{2+2B_2}\right) \quad : B_2 \sim N(0,2)$$

$$= \int_{-\infty}^{\infty} \frac{e^{2+2x} \cdot e^{-\frac{1}{2}\frac{x^2}{2}}}{\sqrt{2\cdot 2\pi}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-x^2/4+2x+2}}{\sqrt{\pi}}$$

$$= \frac{e^6}{2}\int_{-\infty}^{\infty} \frac{e^{-(\frac{x}{2}-2)^2}}{\sqrt{\pi}}dx = \frac{e^6}{\sqrt{2}}\int_{-\infty}^{\infty} \frac{e^{-\left(\frac{x-4}{2}\right)^2}dx}{\sqrt{2\pi}}$$

$x = \sqrt{2}t$: $\frac{e^6}{\cancel{\sqrt{2}}}\int_{-\infty}^{\infty} \frac{e^{-(t-2\sqrt{2})^2/2}\cdot\cancel{\sqrt{2}}\,dt}{\sqrt{2\pi}} \rightarrow 1$

Normal distn w $m = 2\sqrt{2} \Rightarrow$

$\therefore e^6$ remains

Q2: Make your Martingale