Métodos de Integração e Aplicações da Integral Definida

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1 Exercícios 7.4

32) Calcular a integral indefinida.

$$\int \frac{x}{\sqrt{x^2-1}} \tan^3 \sqrt{x^2-1}dx$$

$$u = x^2 - 1$$
$$du = 2xdx$$

$$= \int \frac{\tan^3\left(\sqrt{u}\right)}{2\sqrt{u}}du$$
$$= \frac{1}{2} \cdot \int \frac{\tan^3\left(\sqrt{u}\right)}{\sqrt{u}}du$$

$$v = \sqrt{u}$$
$$dv = \frac{1}{2\sqrt{u}}du$$

$$= \frac{1}{2} \cdot \int 2 \tan^3(v)\, dv$$
$$= \frac{1}{2} \cdot 2 \cdot \int \tan^3(v)\, dv$$
$$= \frac{1}{2} \cdot 2 \cdot \int \tan^2(v) \tan(v)\, dv$$

$$\tan^2(x) = -1 + \sec^2(x)$$

$$= \frac{1}{2} \cdot 2 \cdot \int \left(-1 + \sec^2(v)\right) \tan(v)\, dv$$

$$w = \sec(v)$$
$$dw = \sec(x) \tan(x)\, dv$$

$$= \frac{1}{2} \cdot 2 \cdot \int \frac{-1 + w^2}{w} dw$$
$$= \frac{1}{2} \cdot 2 \cdot \int -\frac{1}{w} + w dw$$
$$= \frac{1}{2} \cdot 2\left(-\int \frac{1}{w} dw + \int w dw\right)$$
$$= \frac{1}{2} \cdot 2\left(-\ln|w| + \frac{w^2}{2}\right)$$

$$w = \sec(v)$$

$$= \frac{1}{2} \cdot 2\left(-\ln|\sec(v)| + \frac{\sec(v)^2}{2}\right)$$

$$v = \sqrt{u}$$

$$= \frac{1}{2} \cdot 2\left(-\ln\left|\sec\left(\sqrt{u}\right)\right| + \frac{\sec\left(\sqrt{u}\right)^2}{2}\right)$$

$$u = x^2 - 1$$

$$\begin{aligned}
&= \frac{1}{2} \cdot\ 2\left(-\ln\left|\sec\left(\sqrt{x^2-1}\right)\right| + \frac{\sec^2\left(\sqrt{x^2-1}\right)}{2}\right) \\
&= -\ln\left|\sec\left(\sqrt{x^2-1}\right)\right| + \frac{1}{2}\sec^2\left(\sqrt{x^2-1}\right) \\
&= -\ln\left|\sec\left(\sqrt{x^2-1}\right)\right| + \frac{1}{2}\sec^2\left(\sqrt{x^2-1}\right) + C
\end{aligned}$$

70) Calcular a integral definida.

$$\int_1^2 \frac{dt}{t^4\sqrt{4+t^2}}$$

$$t = 2\tan(u)$$
$$dt = 2\sec^2(u)$$

$$= \int_{\arctan\left(\frac{1}{2}\right)}^{\frac{\pi}{4}} \frac{\sec(u)}{16\tan^4(u)}du$$
$$= \frac{1}{16}\cdot\int_{\arctan\left(\frac{1}{2}\right)}^{\frac{\pi}{4}} \frac{\sec(u)}{\tan^4(u)}du$$

$$v = \tan\left(\frac{u}{2}\right)$$
$$dv = \sec^2\left(\frac{u}{2}\right)$$

$$= \frac{1}{16}\cdot\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{\left(1-v^2\right)^3}{8v^4}dv$$
$$= \frac{1}{16}\cdot\frac{1}{8}\cdot\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{\left(1-v^2\right)^3}{v^4}dv$$
$$= \frac{1}{16}\cdot\frac{1}{8}\cdot\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{1}{v^4}-\frac{3}{v^2}+3-v^2dv$$
$$= \frac{1}{16}\cdot\frac{1}{8}\left(\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{1}{v^4}dv - \int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{3}{v^2}dv + \int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} 3dv - \int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} v^2dv\right)$$

$$\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{1}{v^4} dv = -\frac{1}{3\left(\sqrt{2}-1\right)^3} + \frac{1}{3\tan^3\left(\frac{1}{2}\arctan\left(\frac{1}{2}\right)\right)}$$

$$\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} \frac{3}{v^2} dv = 3\left(-\sqrt{2}-1+\frac{1}{\tan\left(\frac{1}{2}\arctan\left(\frac{1}{2}\right)\right)}\right)$$

$$\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} 3 dv = 3\left(\sqrt{2}-1\right) - 3\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)$$

$$\int_{\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right)}^{\sqrt{2}-1} v^2 dv = \frac{\left(\sqrt{2}-1\right)^3 - \tan^3\left(\frac{1}{2}\arctan\left(\frac{1}{2}\right)\right)}{3}$$

$$= \frac{1}{16} \cdot \frac{1}{8}\left(-\frac{1}{3\left(\sqrt{2}-1\right)^3} + \frac{1}{3\tan^3\left(\frac{1}{2}\arctan\left(\frac{1}{2}\right)\right)} - 3\left(-\sqrt{2} - 1 + \frac{1}{\tan\left(\frac{1}{2}\arctan\left(\frac{1}{2}\right)\right)}\right) + 3\left(\sqrt{2} - 1\right) - 3\tan\left(\frac{\arctan\left(\frac{1}{2}\right)}{2}\right) - \frac{\left(\sqrt{2}-1\right)^3 - \tan^3\left(\frac{1}{2}\arctan\left(\frac{1}{2}\right)\right)}{3}\right)$$

2 Exercícios 7.9

3) Calcular a integral indefinida.

$$\int \frac{2dx}{sen\,(x) + tan\,(x)}$$

$$u = \tan\left(\frac{x}{2}\right)$$
$$du = \sec^2\left(\frac{x}{2}\right)$$

$$= 2 \cdot \int -\frac{(u+1)(u-1)}{2u} du$$
$$= 2\left(-\frac{1}{2} \cdot \int \frac{(u+1)(u-1)}{u} du\right)$$
$$= 2\left(-\frac{1}{2} \cdot \int u - \frac{1}{u} du\right)$$
$$= 2\left(-\frac{1}{2}\left(\int u du - \int \frac{1}{u} du\right)\right)$$

$$\int u du = \frac{u^2}{2}$$
$$\int \frac{1}{u} du = \ln|u|$$

$$= 2\left(-\frac{1}{2}\left(\frac{u^2}{2} - \ln|u|\right)\right)$$

$$u = \tan\left(\frac{x}{2}\right)$$

$$= 2\left(-\frac{1}{2}\left(\frac{\tan^2\left(\frac{x}{2}\right)}{2} - \ln\left|\tan\left(\frac{x}{2}\right)\right|\right)\right)$$
$$= -\frac{1}{2}\tan^2\left(\frac{x}{2}\right) + \ln\left|\tan\left(\frac{x}{2}\right)\right|$$
$$= -\frac{1}{2}\tan^2\left(\frac{x}{2}\right) + \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

3 Exercícios 8.4

7) Encontrar o comprimento de arco da curva dada.

$$y = \frac{1}{2}\left(e^x + e^{-x}\right), \text{ de } (0,1) \text{ a } \left(1, \frac{e + e^{-}1}{2}\right)$$

Primeiro montamos o sistema:

$$\begin{cases} x &= t \\ y &= \frac{1}{2}\left(e^t + e^{-t}\right) \end{cases}, t \in [0,1]$$

Então calculamos as derivadas:

$$\begin{aligned} t' &= 1 \\ \left(\frac{1}{2}\left(e^t + e^{-t}\right)\right)' &= \frac{1}{2}\left(e^t - e^{-t}\right) \end{aligned}$$

Por fim aplicamos a fórmula:

$$s = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

$$\begin{aligned} s &= \int_0^1 \sqrt{[1]^2 + [\frac{1}{2}(e^x + e^{-x})]^2} dt \\ &= \int_0^1 \sqrt{1 + \left(\frac{1}{2}\left(e^t + e^{-t}\right)\right)^2} dt \\ &= \int_0^1 \sqrt{1 + \frac{(e^t + e^{-t})^2}{4}} dt \\ &= \int_0^1 \sqrt{\frac{e^{-2t} + e^{2t} + 6}{4}} dt \\ &= \int_0^1 \frac{\sqrt{e^{-2t} + e^{2t} + 6}}{\sqrt{4}} dt \\ &= \int_0^1 \frac{\sqrt{e^{-2t} + e^{2t} + 6}}{2} dt \end{aligned}$$

4 Exercícios 8.7

8) Determinar o volume do sólido de revolução gerado pela rotação, em torno do eixo dos y da região R delimitada pelos gráficos da equação dada.

$$x = y^2 + 1, x = \frac{1}{2}, y = -2 \text{ e } y = 2$$

Primeiro definimos $f(y)$:

$$\begin{cases} x &= y^2 + 1 \\ x &= \frac{1}{2} \end{cases}$$

$$\frac{1}{2} = y^2 + 1$$

$$y^2 + \frac{1}{2} = 0$$

$$f(y) = y^2 + \frac{1}{2}$$

Então aplicamos a fórmula:

$$V = \pi \int_a^b [f(y)]^2 dy$$

$$a = -2$$
$$b = 2$$

$$\begin{aligned} V &= \pi \int_{-2}^{2} [y^2 + \frac{1}{2}]^2 dy \\ &= \pi \int_{-2}^{2} y^4 + y^2 + \frac{1}{4} dy \\ &= \pi \int_{-2}^{2} y^4 dy + \int_{-2}^{2} y^2 dy + \int_{-2}^{2} \frac{1}{4} dy \end{aligned}$$

$$\int_{-2}^{2} y^4 dy = \frac{64}{5}$$

$$\int_{-2}^{2} y^2 dy = \frac{16}{3}$$

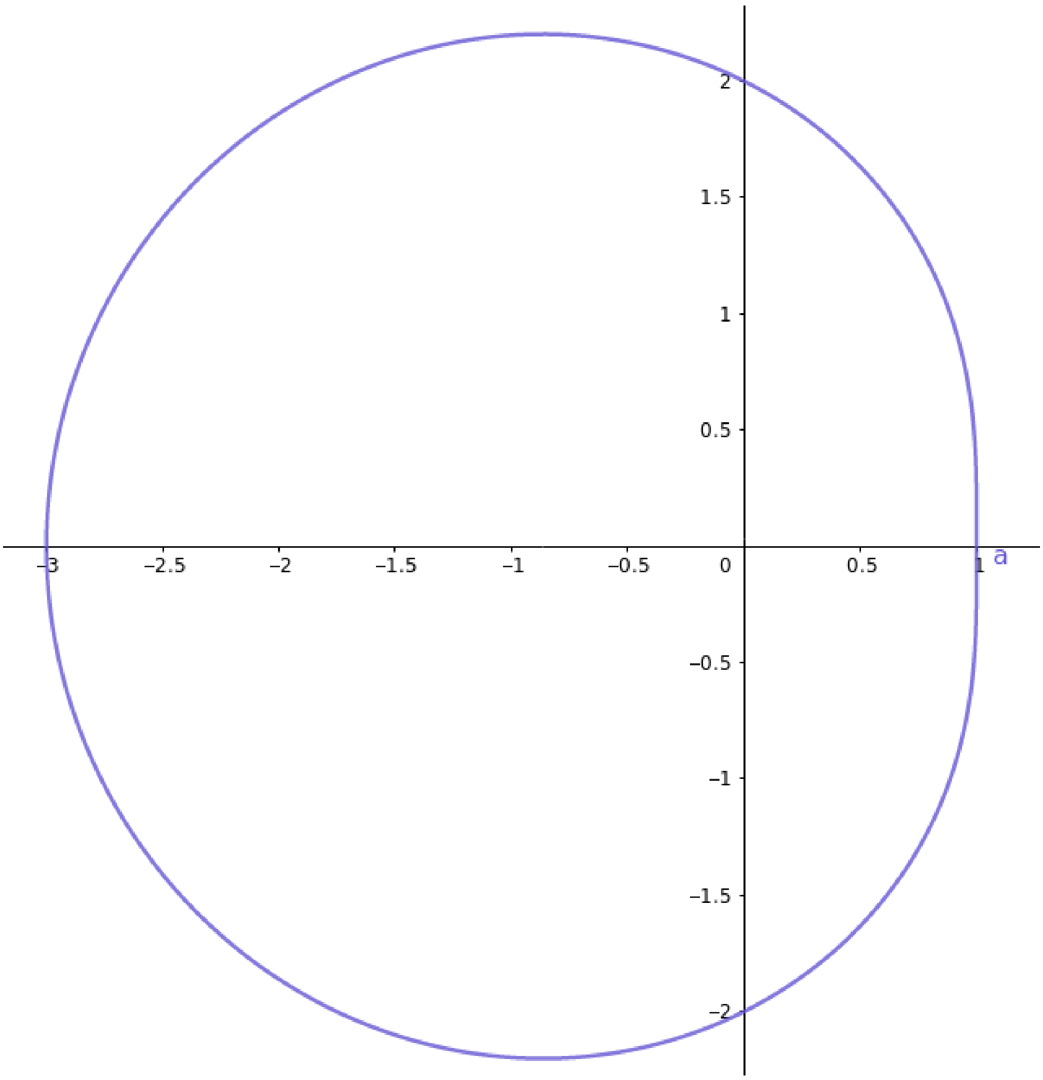
$$\int_{-2}^{2} \frac{1}{4} dy = 1$$

$$
\begin{aligned}
&= \pi\frac{64}{5} + \frac{16}{3} + 1 \\
&= \pi\frac{287}{15} \\
&= \frac{287\pi}{15} \\
&\approx 60.10913\ldots
\end{aligned}
$$

5 Exercícios 8.11

15) Esboçar o gráfico da curva dada em coordenada polar.

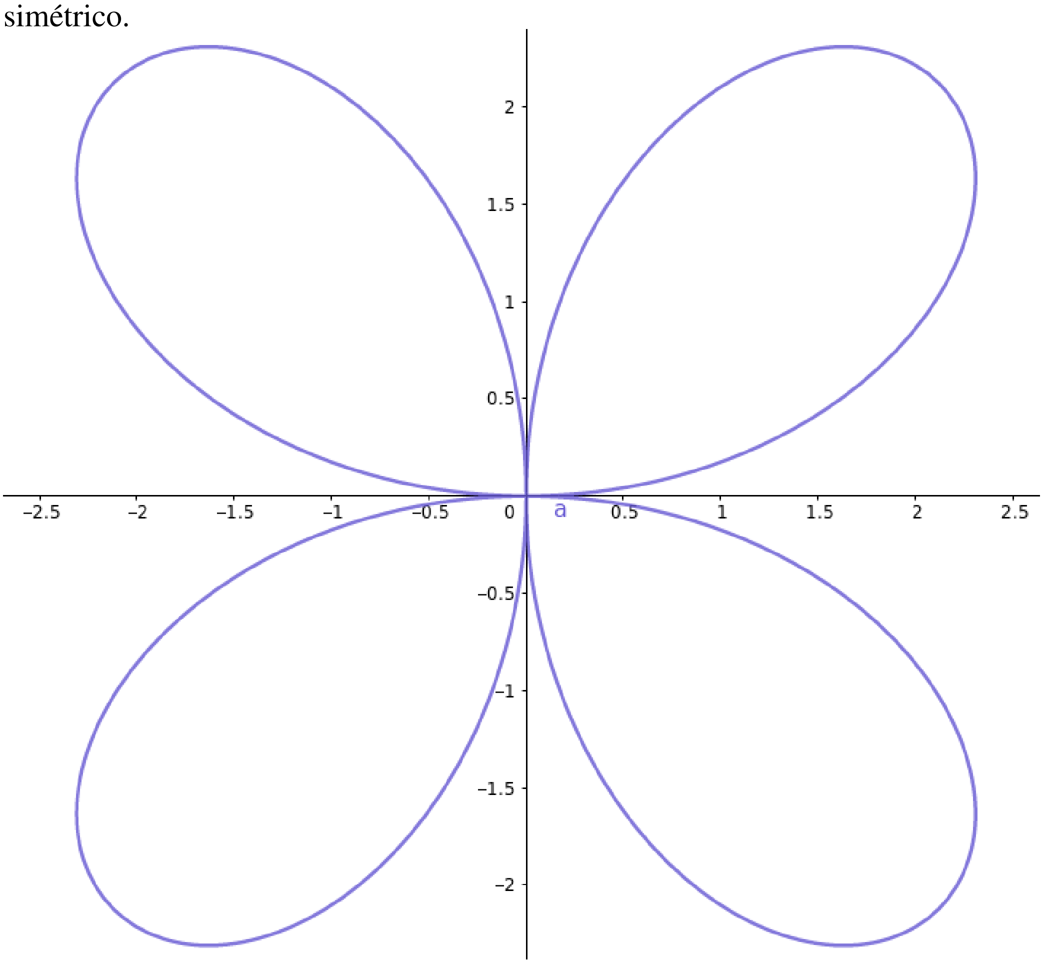
$$r = 2 - \cos\theta$$



51) Calcular a área limitada pela curva dada.

$$r = 3\sin 2\theta$$

Analisando o gráfico abaixo podemos notar que o gráfico dessa função é simétrico.



Portanto, para descobrir a área, basta calcular no primeiro quadrante e multiplicar por quatro. Aplica-se, então, a seguinte fórmula.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$\alpha = 0$$
$$\beta = \frac{\pi}{2}$$

$$
\begin{aligned}
&\frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} (3\sin(2\theta))^2\, d\theta \\
&= \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} 9\sin^2(2\theta)\, d\theta \\
&= \frac{1}{2} \cdot 9 \cdot \int_0^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta
\end{aligned}
$$

$$
\sin^2(x) = \frac{1-\cos(2x)}{2}
$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot 9 \cdot \int_0^{\frac{\pi}{2}} \frac{1-\cos(2\cdot\ 2\theta)}{2} d\theta \\
&= \frac{1}{2} \cdot 9 \cdot \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos(4\theta))\, d\theta \\
&= \frac{1}{2} \cdot 9 \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\, d\theta \\
&= \frac{1}{2} \cdot 9 \cdot \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} 1 d\theta - \int_0^{\frac{\pi}{2}} \cos(4\theta)\, d\theta\right)
\end{aligned}
$$

$$
\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}
$$

$$
\int_0^{\frac{\pi}{2}} \cos(4\theta)\, d\theta = 0
$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot 9 \cdot \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2} \cdot \frac{9\pi}{4} \\
&= \frac{9\pi}{8} \\
&\approx 3.53429\ldots
\end{aligned}
$$

Por fim, multiplicamos o resultado por quatro.

$$
\begin{aligned}
&4 \cdot \frac{9\pi}{8} \\
&= \frac{9\pi}{2} \\
&\approx 14.13716\ldots
\end{aligned}
$$